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A Coupled Method to Solve Reaction-Diffusion-Convection Equation with the Time Fractional Derivative without Singular Kernel

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Abstract: In this work, we find the solution of a class of time fractional reaction-diffusion-convection equations. The time fractional derivatives are described in a new definition of fractional derivative without singular kernel which has been recently introduced by Caputo and Fabrizio. For obtaining the solution, we apply an approach based on a combination of the Laplace transform and the differential transform. Finally, some test problems are discussed to show ability and utility of the proposed method.

Keywords: Fractional calculus, Caputo-Fabrizio fractional derivative, Laplace transform, differential transform method, reaction-diffusion-convection equation.

1 Introduction

The fractional calculus has been focused due to its frequent appearances in different fields of science [1,2,3,4,5]. Many models have been investigated in analytical and numerical frames by a number of authors [6,7,8,9,10]. Various different definitions of the fractional derivatives were suggested [1,11,12]. We recall that all of them are non-local operators related to each other in contrast to the integer order derivative which is a local operator.

Below, the fractional derivative without singular kernel, introduced by Caputo and Fabrizio in [13], is employed, namely

$$D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{f'(\tau)}{(t-\tau)^{\alpha}} d\tau,$$

where $\alpha \in [0,1]$ and $a \in [-\infty, t)$. Also $f \in H^1(a,b)$ for $b > a$. By changing the kernel $\frac{1}{(t-\tau)^{\alpha}}$ with the function $\exp(-\dfrac{\alpha}{1-\alpha}t)$ and $\dfrac{1}{\Gamma(1-\alpha)}$ with $\dfrac{M(\alpha)}{1-\alpha}$, they have suggested the new fractional derivative, named Caputo-Fabrizio (CF) derivative, as

$$\mathscr{D}_t^{\alpha} f(t) = \frac{M(\alpha)}{(1-\alpha)} \int_a^t f'(\tau) \exp\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right] d\tau, \tag{1}$$

where $M(\alpha)$ is a normalization function such that $M(0) = M(1) = 1$. According to (1), it can be concluded that if $f(t)$ is a constant function, then $\mathscr{D}_t^{\alpha} f(t) = 0$ as in the Caputo fractional derivative. But contrary to the Caputo derivative, the kernel does not have singularity for $t = \tau$. Also, $M(\alpha)$ has an explicit formula as [14]

$$M(\alpha) = \frac{2}{2-\alpha}, \qquad 0 \le \alpha \le 1.$$

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Thus, the new definition of fractional derivative can be proposed as [14]

$$\mathscr{D}_t^{\alpha} f(t) = \frac{1}{(1-\alpha)} \int_a^t \exp(-\frac{\alpha(t-\tau)}{1-\alpha}) f'(\tau) d\tau \tag{2}$$

with $0 < \alpha < 1$.

In recent two years, some researchers have used this new fractional derivative to explain some real world applications. For example, the author in [15] applied it to the nonlinear Fisher's reaction-diffusion equation. Also in [16] the nonlinear Baggs and Freedman model is studied. In [17] authors proved uniqueness of a solution for an initial value problem of a nonlinear fractional differential equation with the CF derivative. In [18] this new fractional derivative is used to introduce the mass-spring-damper motion equation. Also, in [19] author applied the CF fractional derivative to KdV–Burgers equation.

In some literature, analytical solutions to problems of time-fractional differential equations have been constructed [3, 7, 11, 20]. In this article, we use the Laplace transform method [1] and the differential transform method [21, 22] simultaneously, to find a solution of the class of time fractional reaction-diffusion-convection equations

$$\mathscr{D}_t^{\alpha} u(x,t) + a_2(x)\frac{\partial^2 u(x,t)}{\partial x^2} + a_1(x)\frac{\partial u(x,t)}{\partial x} + a_0(x)u(x,t) = f(x,t), \tag{3}$$

on $\Omega = \{(x,t) : (x,t) \in [0,L] \times [0,T]\}$, subject to the initial condition

$$u(x,0) = \varphi(x), \tag{4}$$

and the boundary conditions

$$u(0,t) = g(t), \ \frac{\partial u(0,t)}{\partial x} = h(t). \tag{5}$$

In the equation (3), $0 < \alpha \leq 1$ is the order of the fractional derivative and $a_i(x) \neq 0$ are some continuous functions.

The outline of this paper is structured as follows. In Section 2, some preliminaries about the Laplace transform and the differential transform are reviewed. The coupled method is presented in Section 3. Some test problems are solved in Section 4 to show the ability and efficiency of the proposed method. Finally, a conclusion is given in Section 5.

2 The Laplace Transform of the CF Derivative and the Differential Transformation

In present section, some basic definitions and operations of the Laplace transform and the differential transform method will be reviewed.

Definition 1. The Laplace transform of a function $f(t), t \geq 0$, denoted by $\bar{f}(s)$, is defined by

$$\mathscr{L}[f(t)] = \bar{f}(s) = \int_0^{\infty} e^{-st} f(t) dt. \tag{6}$$

Furthermore, the given function $f(t)$ in (6) is called the inverse Laplace transform of $\bar{f}(s)$ and is denoted by $\mathscr{L}^{-1}[\bar{f}]$.

Lemma 1.[13] Let $\alpha \in [0,1]$. The Laplace transform of the CF derivative is

$$\mathscr{L}[\mathscr{D}_t^{\alpha} f(t)] = \frac{s\bar{f}(s) - f(0)}{s + \alpha(1-s)}. \tag{7}$$

Zhou in [21] introduced the differential transform method, based on the Taylor series expansion. This method provides an iterative procedure to generate the derivatives of a function at a point, in terms of the initial or boundary conditions. Thus, this approach can be used to obtain a power series solution of an initial value problem.

Definition 2. [22] The transformation of the kth derivative of a function in one variable is as follows:

$$F(k) = \frac{1}{k!}\left[\frac{d^k f}{dx^k}(x)\right]_{x=x_0}, \tag{8}$$

and the inverse transformation is defined as

$$f(x) = \sum_{k=0}^{\infty} F(k)(x-x_0)^k.$$

Table 1 displays some properties related to the differential transform.

Table 1: Some properties of the differential transformation.

Function Form	Transformed Form
$f(x)=g(x)+h(x)$	$F(k)=G(k)+H(k)$
$f(x)=cg(x)$	$F(k)=cG(k)$ (c is a constant)
$f(x)=g(x).h(x)$	$F(k)=\sum_{k_1=0}^{k} g(k_1)H(k-k_1)$
$f(x)=\frac{d^n g(x)}{dx^n}$	$F(k)=\frac{(k+n)!}{k!}G(k+n)$
$f(x)=x^n$	$F(k)=\delta(k-n)=\begin{cases}1 & k=n\\ 0 & k\neq n\end{cases}$
$f(x)=x^n g(x)$	$F(k)=G(k-n)$
$f(x)=e^{\lambda x}$	$F(k)=\frac{\lambda^k}{k!}$
$f(x)=\sin(\omega x+\upsilon)$	$F(k)=\frac{\omega^k}{k!}\sin(\frac{k\pi}{2}+\upsilon)$
$f(x)=\cos(\omega x+\upsilon)$	$F(k)=\frac{\omega^k}{k!}\cos(\frac{k\pi}{2}+\upsilon)$

3 The Coupled Method

According to Lemma 1, by applying the Laplace transform to Eq. (3) with respect to the initial condition (4), we get

$$\frac{s\bar{u}(x,s)-\varphi(x)}{s+\alpha(1-s)}+a_0(x)\bar{u}(x,s)+a_1(x)\frac{d\bar{u}(x,s)}{dx}$$
$$+a_2(x)\frac{d^2\bar{u}(x,s)}{dx^2}=\bar{f}(x,s), \tag{9}$$

where

$$\bar{u}(x,s)=\mathscr{L}\big[u(x,t)\big], \tag{10}$$

and $\bar{f}(x,s)$ is the Laplace transform of $f(x,t)$. Also from the Laplace transform of the boundary conditions (5), we have

$$\bar{u}(0,s)=\bar{g}(s), \qquad \frac{d\bar{u}(0,s)}{dx}=\bar{h}(s), \tag{11}$$

where $\bar{g}(s)$ and $\bar{h}(s)$ respectively are the Laplace transform of $g(t)$ and $h(t)$.

Then, we obtain the solution of (9)-(11) by the differential transform method as

$$\bar{u}(x,s)=\sum_{k=0}^{\infty}U(k)x^k,$$

where $U(k)$ can be determined by the properties of the differential transformation in Table 1 and the relation (8). Finally, to get the solution of the problem (3)-(5), we use the inverse Laplace transform of $\bar{u}(x,s)$.

4 Test Problems

In this section, the proposed method is implemented on some examples of the initial boundary value problems for the time fractional reaction-diffusion-convection equations with the CF fractional derivative.

Example 1. In this example, we consider the fractional diffusion-convection equation of order $0<\alpha<1$

$$\mathscr{D}_t^{\alpha}u(x,t)+x\frac{\partial u(x,t)}{\partial x}+\frac{\partial^2 u(x,t)}{\partial x^2}=e^t-e^{-\frac{\alpha}{1-\alpha}}+2x^2+2, \quad 0<x<1, 0<t<1, \tag{12}$$

with respect to the conditions

$$u(x,0)=x^2,$$
$$u(0,t)=e^t,\ u_x(0,t)=0. \tag{13}$$

By applying the Laplace transform on the Eqs. (12) and (13), we have

$$\frac{s\bar{u}(x,s)-x^2}{b(s)}+x\frac{d\bar{u}(x,s)}{dx}+\frac{d^2\bar{u}(x,s)}{dx^2}=\frac{1}{s-1}-\frac{1}{s+\frac{\alpha}{1-\alpha}}-\frac{2x^2}{s}+\frac{2}{s}, \tag{14}$$

where $b(S)=s+\alpha(1-s)$, and

$$\bar{u}(0,s)=\frac{1}{s-1},\quad \frac{d\bar{u}}{dx}(0,s)=0. \tag{15}$$

Based on the properties of the differential transformation in Table 1, the transformed form of Eqs. (14) and (15) are as follows:

$$\frac{sU(k)-\delta(k-2)}{b(s)}+kU(k)+(k+1)(k+2)U(k+2)=\left(\frac{1}{s-1}+\frac{\alpha-1}{s+\alpha(1-s)+\frac{2}{s}}\right)\delta(k)+\frac{2}{s}\delta(k-2),$$

or

$$U(k+2)=\frac{1}{(k+1)(k+2)}\left[\left(\frac{1}{b(s)}+\frac{2}{s}\right)\delta(k-2)-\left(\frac{s}{b(s)}+k\right)U(k)+\left(\frac{1}{s-1}+\frac{\alpha-1}{b}(s)+\frac{2}{s}\right)\right], \tag{16}$$

and

$$U(0)=\frac{1}{s-1},\quad U(1)=0. \tag{17}$$

The next terms of $U(k)$, $k\geq 2$, can be obtained by substituting (17) into the recurrence relation (16). Thus, we get

$$U(2)=\frac{1}{s},$$

and

$$U(k)=0,\quad k\geq 3.$$

So

$$\bar{u}(x,s)=\sum_{k=0}^{\infty}U(k)x^k=\frac{1}{s-1}+\frac{1}{s}x^2. \tag{18}$$

Applying the inverse Laplace transform to Eq. (18) gives

$$u(x,t)=e^t+x^2,$$

which is the exact solution of the problem.

Example 2. Consider the equation

$$\mathscr{D}_t^{\alpha}u(x,t)+\frac{\partial u(x,t)}{\partial x}-\frac{\partial^2 u(x,t)}{\partial x^2}=e^t-e^{-\frac{\alpha}{1-\alpha}t}+\sin(x)+\cos(x),\quad 0<x<1, 0<t<1, \tag{19}$$

with respect to the conditions

$$\begin{aligned}&u(x,0)=1+\sin(x),\\&u(0,t)=e^t,\ u_x(0,t)=1.\end{aligned} \tag{20}$$

Similar to Example 1, we can conclude the following transformed form for Eq. (19) as

$$\frac{s\bar{u}(x,s)-(1+\sin x)}{b(s)}+\frac{d\bar{u}(x,s)}{dx}-\frac{d^2\bar{u}(x,s)}{dx^2}=\frac{1}{s-1}-\frac{1}{s+\frac{\alpha}{1-\alpha}}-\frac{1}{s}(\sin x+\cos x),$$

or

$$\frac{s\bar{u}(x,s)}{b(s)}+\frac{d\bar{u}(x,s)}{dx}-\frac{d^2\bar{u}(x,s)}{dx^2}=\left(\frac{1}{b(s)}+\frac{1}{s}\right)\sin x+\frac{1}{s}\cos x+\frac{\alpha}{b(s)}+\frac{1}{s-1}, \tag{21}$$

and Eq. (20) as

$$\bar{u}(0,s)=\frac{1}{s-1},\ \frac{d\bar{u}(0,s)}{dx}=\frac{1}{s}. \tag{22}$$

Also, for Eqs. (21) and (22), we get the differential transforms

$$\begin{aligned}U(k+2)=\frac{1}{(k+1)(k+2)}\Big[\frac{s}{b(s)}U(k)+(k+1)U(k+1)-\left(\frac{1}{b(s)}+\frac{1}{s}\right)\frac{\sin\frac{k\pi}{2}}{k!}-\\ \frac{\cos\frac{k\pi}{2}}{sk!}-\left(\frac{\alpha}{b(s)}+\frac{1}{s-1}\right)\delta(k)\Big],\end{aligned} \tag{23}$$

and

$$U(0)=\frac{1}{s-1},U(1)=\frac{1}{s}. \tag{24}$$

Substituting (24) into (23) gives

$$U(2)=0,\quad U(3)=\frac{1}{6s},\quad U(4)=0,\quad U(5)=\frac{1}{120s},\cdots,$$

as a result, by mathematical induction, we get

$$U(2i)=0,U(2i+1)=\frac{1}{(2i+1)!s},\quad i\geq 1.$$

Then

$$\bar{u}(x,s)=\lim_{n\to\infty}\sum_{i=0}^{n}U(i)x^i=\frac{1}{s-1}+\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{s}\frac{x^{2i+1}}{(2i+1)!}. \tag{25}$$

Applying the inverse Laplace transform to Eq. (25) gives

$$u(x,t)=e^t+\sin x.$$

Example 3. Let us consider the following problem

$$D_t^{\alpha}u(x,t)-\frac{\partial^2u(x,t)}{\partial x^2}+e^{-\frac{\alpha}{1-\alpha}t}+1=0,\quad 0<x<1,\quad 0<t<1, \tag{26}$$

$$u(x,0)=x^2,\qquad 0<x<1, \tag{27}$$

$$u(0,t)=\alpha t,\ \ u_x(0,t)=0,\quad 0<t<1. \tag{28}$$

According to the Laplace transform of the Eq. (26) and the initial condition (27), we get

$$\frac{s\bar{u}(x,s)-x^2}{b(s)}-\frac{d^2\bar{u}(x,s)}{dx^2}+\frac{1}{s}+\frac{1}{s+\frac{\alpha}{1-\alpha}}=0, \tag{29}$$

Now, by taking the differential transform of (29), we have

$$U(k+2)=\frac{1}{(k+1)(k+2)}\left[\frac{s}{b(s)}U(k)-\frac{\delta(k-2)}{b(s)}+\left(\frac{1}{s}+\frac{1-\alpha}{b(s)}\right)\delta(k)\right]. \tag{30}$$

Substituting the Laplace transform of the boundary conditions, *i.e.* $U(0)=\frac{\alpha}{s^2}$ and $U(1)=0$, into the Eq. (30) gives

$$U(2)=\frac{1}{s},$$

and

$$U(n)=0,\quad n\geq 3.$$

Hence,

$$\bar{u}(x,s) = \lim_{n\to\infty}\sum_{i=0}^{n} U(i)x^i = \frac{\alpha}{s^2} + \frac{1}{s}x^2.$$

Finally, by applying the inverse Laplace transform, we obtain the exact solution of the problem (26)-(28) as

$$u(x,t) = \alpha t + x^2.$$

Example 4. Consider the time fractional equation

$$\mathscr{D}_t^{\alpha} u(x,t) - \frac{\partial^2 u(x,t)}{\partial x^2} + e^{x-\frac{\alpha}{1-\alpha}t} = 0, \quad 0 < x < 1, 0 < t < 1, \tag{31}$$

with conditions:

$$\begin{aligned} &u(x,0) = e^x, \\ &u(0,t) = u_x(0,t) = e^t. \end{aligned} \tag{32}$$

Here, we have the Laplace transform of the Eq. (31) as

$$\frac{s\bar{u}(x,s) - e^x}{b(s)} - \frac{d^2\bar{u}(x,s)}{dx^2} + \frac{e^x}{s+\frac{\alpha}{1-\alpha}} = 0, \tag{33}$$

or

$$\frac{s\bar{u}(x,s)}{b(s)} - \frac{d^2\bar{u}(x,s)}{dx^2} - \frac{\alpha}{b(s)}e^x = 0. \tag{34}$$

and the Laplace transform of the boundary conditions (32) as

$$\bar{u}(0,s) = \frac{d\bar{u}(0,s)}{dx} = \frac{1}{s-1}. \tag{35}$$

Differential transform of the Eq. (34) becomes

$$(k+1)(k+2)U(k+2) = \frac{s}{b(s)}U(k) - \frac{\alpha}{k!b(s)}.$$

So, the next terms of $U(k)$, $k \geq 2$ can be obtained as

$$U(2) = \frac{1}{2(s-1)}, U(3) = \frac{1}{6(s-1)}, U(4) = \frac{1}{24(s-1)}, \cdots, U(n) = \frac{1}{n!(s-1)}.$$

Thus

$$\bar{u}(x,s) = \lim_{n\to\infty}\sum_{i=0}^{n} U(i)x^i = \lim_{n\to\infty}\sum_{i=0}^{n} \frac{1}{s-1}\frac{x^i}{i!} = \frac{1}{s-1}e^x.$$

Applying the inverse Laplace transform gives

$$u(x,t) = e^{x+t},$$

which is the exact solution of the problem (31)-(32).

5 Conclusion

In this paper, an approach is presented to solve the initial-boundary value problems for a class of time fractional reaction-diffusion-convection equations with a fractional derivative without singular kernel. A combination of the Laplace transform and the differential transform is used to obtain the exact solution of the problem. First, the Laplace transform is implemented on the time fractional equation and its boundary conditions. Afterwards, the differential transform method is employed to find the series solution of the transformed problem. Finally, taking the inverse Laplace transform will get the exact solution of the initial boundary value problem. The results verify that the proposed coupled method is an applicable and efficient technique to solve such problems.

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