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Computing The Moments Of Order Statistics From Nonidentically Distributed Marshall-Olkin Extended Burr XII Random Variables

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Abstract: Order statistics (os) for independent non-identically distributed (inid) random variables (rvs) is widely discussed in the literature, see, for example, Balakrishnan [3], Balakrishnon and Subramanian [4], Barakat and Abdelkader [8] and Jamjoom and Al-Saiary [14]. In this paper a recurrence relation is established for computing all single moments of all os arising from inid Marshall-Olkin extended Burr XII (MOEB XII) rvs. Another proof for the independent identical distributed (iid) rvs case is also presented and numerical examples are given.

Keywords: Order statistics; Moments; Marshall-Olkin extended Burr XII distribution.

1 Introduction

Order statistics is an important branch of statistics which deals with theory and applications of ordered rvs and functions involving them. The subject of os from inid rvs is discussed widely in the literature see for example David [10], Bapat and Beg [5] and David and Nagaraja [11]. Barakat [6] found the limit behavior of bivariate os from inid rvs. Gungor et al. [17] expressed the multivariate os by marginal ordering of inid rvs under discontinuous distribution functions. The moments of order statistics of inid rvs have been treated using three different approaches. The first approach is used when there exists a basic relation between the probability density function (pdf) and the cumulative distribution function (cdf) see Balakrishnan [3]. Applications of this approach are found in the literature for several continuous distributions see Jamjoom and Al-Saiary [15] and the references therein. In particular, Balakrishnan [3] applied this approach to derive recurrence relations for single and product moments of os from inid rvs for the exponential and right truncated exponential distributions. Childs and Balakrishnan [9] found the moments of os from inid rvs for the logistic distribution.

The second approach was introduced by Barakat and Abdelkader [8]. Although this approach is an easier manner to evalute the moments of os of inid rvs but its application is restricted to distributions having cdfs $F(x)$ that can be written as $F(x) = 1-\lambda(x)$. Of course this approach can also be applied if the survival function of the considered distribution has an excplicit form. The first application of this second approach was by Barakat and Abdelkader [7] to Weibull distribution and then a generalized version of the approach was given by Barakat and Abdelkader [8] where they applied it to Erlang, positive exponential, pareto and laplace distributions. Later this approach is applied by Abdelkader [1,2] to compute the moments of os using the survival function of inid rvs having, respectively, Gamma and Beta distributions. Further, Jamjoom [12], Jamjoom and Al-Saiary [13] have applied this technique to compute the moments of os of inid Burr(XII) distribution as well as Beta three-parameter type I distribution.

The third approach, which referred to as the moment generating function technique, is established by Jamjoom and Al-Saiary [14] and depends mainly on the second approach. The moments of inid os for Burr type II, exponential and Erlang truncated exponential distributions, are computed using this third approach by Jamjoom and Al-Saiary [14].

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A rv X is said to be has a MOEB(XII) distribution if its cdf is given by

$$F(x) = 1 - \frac{\alpha}{(1+x^c)^m - \bar{\alpha}},\ x > 0,\ \alpha,\ c,\ m > 0,\ \bar{\alpha} = 1 - \alpha, \tag{1}$$

In fact the MOEB(XII) distribution is an extended class that includes some distributions as special cases Burr(XII) ($\alpha = 1$), Lomax ($\alpha = 1, c = 1$) and log-logistic or weibull exponential distribution ($\alpha = 1,\ m = 1$). For the details of the mathematical statistical properties and application fields of the MOEB(XII) distribution see Gharib et al. [16].

In the present paper the problem of computing the moments of os from inid rvs having MOEB(XII) distribution is discussed using the second approach.

Let $X_1, X_2, \ldots, X_n$ be independent rvs and let $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ denote the corresponding os. Bapat and Beg [5] have shown that the cdf of the rth os $X_{r:n}$ $(1 \leq r \leq n)$ can be expressed in terms of permanents, that is

$$F_{r:n}(x) = \sum_{i=r}^{n} \frac{1}{i!\ (n-i)!} per\underset{i\quad n-i}{\left[F(x)\ \bar{F}(x) \right]},\ -\infty < x < \infty, \tag{2}$$

where $F(x)$ and $\bar{F}(x) = 1 - F(x)$ denote the column vectors $(F_1(x),\ F_2(x),\ \ldots,\ F_n(x))^{\grave{}}$ and $(\bar{F}_1(x),\ \bar{F}_2(x), \ldots, \bar{F}_n(x))^{\grave{}}$ respectively. Moreover if $a_1,\ a_2,\ \ldots$ are column vectors then

$$\underset{i_1\quad i_2\ \ldots}{\left[a_1,\ a_2,\ \ldots \right]},$$

will denote the matrix obtaind by taking i_1 copies of a_1, i_2 copies of a_2 and so on. Also, in (2) $per(A)$ denotes the permanent of a square matrix A which is defined similar to the determinants except that all terms in the expansion have a positive sign, see Mine [18].

Assume that the rvs $X_i,\ 1,\ 2,\ \ldots,\ n$ are inid having MOEB(XII) distribution with cdf (1).

In the next section, we derive the kth moments $\mu_{n::n}^{(k)}$ and $\mu_{1:n}^{(k)}$ of the maximum and minimum of a random sample of size n from MOEB(XII) distribution.

2 Main result

Relation (3). For $n = 1, 2, \ldots$ and $k = 1, 2, \ldots,$

$$\mu_{n:n}^{(k)} = \frac{k}{c} \sum_{j=1}^{n} (-1)^{j+1} I_j, \tag{3}$$

where

$$I_j = \alpha^j \sum_{1<i_1<i_2<\ldots<i_n \leq n} \sum \sum \ldots \sum \left(\sum_{u_1=0}^{\infty} \sum_{u_2=0}^{\infty} \ldots \sum_{u_n=0}^{\infty} \bar{\alpha}^{\sum_{j=1}^{n} u_j} \right) B\left(\sum_{j=1}^{n} m_{i_j}(1+u_j) - \frac{k}{c}, \frac{k}{c} \right), \tag{4}$$

and

$$\mu_{1:n}^{(k)} = \frac{k}{c} I_n, \tag{5}$$

where I_n is defined in (4) when $j = n$.

*Proof.*By definition

$$\mu_{n:n}^{(k)} = k \int_0^{\infty} x^{k-1} \left(1 - F_{n:n}(x)\right) dx,$$

where $F_{n:n}(x)$ is the cdf of the maximum os from inid rvs X_i, $i = 1, 2, \ldots, n$ defined by

$$F_{n:n}(x) = \prod_{i=1}^{n} F_i(x),$$

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and for MOEB(XII) distribution we have

$$F_{n:n}(x) = \prod_{i=1}^{n} \left(1 - \frac{\alpha}{(1+x^c)^{m_i} - \bar{\alpha}}\right),$$

then,

$$\begin{aligned}
\mu_{n:n}^{(k)} &= k \int_0^\infty x^{k-1} \left\{ 1 - \prod_{i=1}^{n} \left(1 - \frac{\alpha}{(1+x^c)^{m_i} - \bar{\alpha}}\right) \right\} dx, \\
&= k \int_0^\infty x^{k-1} \left\{ \sum_{i=1}^{n} \frac{\alpha}{((1+x^c)^{m_i} - \bar{\alpha})} - \sum\sum_{1\le i_1<i_2\le n} \left[\frac{\alpha^2}{((1+x^c)^{m_{i_1}} - \bar{\alpha})((1+x^c)^{m_{i_2}} - \bar{\alpha})} \right] \right. \\
&\quad + \sum\sum\sum_{1\le i_1<i_2<i_3\le n} \left[\frac{\alpha^3}{((1+x^c)^{m_{i_1}} - \bar{\alpha})((1+x^c)^{m_{i_2}} - \bar{\alpha})((1+x^c)^{m_{i_3}} - \bar{\alpha})} \right] + \dots \\
&\quad \left. + (-1)^{n+1} \left[\frac{\alpha^n}{((1+x^c)^{m_{i_1}} - \bar{\alpha})((1+x^c)^{m_{i_2}} - \bar{\alpha})\dots((1+x^c)^{m_{i_n}} - \bar{\alpha})} \right] \right\} dx,
\end{aligned}$$

putting $(1+x^c) = y^{-1}$, we get,

$$\begin{aligned}
\mu_{n:n}^{(k)} &= \frac{k}{c} \left\{ \alpha \sum_{i=1}^{n} \left(\sum_{u=0}^{\infty} (\bar{\alpha})^u \right) B\left(m_i\,(1+u) - \frac{k}{c}, \frac{k}{c} \right) - \right. \\
&\quad -\alpha^2 \sum\sum_{1\le i_1<i_2\le n} \left(\sum_{u_1=0}^{\infty} \sum_{u_2=0}^{\infty} (\bar{\alpha})^{u_1+u_2} \right) B\left(m_{i_1}(1+u_1) + m_{i_2}(1+u_2) - \frac{k}{c}, \frac{k}{c} \right) \\
&\quad + \dots + (-1)^{n+1} \alpha^n \sum\sum_{1\le i_1<i_2<\dots<i_n\le n} \dots \sum \left(\sum_{u_1=0}^{\infty} \sum_{u_2=0}^{\infty} \dots \sum_{u_n=0}^{\infty} (\bar{\alpha})^{\sum_{i=1}^{n} u_i} \right) \\
&\quad \left. B\left(\sum_{j=1}^{n} m_{i_j}(1+u_j) - \frac{k}{c}, \frac{k}{c} \right) \right\}.
\end{aligned}$$

This can be written as

$$\mu_{n:n}^{(k)} = \frac{k}{c} \sum_{j=1}^{n} (-1)^{j+1} I_j,$$

where

$$I_j = \alpha^j \sum\sum_{1<i_1\le i_2\le\dots\le i_n<n} \dots \sum \left(\sum_{u_1=0}^{\infty} \sum_{u_2=0}^{\infty} \dots \sum_{u_n=0}^{\infty} (\bar{\alpha})^{\sum_{i=1}^{n} u_i} \right) B\left(\sum_{j=1}^{n} m_{i_j}(1+u_j) - \frac{k}{c}, \frac{k}{c} \right)$$

The proof of (5) follows by using the relation

$$\mu_{1:n}^{(k)} = k \int_0^\infty x^{k-1} (1 - F_{1:n}(x))\, dx,$$

where

$$F_{1:n}(x) = 1 - \prod_{i=1}^{n} (1 - F_i(x)),$$

is the cdf of the smallest os from inid rvs.

Thus for MOEB(XII) distribution we have

$$\begin{aligned}
\mu_{1:n}^{(k)} &= k \int_0^\infty x^{k-1} \prod_{i=1}^{n} \left(\frac{\alpha}{(1+x^c)^{m_i} - \bar{\alpha}} \right) dx \\
&= k \int_0^\infty x^{k-1} \left(\frac{\alpha}{((1+x^c)^{m_1} - \bar{\alpha})} \frac{\alpha}{((1+x^c)^{m_2} - \bar{\alpha})} \dots \frac{\alpha}{((1+x^c)^{m_n} - \bar{\alpha})} \right) dx
\end{aligned}$$

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putting $(1+x^c) = y^{-1}$, then,

$$\mu_{1:n}^{(k)} = \frac{k}{c}\alpha^n \int_0^1 y^{\sum_{i=1}^{n} m_i - \frac{k}{c} - 1} (1-y)^{\frac{k}{c}-1}$$
$$\left(\sum_{u_1=0}^{\infty} (\bar{\alpha} y^{m_1})^{u_1} \sum_{u_2=0}^{\infty} (\bar{\alpha} y^{m_2})^{u_2} \ldots \sum_{u_n=0}^{\infty} (\bar{\alpha} y^{m_n})^{u_n} \right) dy$$
$$= \frac{k}{c}\alpha^n \left(\sum_{u_1=0}^{\infty} \sum_{u_2=0}^{\infty} \ldots \sum_{u_n=0}^{\infty} (\bar{\alpha})^{\sum_{i=1}^{n} u_i} \right) B\left(\sum_{i=1}^{n} m_i(1+u_i) - \frac{k}{c}, \frac{k}{c} \right) = \frac{k}{c} I_n,$$

where

$$I_n = \alpha^n \left(\sum_{u_1=0}^{\infty} \sum_{u_2=0}^{\infty} \ldots \sum_{u_n=0}^{\infty} (\bar{\alpha})^{\sum_{i=1}^{n} u_i} \right) B\left(\sum_{i=1}^{n} m_i(1+u_i) - \frac{k}{c}, \frac{k}{c} \right)$$

which can also be written as

$$I_n = \sum\sum_{1 \le i_1 < i_2 < \ldots < i_n \le n} \ldots \sum \alpha^n \left(\sum_{u_1=0}^{\infty} \sum_{u_2=0}^{\infty} \ldots \sum_{u_n=0}^{\infty} (\bar{\alpha})^{\sum_{i=1}^{n} u_i} \right) B\left(\sum_{i=1}^{n} m_i(1+u_i) - \frac{k}{c}, \frac{k}{c} \right)$$

which completes the proof.

Theorem 2.1. For $r = 1,2,\ldots,n$ and $k = 1,2,\ldots$

$$\mu_{r:n}^{(k)} = \mu_{r-1:n}^{(k)} + \sum_{j=1}^{r} (-1)^{j-1} \binom{n-r+j}{j-1} I_{n-r+j},$$

where $I_j, j = 1,2,\ldots,r$ is given by (4) and with the convention that $\mu_{0:n}^{(k)} = 0$.

Proof. Equation (2) can be rewritten as

$$F_{r-1:n}(x) = F_{r:n}(x) + \frac{1}{(r-1)!\,(n-r+1)!} per \quad \underset{r-1 \quad n-r+1}{\left[F(x)\ \bar{F}(x) \right]},$$

which is equivalent to

$$F_{r-1:n}(x) = F_{r:n}(x) + \sum_{p} \prod_{j=1}^{n} F_{i_j}(x) \prod_{j=r}^{n} F_{i_{n-j+1}}(x),$$

where the summation p extends over all permutations $(i_1, i_2, \ldots, i_n)$ of $(1,2,\ldots,n)$ for which $1 \le i_1 < i_2 < \ldots < i_{r-1} \le n$ and $1 \le i_r < i_{r+1} < \ldots < i_{n-1} \le n$. Now let

$$x_{i_0} = \inf\{x : F_i(x) > 0\} \ge 0, \text{ for all } i.$$

Then

$$\mu_{r:n}^{(k)} = E(X_{r:n}^k) = k \int_0^{\infty} x^{k-1} \bar{F}_{r:n}(x) dx$$
$$= \mu_{r-1:n}^{(k)} + Q_{r:n}^{(k)},$$

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where

$$
\begin{aligned}
Q_{r:n}^{(k)} &= k\int_0^\infty x^{k-1}\sum_p \prod_{j=1}^{r-1}(1-\bar{F}_{i_j}(x))\prod_{j=r}^{n}\bar{F}_{i_j}(x)dx \\
&= k\int_0^\infty x^{k-1}\sum_p \prod_{j=1}^{r-1}\left(1-\frac{\alpha}{(1+x^c)^{m_{i_j}}-\bar{\alpha}}\right)\prod_{j=r}^{n}\frac{\alpha}{(1+x^c)^{m_{i_j}}-\bar{\alpha}}dx \\
&= k\int_0^\infty x^{k-1}\sum_p \left[1-\sum_{j_1=1}^{r-1}\frac{\alpha}{(1+x^c)^{m_{i_{j_1}}}-\bar{\alpha}}+\frac{\alpha^2}{((1+x^c)^{m_{i_{j_1}}}-\bar{\alpha})((1+x^c)^{m_{i_{j_2}}}-\bar{\alpha})}\right. \\
&\quad \left.+\ldots+(-1)^{r-1}\frac{\alpha^{r-1}}{((1+x^c)^{m_{i_{j_1}}}-\bar{\alpha})((1+x^c)^{m_{i_{j_2}}}-\bar{\alpha})\ldots((1+x^c)^{m_{i_{j_{r-1}}}}-\bar{\alpha})}\right] \\
&\quad \prod_{j=r}^{n}\frac{\alpha}{(1+x^c)^{m_{i_j}}-\bar{\alpha}}dx.
\end{aligned}
$$

Putting $(1+x^c)=y^{-1}$, we get

$$
\prod_{j=r}^{n}\frac{\alpha}{(1+x^c)^{m_{i_j}}-\bar{\alpha}}=\alpha^{n-(r-1)}\left(\sum_{u_r=0}^{\infty}\sum_{u_{r+1}=0}^{\infty}\ldots\sum_{u_n=0}^{\infty}(\bar{\alpha})^{\sum_{i=r}^{n}u_j}\right)y^{\sum_{j=r}^{n}m_{i_j}(1+u_j)}.
$$

Therefor,

$$
\begin{aligned}
Q_{r:n}^{(k)} &= \frac{k}{c}\sum_p\int_0^1 \alpha^{n-(r-1)}\left(\sum_{u_r=0}^{\infty}\sum_{u_{r+1}=0}^{\infty}\ldots\sum_{u_n=0}^{\infty}(\bar{\alpha})^{\sum_{i=r}^{n}u_j}\right)\left[y^{\sum_{j=r}^{n}m_{i_j}(1+u_j)-\frac{k}{c}-1}-\right. \\
&\quad -\alpha\sum_{j_1=1}^{r-1}\left(\sum_{u_1=0}^{\infty}\bar{\alpha}^{u_1}\right)y^{m_{i_{j_1}}(1+u_1)+\sum_{j=r}^{n}m_{i_j}(1+u_j)-\frac{k}{c}-1}+ \\
&\quad +\alpha^2\sum\sum_{1\le i_1<i_2\le r-1}\left(\sum_{u_1=0}^{\infty}\sum_{u_2=0}^{\infty}(\bar{\alpha})^{u_1+u_2}\right)y^{m_{i_{j_1}}(1+u_1)+m_{i_{j_2}}(1+u_2)+\sum_{j=r}^{n}m_{i_j}(1+u_j)-\frac{k}{c}-1}+ \\
&\quad +\ldots+(-1)^{r-1}\alpha^{r-1}\left(\sum_{u_r=0}^{\infty}\sum_{u_{r+1}=0}^{\infty}\ldots\sum_{u_{r-1}=0}^{\infty}(\bar{\alpha})^{\sum_{i=1}^{r-1}u_j}\right) \\
&\quad \left. y^{\sum_{j=1}^{r-1}m_{i_j}(1+u_j)+\sum_{j=r}^{n}m_{i_j}(1+u_j)-\frac{k}{c}-1}(1-y)^{\frac{k}{c}-1}\right]dy
\end{aligned}
$$

hence, applying integration we get

$$Q_{r:n}^{(k)} = \frac{k}{c}\sum_{p}\Bigg[\alpha^{n-(r-1)}\left(\sum_{u_r=0}^{\infty}\sum_{u_{r+1}=0}^{\infty}...\sum_{u_n=0}^{\infty}(\bar{\alpha})^{\sum_{i=r}^{n}u_j}\right)B\left(\sum_{i=r}^{n}m_{i_j}(1+u_j)-\frac{k}{c},\frac{k}{c}\right)-$$
$$-\alpha^{n-r+2}\sum_{j_1}^{r-1}\left(\sum_{u_1=0}^{\infty}\bar{\alpha}^{u_1}\right)\left(\sum_{u_r=0}^{\infty}\sum_{u_{r+1}=0}^{\infty}...\sum_{u_n=0}^{\infty}(\bar{\alpha})^{\sum_{i=r}^{n}u_j}\right)$$
$$B\left(\sum_{i=r}^{n}m_{i_j}(1+u_j)+m_{i_{j_1}}(1+u_1)-\frac{k}{c},\frac{k}{c}\right)$$
$$+\alpha^{n-r+3}\sum\sum_{1\le i_1<i_2\le r-1}\left(\sum_{u_1=0}^{\infty}\sum_{u_2=0}^{\infty}(\bar{\alpha})^{u_1+u_2}\right)\left(\sum_{u_r=0}^{\infty}\sum_{u_{r+1}=0}^{\infty}...\sum_{u_n=0}^{\infty}(\bar{\alpha})^{\sum_{i=r}^{n}u_j}\right)$$
$$B\left(\sum_{i=r}^{n}m_{i_j}(1+u_j)+m_{i_{j_1}}(1+u_1)+m_{i_{j_2}}(1+u_2)-\frac{k}{c},\frac{k}{c}\right)+...+$$
$$+(-1)^{r-1}\alpha^{n}\left(\sum_{u_r=0}^{\infty}\sum_{u_{r+1}=0}^{\infty}...\sum_{u_n=0}^{\infty}(\bar{\alpha})^{\sum_{i=1}^{n}u_j}\right)B\left(\sum_{i=1}^{n}m_{i_j}(1+u_j)-\frac{k}{c},\frac{k}{c}\right)\Bigg].$$

Now using the facts that $\sum_{p}(1)=\binom{n}{r-1}$ and that $\sum\sum...\sum_{1\le i_1<i_2<...<i_m\le n}(1)=\binom{n}{m}$, for all $n\ge m$, the above relation reduces to:

$$Q_{r:n}^{(k)} = \frac{k}{c}\alpha^{n}\sum_{j=1}^{n}(-1)^{j-1}a_j\, I_{n-r-j}$$

where $I_j, j=1,2,...,r$ is given by (4) and $a_j=\frac{(n-r+j)!}{(n-r+1)!\,(j-1)!}$ since,

$$\binom{n}{r-1}\binom{r-1}{j-1}=a_j\binom{n}{r-1}.$$

This completes the proof of the theorem.

To sum up the computations for obtaining the kth moments of all os, one needs to compute the sequence $\{I_j\}_{j=1}^{j=n}$ which is given by (4). Then recursively applying theorem 2.1, starting with the maximum $\mu_{n:n}^{(k)}$ in (3) one can obtain all moments of all os $\mu_{r:n}^{(k)}$, $r\le n$ from MOEB(XII) distribution. For example if $n=3$, we get

$$\mu_{3:3}^{(k)} = \frac{k}{c}\left(I_1-I_2+I_3\right),$$

where

$$I_1=\alpha^3\left(\sum_{u_1=0}^{\infty}\sum_{u_2=0}^{\infty}\sum_{u_3=0}^{\infty}(\bar{\alpha})^{\sum_{i=1}^{3}u_i}\right)\Big[B\left(m_1(1+u_1)-\frac{k}{c},\frac{k}{c}\right)+B\left(m_2(1+u_2)-\frac{k}{c},\frac{k}{c}\right)$$
$$+B\left(m_3(1+u_3)-\frac{k}{c},\frac{k}{c}\right)\Big]$$
$$I_2=\alpha^2\left(\sum_{u_1=0}^{\infty}\sum_{u_2=0}^{\infty}\sum_{u_3=0}^{\infty}(\bar{\alpha})^{\sum_{i=1}^{3}u_i}\right)\Bigg[B\left(m_1(1+u_1)+m_2(1+u_2)-\frac{k}{c},\frac{k}{c}\right)$$
$$+B\left(m_1(1+u_1)+m_3(1+u_3)-\frac{k}{c},\frac{k}{c}\right)+B\left(m_2(1+u_2)+m_3(1+u_3)-\frac{k}{c},\frac{k}{c}\right)\Bigg]$$
$$I_3=\alpha^3\left(\sum_{u_1=0}^{\infty}\sum_{u_2=0}^{\infty}\sum_{u_3=0}^{\infty}(\bar{\alpha})^{\sum_{i=1}^{3}u_i}\right)B\left(m_1(1+u_1)+m_2(1+u_2)+m_3(1+u_3)-\frac{k}{c},\frac{k}{c}\right) \quad (6)$$

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$\mu_{1:3}^{(k)} = \frac{k}{c} I_3$
$\mu_{2:3}^{(k)} = \frac{k}{c} (I_2 - 2I_3)$.
These results can be put in the following table.

$\mu_{3:3}^{*(k)}$	I_1	$-I_2$	$+I_3$
$\mu_{2:3}^{*(k)}$		I_2	$-2I_3$
$\mu_{1:3}^{*(k)}$			I_3

The moments $\mu_{r:n}^{(k)}, r \leq n$ of order statistics arising from non-identically MOEB(XII) random variables with $n = 3$.

Where $\mu_{r:n}^{*(k)} = \frac{c}{k} \mu_{r:n}^{(k)}$.
For a general form of this table see Barakat and Abdelkader (2000).

3 Independent identically distributed case

In this section, the moments of the os arising from iid MOEB(XII) rvs are derived in the following theorem.

Theorem 3.1. For the case of a sample of n iid arising from MOEB(XII) distributin the kth moment $(k = 1, 2, ...)$ of the rth $(1 \leq r \leq n)$ os is given by

$$\mu_{r:n}^{(k)} = k \sum_{j=1}^{r} (-1)^{j-(n-r+1)} \binom{j-1}{n-r} I_j,$$

where

$$I_j = \alpha^j \binom{n}{j} \left(\sum_{u_1=0}^{\infty} \sum_{u_2=0}^{\infty} ... \sum_{u_n=0}^{\infty} (\bar{\alpha})^{\sum_{i=1}^{j} u_i} \right) B\left(m\left(\sum_{i=1}^{j} u_i + 1 \right) - \frac{k}{c}, \frac{k}{c} \right).$$

Proof.

$$\begin{aligned} I_j &= \binom{n}{j} \int_0^\infty x^{k-1} [\bar{F}(x)]^j dx \\ &= \binom{n}{j} \int_0^\infty x^{k-1} \left[\frac{\alpha}{(1+x^c)^m - \bar{\alpha}} \right]^j dx \\ &= \alpha^j \binom{n}{j} \left(\sum_{u_1=0}^{\infty} \sum_{u_2=0}^{\infty} ... \sum_{u_n=0}^{\infty} (\bar{\alpha})^{\sum_{i=1}^{j} u_i} \right) B\left(m\left(\sum_{i=1}^{j} u_i + 1 \right) - \frac{k}{c}, \frac{k}{c} \right). \end{aligned}$$

Corollary 3.1. For iid MOEB(XII) rvs, $\mu_{1:n}^{(k)}$ becomes

$$\begin{aligned} \mu_{1:n}^{(k)} &= k \int_0^\infty x^{k-1} \left(\frac{\alpha}{(1+x^c)^{m_i} - \bar{\alpha}} \right)^n dx \\ &= k\, \alpha^n \left(\sum_{u_1=0}^{\infty} \sum_{u_2=0}^{\infty} ... \sum_{u_n=0}^{\infty} (\bar{\alpha})^{\sum_{i=1}^{j} u_i} \right) B\left(m\left(\sum_{i=1}^{j} u_i + 1 \right) - \frac{k}{c}, \frac{k}{c} \right). \end{aligned}$$

4 Numericall application

The following examples are computed when k=1.

Case 1: independent identically distributed

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Example 4.1. Let n=3 and m=2, 3, 4, and 5 table 2 shows the results:

m	2	3	4	5
$\mu_{3:3}$	4.88419	1.65829	0.926742	0.62209

For example when m=3,

$$\mu_{3:3} = \frac{1}{c}(I_1 - I_2 + I_3),$$

$$I_1 = \alpha^n \left(\sum_{u_1=0}^{\infty} \sum_{u_2=0}^{\infty} \sum_{u_3=0}^{\infty} (\bar{\alpha})^{\sum_{i=1}^{3} u_i} \right) B\left(m\left(\sum_{i=1}^{3} u_i + 1 \right) - \frac{1}{c}, \frac{1}{c} \right) = 0.994973$$

$$I_2 = \alpha^2 \left(\sum_{u_1=0}^{\infty} \sum_{u_2=0}^{\infty} \sum_{u_3=0}^{\infty} (\bar{\alpha})^{\sum_{i=1}^{3} u_i} \right) B\left(m\left(\sum_{i=1}^{3} u_i + 1 \right) - \frac{1}{c}, \frac{1}{c} \right) = 0.663315$$

$$I_3 = \alpha^3 \left(\sum_{u_1=0}^{\infty} \sum_{u_2=0}^{\infty} \sum_{u_3=0}^{\infty} (\bar{\alpha})^{\sum_{i=1}^{3} u_i} \right) B\left(m\left(\sum_{i=1}^{3} u_i + 1 \right) - \frac{1}{c}, \frac{1}{c} \right) = 0.994973$$

Therefor, $\mu_{3:3} = 1.65829$.

Case 2: independent nonidentically distributed

Example 4.2.

(a) Setting n=2, α=1.5, c=0.8 and m_1=2, m_2=3, in (3), (4) we get

$$\mu_{2:2} = \frac{1}{c}(I_1 - I_2),$$

where

$$I_1 = \alpha^2 \left(\sum_{u_1=0}^{\infty} \sum_{u_2=0}^{\infty} (\bar{\alpha})^{\sum_{i=1}^{2} u_i} \right) \left[B\left(m_1(1+u_1) - \frac{1}{c}, \frac{1}{c} \right) + B\left(m_2(1+u_2) - \frac{1}{c}, \frac{1}{c} \right) \right] = 2.07087,$$

$$I_2 = \alpha^2 \left(\sum_{u_1=0}^{\infty} \sum_{u_2=0}^{\infty} (\bar{\alpha})^{\sum_{i=1}^{2} u_i} \right) \left[B\left(m_1(1+u_1) + m_2(1+u_2) - \frac{1}{c}, \frac{1}{c} \right) \right] = 0.240458.$$

Therefor $\mu_{2:2} = 2.28802$.

(b) Let n=3, α=1.5, c=0.8 and m_1=2, m_2=3, m_3=4 in (3), (4) we get

$$\mu_{3:3} = \frac{1}{c}(I_1 - I_2 + I_3) = 0.62209.$$

Where I_1, I_2 and I_3 are given by (6).

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