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Optimal control of fredholm integral equations

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In this paper a numerical method is proposed for solving optimal control problems governed by Fredholm integral equations (OCF). The method is based upon sinc wavelet and parametrization method and transforms the problem to a nonlinear programming problem. Control function $u(t)$ and state function are approximated by a finite combination of elements of a basis and by a finite combination of sinc wavelet respectively. Numerical examples show the validity and applicability of the proposed method.

Keywords: Sinc function, Fredholm integral equation, Optimal control problem, Approximation.

1 Introduction

The optimal control problem governed by Fredholms integral equations is used in the modeling of a wide variety of real process in science and technology. Existence and uniqueness of the optimal control problem governed by Fredholm integral equations is presented in [7]. Several numerical methods for approximating the solution of the (OCF) are existed in the literature. A method of successive approximations which introduced in [13] are extended in [3, 14]. Recently by using a collective Gauss-Seidel scheme and a multigrid scheme, an iterative method is proposed for (OCF) of second kind.
Nowadays, wavelet theory has attracted considerable attention due to the advantages wavelets have over traditional Fourier transforms in accurately approximating functions that have discontinuities and sharp peaks. Wavelets have become a popular tool for speech processing, identification and the modeling and control of the dynamical behavior of systems.
Recently optimal control problems described by ODE is recently solved by a Haar wavelets method in [4], but no attempts have been made to apply the wavelet to solve (OCF).
Since the sinc method is a highly efficient numerical method developed by Frank Stenger,

we use an approximation of state function by a combination of sinc function with unknown parametrization and control function by a combination of elements of basis [8] .
In this paper, we first introduce the sinc wavelets properties, then we assume that the state variables in the (OCF) be expressed in the form of sinc function with unknown coefficients and the control variables is approximated in a finite dimension space. This method is based on reducing the (OCF) to an optimal problem governed by fredholm integral equations.
This paper develops in six section as follows: Section 2 defines the optimal control problem of ferdholm integration. Section 3 is the introduction of Sinc function properties. In Section 4 we describe our approach. In Section 5 we introduce our algorithm. In Section 6 we report our numerical finding. Section 7 is conclusion.

2 Problem statement

Optimal control problem of nonlinear Fredholm integral equations is formulated as the following:

$$\text{Minimize} \;\; J(x,u)=\int_{t_0}^{t_f} f(t,x(t),u(t))dt, \tag{2.1}$$

subject to:

$$x(t)=G(t)+\int_{t_0}^{t_f} K(\xi,t,x(\xi),u(\xi))d\xi, \tag{2.2}$$

where

$$x(t)=(x_1(t),x_2(t),\cdots,x_l(t))^t,$$

$$G(t)=(g_1(t),g_2(t),\cdots,g_l(t))^t,$$

$$K(\xi,t,u(\xi),x(\xi))=(k_1(\xi,t,u(\xi),x(\xi)),k_2(\xi,t,u(\xi),x(\xi)),\cdots,k_l(\xi,t,u(\xi),x(\xi)))^t,$$

and $u(t)=(u_1(t),u_2(t),\cdots,u_m(t))^t$, and $f\in C([t_0,t_f]\times\mathbb{R}^l\times\mathbb{R}^m)$.

3 Sinc function properties

The sinc function is defined on the whole real line by

$$sinc(t)=\begin{cases}\frac{\sin(\pi t)}{\pi t}, & t\neq 0,\\ 1, & t=0.\end{cases} \tag{3.1}$$

For $h>0$, the translated sinc functions with evenly spaced nodes are given as

$$S(j,h)(t)=sinc(\frac{t-jh}{h}), \qquad j=0,\pm1,\pm2,\cdots . \tag{3.2}$$

If f is defined on the real line, then for $h > 0$ the series

$$C(f,h) = \sum_{j=-\infty}^{\infty} f(jh) sinc(\frac{t-jh}{h}),$$

is called the Whittaker cardinal expansion of f whenever this series converges. These properties are derived in the infinite strip D_d of the complex plane where for $d > 0$

$$D_d = \{\zeta = \xi + i\eta \ : |\eta| < d \leq \frac{\pi}{2}\}.$$

In addition, we choose

$$h = \sqrt{\frac{\pi d}{\alpha N}}, \quad 0 < \alpha \leq 1. \tag{3.3}$$

To construct approximation on interval (a, b), which are used in this paper, consider the conformal map

$$\phi(t) = \ln(\frac{t-a}{b-t}). \tag{3.4}$$

The map ϕ carries the eye-shaped region

$$D_E = \{z = x + iy : |arg(\frac{z-a}{b-z})| < d \leq \frac{\pi}{2}\},$$

onto D_d. The basis functions on (a, b) are then given by

$$S(j,h) \circ \phi(t) = sinc(\frac{\phi(t) - jh}{h}).$$

Notice that these functions exhibit Kronecker delta behavior on the grid points $t_j \in (a, b)$ defined by

$$t_j = \phi^{-1}(jh) = \frac{a + be^{jh}}{1 + e^{jh}}. \tag{3.5}$$

The mesh size h represents the mesh size in D_d for the unform grids $\{jh\}$, $j = 0, \pm 1, \pm 2, \ldots$. The sinc grid points $t_j \in (a, b)$ in D_E will be denoted by t_j because they are real, let us also define ρ by $\rho(z) = e^{\phi(z)}$, and Γ be defined by $\Gamma = \{z \in C : z = \phi^{-1}(t), t \in \mathbb{R}\}$, we need the following definitions and theorems in [2].

Theorem 3.1. *If $\phi F \in L_\alpha(D)$ then for all $x \in \Gamma$*

$$|F(z) - \sum_{k=-\infty}^{\infty} F(z_k) S(k,h) \circ \phi(z)| \leq \frac{2N(F\phi^{'})}{\pi d} e^{-\pi d/h}.$$

Moreover, if $|F(z)| \leq Ce^{-\alpha|\phi(z)|}$, $z \in \Gamma$, for some positive constants C and α, and if the selection $h = \sqrt{\pi d/\alpha N} \leq 2\pi d/\ln 2$, then

$$|F(z) - \sum_{k=-N}^{N} F(z_k) S(k,h) \circ \phi(z)| \leq C_2 \sqrt{N} e^{-\sqrt{\pi d \alpha N}}, z \in \Gamma,$$

where C_2 depends only on F, d and α.

The above expressions show sinc interpolation on $L_\alpha(D)$ converges exponentially [10].

Theorem 3.2. *Let $L_\alpha(D)$ be the set of all analytic functions, let $\frac{F}{\phi'} \in L_\alpha(D)$, let N be a positive integer and let h be selected by the formula*

$$h = (\frac{\pi d}{\alpha N})^{\frac{1}{2}},$$

then there exist positive constant c_1, independent of N, such that

$$|\int_\Gamma F(z)dz - h\sum_{k=-N}^{N}\frac{F(z_k)}{\phi'(z_k)}| \le c_1 e^{(-\pi d\alpha N)^{\frac{1}{2}}}.$$

Corollary 3.1. *Let $\frac{F}{\phi'} \in L_\alpha(D)$, and let h be selected by (3.3), then*

$$|\int_\Gamma F(z)S(k,h)\circ\phi(z)dz - h\frac{F(z_k)}{\phi'(z_k)}| \le c_2 e^{(-\pi d\alpha N)^{\frac{1}{2}}}. \tag{3.6}$$

Lemma 3.1. *Let ϕ be the conformal one-to-one mapping of the simply connected domain D_E onto D_E, given by (3.1). Then*

$$\delta_{ji}^{(0)} = [S(j,h)\circ\phi(t)]\mid_{t=t_i} = \begin{cases} 1, & j = i, \\ 0, & j \ne i. \end{cases} \tag{3.7}$$

4 Wavelet approach

In this section we construct continuous control function $u(t)$ by a finite combination of elements of a basis [8], $\{q_j\}$ which is dense in $C([t_0, t_f])$ as follow

$$u(t) = \sum_{j=0}^{k} w_j q_j, \tag{4.1}$$

by using sinc collocation, we approximate $x(t)$ as

$$x(t) = \sum_{j=-N}^{N} c_j S(k,h)\circ\phi(t). \tag{4.2}$$

Obviously by using (3.5), (3.7) in (4.2) we have

$$x(t_j) = c_j, \quad j = -N \dots N.$$

Replacing approximation defined in (4.1), (4.2), in cost function (2.1) we obtain

$$J(x,u) = \int_{t_0}^{t_f} f(t, \sum_{j=-N}^{N} c_j S(k,h)\circ\phi(t), \sum_{j=0}^{k} w_j q_j)dt, \tag{4.3}$$

we rewrite the fredholm integration equation (2.2) as

$$\sum_{j=-N}^{N} c_j S(j,h)\circ\phi(t) = G(t) + \int_{t_0}^{t_f} K(\xi, t, \sum_{j=-N}^{N} c_j S(j,h)\circ\phi(\xi), \sum_{j=0}^{k} w_j q_j) d\xi, \quad (4.4)$$

substituting $t = t_i,\ i = -N \dots N$ and by using theorem 3.1 in (4.4) we obtain

$$c_i - h\sum_{j=-N}^{N} \frac{K(\xi_j, t_i)c_j}{\phi'(\xi_j)} = G(t_i), \qquad i = -N \dots N, \quad (4.5)$$

equation (4.5) consists 2N + 1 nonlinear algebraic equations with 2N + k + 2 unknown $\{c_j\}_{j=-N}^{N}$ and $\{w_j\}_{j=0}^{k}$. The approximate control function and state function could be obtained by solution of the following optimal problem

$$\min_{(c_{-N},\cdots,c_N,w_0,\cdots,w_k)} \quad J_k(c_{-N},\cdots,c_N,w_0,\cdots,w_k) = \int_{t_0}^{t_f} f(t, \textstyle\sum_{j=-N}^{N} c_j S(k,h)\circ\phi(t), \sum_{j=0}^{k} w_j q_j) dt,$$

$$\text{subject to:} \qquad \begin{cases} c_{-N} - h\sum_{j=-N}^{N} \frac{K(\xi_j, t_{-N})c_j}{\phi'(\xi_j)} = G(t_{-N}) \\ c_{-N+1} - h\sum_{j=-N}^{N} \frac{K(\xi_j, t_{-N+1})c_j}{\phi'(\xi_j)} = G(t_{-N+1}) \\ \vdots \\ c_N - h\sum_{j=-N}^{N} \frac{K(\xi_j, t_N)c_j}{\phi'(\xi_j)} = G(t_N). \end{cases} \quad (4.6)$$

Assuming J_k^* as optimal value of (4.6) in kth iteration, a stopping criteria may be considered as follows

$$|J_k^* - J_{k-1}^*| < \epsilon, \quad (4.7)$$

small positive number ϵ could be chosen according to the accuracy desired. The above results have been summarized in an algorithm.

5 Algorithm of the method

In this section, we propose an algorithm basis on the above discussions. This algorithm is presented in two stages, initialization step and main step.

Initialization step: Choose $\epsilon > 0$ for the accuracy desired and set $k = 1$, and go to the main step.

Main step:

Step 1. Set $u(t)$ and $x(t)$ by (4.1), (4.2) and go to Step 2.

Step 2. Compute t_j and ξ_j by (3.5) and go to Step 3.

Step 3. Then compute nonlinear algebraic equations by (4.5).

Step 4. Finally compute optimization problem (4.6).

Step 5. Compute $J_k^* = infJ_k$ in (4.6) if $k = 1$ go to step (7) otherwise go to step 6.

Step 6. If the stopping criteria (4.7) holds, stop; Otherwise, go to step 7.

Step 7. $k = k + 1$ and go step 1.

6 Convergency of the approach

In this section we are going to study the convergency of the proposed method in section 3.

Definition 6.1. Pair $(x(t), u(t))$ is called an admissible, if it satisfies in (2.2). We define ξ as the set of admissible control functions and φ as the set of admissible pairs.
Define φ_N^r and φ^r as follows:

$$\varphi_N^r = \{(x_N(t), u_r(t)) \mid u_r(t) = \sum_{j=0}^{r} c_j q_j(t),\ u_r(t) \in \xi,\ x_N(t) = \sum_{j=-N}^{N} c_j S(k,h) \circ \phi(t)\}$$

$$\varphi^r = \{(x(t), u_r(t)),\ u_r(t) \in \xi,\ x(t) = \sum_{j=-\infty}^{\infty} c_j S(k,h) \circ \phi(t)\}$$

At first we express the following theorems.

Theorem 6.1. *Assume that the exact solution of equation (2.2), is $x_r(t)$ with approximation $u_r(t)$ for control function $u(t)$, and $x_{r,N}(t)$ is the numerical solution of the equation (2.2), then*

$$\sup_{N \in \mathbb{N}} |x_r(t) - \sum_{j=-N}^{N} c_j S(k,h) \circ \phi(\xi)| \leq C_1 \sqrt{N} e^{-\sqrt{\pi d \alpha N}}$$

where C_1 is a positive constant.
Proof:
Assume that the exact solution is $x_r(t)$, i.e.

$$x_r(t) = G(t) + \int_{t_0}^{t_f} K(\xi, t, x(\xi), u_r(\xi)) d\xi,$$

also assume that the numerical solution is $x_{r,N}(t)$, then

$$x_{r,N}(t) = G(t) + \int_{t_0}^{t_f} K(\xi, t, \sum_{j=-N}^{N} c_j S(k,h) \circ \phi(\xi), u_r(\xi)) d\xi,$$

$$\sup_{N\in\mathbb{N}} |x_r(t) - x_{r,N}(t)| \leq |\int_{t_0}^{t_f} K(\xi, t, x(\xi), u_r(\xi)) d\xi$$

$$- \int_{t_0}^{t_f} K(\xi, t, \sum_{j=-N}^{N} c_j S(k,h) \circ \phi(\xi), u_r(\xi)) d\xi|$$

$$\leq \nabla K(\xi, t, x(\xi), u_r(\xi)) |x_r(t) - \sum_{j=-N}^{N} c_j S(k,h) \circ \phi(\xi)|,$$

assume ∇K is bounded

$$\nabla K(\xi, t, x(\xi), u_r(\xi)) \leq M,$$

by using theorem (3.1), we have

$$|x_r(t) - \sum_{j=-N}^{N} c_j S(k,h) \circ \phi(\xi)| \leq C_2 \sqrt{N} e^{-\sqrt{\pi d \alpha N}},$$

then

$$\sup_{N\in\mathbb{N}} |x_r(t) - \sum_{j=-N}^{N} c_j S(k,h) \circ \phi(\xi)| \leq M C_2 \sqrt{N} e^{-\sqrt{\pi d \alpha N}}.$$

Letting $C_1 = MC_2$ completes proof of the theorem.

If we define $\alpha_N^r = \inf \varphi_N^r J$ and also assume $\inf \varphi^r J$ is finite, unique and equal to α^r, then we will have

Theorem 6.2. *Show the following relation is hold:*

$$\alpha^1 \geq \alpha^2 \geq \cdots \geq \alpha^r \geq \cdots \geq \alpha = \inf_{(x,u)\in\varphi^r} J(x,u).$$

Proof: *By definition φ_r*

$$\varphi^1 \subseteq \varphi^2 \subseteq \cdots \subseteq \varphi^r \subseteq \cdots \subseteq \varphi.$$

Theorem 6.3. *Show $lim_{r\to\infty}\ \alpha^r = \alpha$ in which $\alpha^r = \inf_{(x,u)\in\varphi^r} J(x,u)$.*
Proof: *It can be concluded that $\{\alpha^r\}$ is convergent, because it is a non-increasing and bounded sequence. By the continuity of f_0 and density of polynomials in $C(I)$, the theorem holds.*

Theorem 6.4. *([11]). Suppose that X and Y be metric space and $E \subset X$, $f : E \longrightarrow Y$, and p is limit point E*

$$lim_{x \to p} f(x) = q,$$

if and only if relation follow

$$lim_{n \to \infty} f(p_n) = q,$$

for all sequence $\{p_n\}$ in E that

$$p_n \neq p, \quad lim_{n \to \infty} p_n = p$$

Theorem 6.5. *Now we prove $lim_{k \to \infty} \ lim_{n \to \infty} \alpha_N^r = \alpha$, in which $\alpha = inf_{(x,u) \in \varphi} J(X,U) = J(X^*, U^*)$.*

Proof: *By above theorems the theorem holds.*

7 Numerical examples

In this section we three (OCF) problems will be tested by using the method discussed in section 4. In both examples we consider the monomial functions $\{t^j\}$ as dense basis of $C([t_0, t_f])$ and we choose $\alpha = 1/2$ and $d = \pi/2$ which lead to $h = \pi/\sqrt{N}$. All computations were carried out by MAPLE programming. In examples, the maximum absolute error at sinc grid points is taken as

$$\parallel E_x \parallel = \max_{-N \leq i \leq N} |x_{\text{exact solution}}(t_i) - x_{\text{our method}}(t_i)\ |.$$

Example 1. Consider the following optimal control problem which is minimization of the functional

$$J = \int_1^2 (x(t) - \cosh(2t-3) + \cosh(1))^2 + (u(t) - \frac{t}{3})^2 dt, \tag{7.1}$$

subject to:

$$x(t) = cosh(2t-3) + (\frac{t}{2} - 1)\cosh(1) - \frac{t}{2}\sinh(1) + \int_1^2 tu(\xi)x(\xi)d\xi,$$

With the optimal control function $u(t) = \frac{t}{3}$ and exact state function $x(t) = \cosh(2t-3) - \cosh(1)$.

By applying the proposed method the computed results have been shown in Table 1. We report the absolute value of the errors of our method for $N = 10$ and $N = 20$ in Tables 2.

Table.1. Numerical results in Example 1.

k	J_k^*	
	$N=10$	$N=20$
0	0.56587	0.56575
1	0.45932	2.74037×10^{-17}

Table.2. Absolute errors for $x(t)$ and $u(t)$ in Example 1 ($k = 1$).

N	E_x	E_u
10	0.700531	0.73925
20	5.22537×10^{-9}	6.77317×10^{-9}

Example 2. Consider the following optimal control of fredholm equation problem

$$\text{Minimize } \ J = \int_0^{\frac{\pi}{2}} (x(t) - sin(t))^2 + (u(t) - cos(t))^2 dt,$$

subject to:

$$x(t) = sin(t) - \frac{t^2}{3} + \int_0^{\frac{\pi}{2}} t^2 u(\xi)^2 x(\xi) d\xi.$$

The exact optimal trajectory and control functions are $x(t) = \sin(t)$ and $u(t) = \cos(t)$, respectively. Results of applying the given algorithm are presented in Table 3. Also, reported the absolute value of the errors of our method for $N = 10$ and $N = 20$ in Tables 4.

Table.3. Numerical results in Example 2.

k	J_k^*	
	$N{=}10$	$N{=}20$
4	7.3341×10^{-9}	6.8388×10^{-9}
5	5.2484×10^{-10}	2.9656×10^{-11}

Table.4. Absolute errors for $x(t)$ and $u(t)$ in Example 2 ($k = 5$).

N	E_x	E_u
10	2.0653×10^{-5}	2.6408×10^{-5}
20	5.1497×10^{-8}	1.6535×10^{-5}
30	1.4172×10^{-8}	1.6536×10^{-5}

Example 3. Consider the following (OCF) problem

$$\text{Minimize } \ J = \int_{-1}^{1} (x_1(t) - t^2 + 1)^2 + (x_2(t) - 1 + t^2)^2 + (u(t) - sin(t))^2 \, dt,$$

subject to:

$$x_1 = t^2 - 1 + \int_{-1}^{1} t^3 u(\xi)(x_1(\xi) + x_2(\xi)) \, d\xi,$$

$$x_2 = 1 - t^2 + \int_{-1}^{1} (\xi - u(\xi))(x_1(\xi) + x_2(\xi)) \, d\xi.$$

The exact control function and exact state functions of this problem are $u(t) = sin(t)$, $x_1(t) = t^2 - 1$ and $x_2(t) = 1 - t^2$, respectively. Results of applying the given algorithm are presented in Table 5. Also, reported the absolute value of the errors of our method for $N = 10$ and $N = 20$ in Tables 6.

Table.5. Numerical results in Example 3.

k	J_k^*	
	$N=10$	$N=20$
4	$1.88493383 \times 10^{-7}$	$1.88493381 \times 10^{-7}$
5	6.88×10^{-12}	6.88×10^{-12}

Table.6. Absolute errors for $x(t)$ and $u(t)$ in Example 3 ($k = 5$).

N	E_{x_1}	E_{x_2}	E_u
20	3.0945×10^{-14}	5.1672×10^{-9}	7.1×10^{-6}

8 Conclusion

In this article, the sinc functions and parametrization approach are used to solve the optimal control problem governed by Fredholms integral equations. Numerical results given in tables show high accuracy of the proposed method, with increasing the N, errors are decreased more rapidly. We can get much better results with increasing the N.

References

[1] T. Abualrub, I. Sadek, M. Abukhaled, Optimal Control Systems by Time-Dependent Coefficients Using CAS Wavelets. Hindawi Publishing Corporation Journal of Applied Mathematics. (2009).

[2] M. Annunziato, A. Borz‘, Fast solvers of Fredholm optimal control problems. Accepted for publication to:Numer. Math.: Th., Meth and Appl. (2010).

[3] S. A. Belbas, Iterative schemes for optimal control of Volterra integral equations, Nonlinear Analysis, 37 (1999) 57-79.

[4] R. Dai ,J. E. Cochran, Jr, Wavelet Collocation Method for Optimal Control Problems, J Optim Theory Appl 143 (2009) 265278.

[5] M. Dehghana, A.Saadatmandi, The numerical solution of a nonlinear system of second-order boundary value problems using the sinc-collocation method. Mathematical and Computer Modelling. 46 (2007) 1434-1441.

[6] M. El.Gamel Sinc and the numerical solution of fifth-order boundary value problems. Applied Mathematics and Computation. 187 (2007) 1417-1433.

[7] T. Rubiček, Optimal control of nonlinear Fredholm integral equations. Journal of Optimization and its Applications. 19 (1998) 707-729.

[8] W. Rudin, Principles of mathematical analysis, 3rd edn. McGraw-Hill. (1976).
[9] F.Stenger, Numerical methods based on sinc and analytic functions. Springer. New York. (1993). Program. 122 (2010) 301-347.
[10] D.F. Winter, K. Bowers, J. Lund, Wind-driven currents in a sea with a variable eddy viscosity calculated via a SincGalerkin technique, Internat. J. Numer. Methods Fluids 33 (2000) 10411073.
[11] W. Rudin, Principles of Mathematical Analysis, 3rd edn. McGraw-Hill, 1976.
[12] A. Adawi , F. Awawdeh, A Numerical Method for Solving Linear Integral Equations,Int. J. Contemp. Math. Sciences, Vol. 4, (2009), no. 10, 485 - 496.
[13] F. L. Chernousko, A. A. Lyubushin, Method of successive approximations for solution of optimal control problems, Optimal Control Appl. Methods, 3,2 (1982) 101114.
[14] W.H. Schmidt, Iterative methods for optimal control processes governed by integral equations, International Series of Numerical Mathematics 111 (1993), 6982.

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