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Gevrey regularity for the linear spatially homogeneous Boltzmann equation II. Non Maxwellian case with strong singularity

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Abstract: This paper is the follow up study of [4], it shows that in the non Maxwellian case with strong singularity, the solutions of the linearized spatially homogeneous Boltzmann equation also have the Gevrey regularity in any local space.

Keywords: Gevrey class regularity, Non-Maxwellian molecules, Non-cutoff, pseudo-differential operators, strong singularity.

1. Introduction

In this paper, we also study the Gevrey regularity of solutions for the following linearized Cauchy problem which has appeared in [4]:

$$\begin{cases} \frac{\partial f}{\partial t} = Lf = Q(\mu, f) + Q(f, \mu), \ v \in \mathbb{R}^3, \ t > 0; \\ f|_{t=0} = f_0 \end{cases} \tag{1.1}$$

where $\mu(v) = (2\pi)^{-\frac{3}{2}} e^{-\frac{|v|^2}{2}}$,

$$f_0 \geq 0, \int_{\mathbb{R}^3} f_0(v)\{1 + |v|^2 + \log(1 + f_0(v))\} dv < +\infty, \tag{1.2}$$

and

$$Q(g,f) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - v_*, \sigma)\{g(v'_*)f(v') - g(v_*)f(v)\} d\sigma dv_*$$

Here, $\sigma \in \mathbb{S}^2$ (unit sphere of $\mathbb{R}^3$), and the collisional velocities:

$$v' = \frac{v + v_*}{2} + \frac{|v + v_*|}{2}\sigma, \ v'_* = \frac{v + v_*}{2} - \frac{|v + v_*|}{2}\sigma.$$

The Boltzmann collision cross section $B(|z|, \sigma)$ is a non-negative function which described as follows:

$$B(|v - v_*|, \sigma) = \Phi(|v - v_*|) b(\cos\theta) \tag{1.3}$$

where $\cos\theta = < \frac{v - v_*}{|v - v_*|}, \sigma >, \ \theta \in [0, \frac{\pi}{2}]$,

$$\begin{cases} \Phi(|v - v_*|) = (1 + |v - v_*|^2)^{\frac{\gamma}{2}}, \ 0 < \gamma < 1, \\ b(\cos\theta) \approx \frac{K}{\theta^{2+\nu}}, \ \theta \to 0, \ 0 < \nu < 2 \end{cases} \tag{1.4}$$

We have studied the mild singularity case $0 < \nu < 1$ in [4]. In this paper, we move on to discuss the strong singularity case $1 \leq \nu < 2$. Setting

$$\begin{cases} ||f||_{L^p_r} = || < |v| >^r f(v)||_{L^p} \\ ||f||_{H^s_r} = || < |D| >^s < |v| >^r f(v)||_{L^2} \end{cases}$$

where $< |v| >= (1 + |v|^2)^{\frac{1}{2}}$ and $< |D| >$ is the corresponding pseudo-differential operator. We first list the following definitions:

Definition 1.1 (cf. [6]). For an initial datum $f_0(v) \in L^1_2(\mathbb{R}^3)$, $f(t, v)$ is called a weak solution of the Cauchy problem (1.1) if it satisfies $f(0, v) = f_0$ and

$$\int_{\mathbb{R}^3} f(t,v)\varphi(t,v)dv - \int_{\mathbb{R}^3} f(0,v)\varphi(0,v)dv$$

$$- \int_0^t d\tau \int_{\mathbb{R}^3} f(\tau,v)\partial_\tau \varphi(\tau,v)dv$$

$$= \int_0^t d\tau \int_{\mathbb{R}^3} L(f)(\tau,v)\varphi(\tau,v)dv,$$

for any test function $\varphi \in L^\infty([0,T]; W^{2,\infty}(\mathbb{R}^3))$.

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Definition 1.2 (cf. [2, 3, 5, 6, 7]). Suppose that U is a bounded open set on $\mathbb{R}^3$, for $s \geq 1$, $u \in G^s(U)$ which is the Gevrey class function space with index s, if there exists a constant $C = C(U) > 0$ such that for any $k \in \mathbb{N}$,

$$||D^k u||_{L^2(U)} \leq C^{k+1}(k!)^s,$$

or equivalently,

$$|| < |D| >^k u||_{L^2(U)} \leq C^{k+1}(k!)^s$$

where

$$||D^k u||^2_{L^2(U)} = \sum_{|\beta|=k} ||D^\beta u||^2_{L^2(U)}, \; < |D| >= (1+|D_v|^2)^{\frac{1}{2}}.$$

Particularly, $u \in G^s(\mathbb{R}^3)$, i.e., $||D^k u||_{L^2(\mathbb{R}^3)} \leq C^{k+1}(k!)^s$, is equivalent to the fact that there exists $\varepsilon_0 > 0$ such that $e^{\varepsilon_0 <|D|>^{1/s}} u \in L^2(\mathbb{R}^3)$.

Now we state our main result as below:

Theorem 1.1. Suppose that Φ, b have the forms in (1.4), $1 < \nu < 2$. Let U be a bounded open set of $\mathbb{R}^3$, and $f(t,v)$ be the weak solution of the Cauchy problem (1.1) satisfying

$$\sup_{t\in(0,T]} ||f(t,\cdot)||_{L^2(U)} < +\infty. \tag{1.5}$$

Then for any $t \in (0,T]$, there exists a number $s = s(t) > 3$ satisfying $f(t,\cdot) \in G^s(U)$. More precisely, for any fixed $0 < t_0 \leq T$, there exist a constant $C = C(U) > 0$ and a number $s > 3$ such that for any $k \in \mathbb{N}$,

$$\sup_{t\in[t_0,T]} ||D^k f(t,\cdot)||_{L^2(U)} \leq C^{k+1}(k!)^s.$$

2. Proof of main result

We will prove Theorem 1.1 in this section. Let us start with some preliminaries which are used throughout this paper, cf. [4]:

Lemma 2.1. Suppose that $\Phi(v) =< |v| >^\gamma = (1+|v|^2)^{\frac{\gamma}{2}}$ where $\gamma \in (0,1)$, $v \in \mathbb{R}^n$ and $n \in \mathbb{N}$. Then the k-th order derivative of Φ satisfying

$$|\Phi^{(k)}(v)| \leq 4^k k! \Phi(v) < |v| >^{-k}.$$

Remark 2.1. Let $M_N(\xi) = (1+|\xi|^2)^{\frac{Nt}{2}}, \xi \in \mathbb{R}^3, N \in \mathbb{N}$. Then for any $t \in (0,1]$,

$$\begin{aligned}|\partial^k_\xi M_N(\xi)| &\leq 4^k < |\xi| >^{Nt-k} |N(N-1)\cdots(N-k+1)| \\ &\leq 4^k < |\xi| >^{(N-k)t} |N(N-1)\cdots(N-k+1)|,\end{aligned}$$

where $k \in \mathbb{N}, 1 \leq k \leq N$.

Lemma 2.2. Suppose the Fourier transform for v_*,

$$\mathscr{F}(\Phi(|v-v_*|)\mu(v_*))(\xi) = h(v,\xi)\hat{\mu}(\xi),$$

where μ is the absolute Maxwellian distribution in (1.1), and $\hat{\mu}(\xi)$ is its Fourier transform. Then we have

$$\begin{aligned}h(v,\xi) = (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} e^{-\frac{|v_*|^2}{2}} [1+|v-v_*|^2 - |\xi|^2 \\ +2i(v-v_*)\cdot\xi]^{\frac{\gamma}{2}} dv_*.\end{aligned}$$

Lemma 2.3. For the expression of $h(v,\xi)$ in Lemma 2.4, we have

$$|h(v,\xi)| \leq C\cdot < |v| >^\gamma < |\xi| >^\gamma$$

and

$$|\nabla^2_\xi h(v,\xi)| \leq C\cdot < |v| >^\gamma < |\xi| >^\gamma$$

and

$$|h(v,\xi^+) - h(v,\xi)| \leq C\cdot < |v| >^\gamma < |\xi| >^{1+\gamma} \sin\frac{\theta}{2}$$

where $\theta = \arccos < \frac{\xi}{|\xi|}, \sigma >, \xi^+ = \frac{\xi+|\xi|\sigma}{2}$, C is a constant independent of v and ξ.

And then we adopt the same assumption as in [4]. That is, without loss of generality, we suppose that f has compact support in $\overline{U}$, and let

$$(E_k): \text{for any } i \in [0,k-1], \sup_{t\in(0,T]} ||f(t,\cdot)||_{H^{it}} \leq C_0^{i+1}(i!)^s$$

where $T \leq 1$, C_0 is a sufficiently large constant satisfying

$$C_0 \geq 16^6 \max(\sup_{t\in(0,T]} ||f||_{L^i}, i=1,2)).$$

Setting $\xi^\pm = \frac{\xi\pm|\xi|\sigma}{2}$. By [1], we have

$$\left(Lf, M_k^2 f\right)_{L^2} = I + I_0 + I_1 + I_2 + I_3. \tag{2.1}$$

Here,

$$I = \left(Q(\mu, M_k f), M_k f\right)_{L^2},$$

$$I_0 = \left(Q(f,\mu), < |D| >^r f\right)_{L^2},$$

$$\begin{aligned}I_1 = \int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b(\cos\theta)\mu(v_*)\left(M_k(\xi) - M_k(\xi^+)\right)\widehat{\Phi^* f}(\xi^+) \\ e^{-iv_*\xi^-}\overline{M_k(\xi)\hat{f}(\xi)} d\sigma dv_* d\xi,\end{aligned}$$

$$\begin{aligned}I_2 = \int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b(\cos\theta)\mu(v_*)\{[M_k,\Phi^*]f(v')\cdot M_k f(v') \\ - [M_k,\Phi^*]f(v)\cdot M_k f(v)\} d\sigma dv_* dv\end{aligned}$$

and

$$\begin{aligned}I_3 = \int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b(\cos\theta)\mu(v_*)\left([M_k,\Phi^*]f(v) - [M_k,\Phi^*]f(v')\right) \\ M_k f(v') d\sigma dv_* dv\end{aligned}$$

where $M_k(D_v) =< |D_v| >^{kt}, \Phi^*(v) =< |v-v_*| >^\gamma$. The following inequalities come from [4]:

$$\begin{cases}|I| \leq -C||M_k f||^2_{H^{\frac{v}{2}}} + C'||M_k f||^2_{L^2}; \\ |I_1| \leq C\cdot\left(k^2||M_k f||^2_{L^2} + [C_0^{k+1}(k!)^s]^2\right); \\ |I_2| \leq C\{[C_0^{k+1}(k!)^s]^2 + ||M_k f||^2_{L^2}\}.\end{cases} \tag{2.2}$$

In the case $1 < \nu < 2$, the major difficulty comes from the estimation of I_0 and I_3. The way of [4] can not be used directly because of the fact that

$$\int_{\mathbb{S}^2} b\theta d\sigma = \infty.$$

Therefore, we must use some new methods which are different from that of [4] to overcome this difficulty. we state the following lemmas here and prove them in Section 4:

Lemma 2.4. For any $r > 0, f \in L^1_{2+\gamma}(\mathbb{R}^3) \cap H^{+\infty}(\mathbb{R}^3)$, there exists a constant C independent of r satisfying

$$I_0 \le C||f||_{L^1_{2+\gamma}}||f||_{L^1}(r+3)!$$

Lemma 2.5. Under the assumption (E_k), for any fixed number $\varepsilon \in (0, 1-\frac{\nu}{2})$, there exists a constant C satisfying

$$I_3 \le C\big\{[C_0^{k+1}(k!)^s]^2 + k^4||M_k f||^2_{H^{(\nu-\varepsilon)/2}}\big\}.$$

Therefore, By using Lemma 3.1, Lemma 3.2 and (3.1), (3.2), we conclude that

$$\big|\big(Lf, M_k^2 f\big)_{L^2}\big| \le C_1\big\{[C_0^{k+1}(k!)^s]^2 + k^4||M_k f||^2_{H^{(\nu-\varepsilon)/2}}\big\}$$
$$-C_2||M_k f||^2_{H^{\frac{\nu}{2}}}$$

which implying

$$\big|\big(\frac{\partial f}{\partial t}, M_k^2 f\big)_{L^2}\big| \le C_1\big\{[C_0^{k+1}(k!)^s]^2 + k^4||M_k f||^2_{H^{(\nu-\varepsilon)/2}}\big\}$$
$$-C_2||M_k f||^2_{H^{\frac{\nu}{2}}}.$$

That is,

$$\begin{aligned} &||M_k f(t,v)||^2_{L^2} + C_2\int_0^t ||M_k f||^2_{H^{\frac{\nu}{2}}} d\tau \\ &\le 2k\int_0^t ||(\log < D_v >)^{\frac{1}{2}}(M_k f)(\tau)||^2_{L^2} d\tau \\ &+ C_1\int_0^t k^4||M_k f||^2_{H^{(\nu-\varepsilon)/2}} d\tau \\ &+ C_1[C_0^{k+1}(k!)^s]^2 + ||f_0(v)||^2_{L^2}. \end{aligned}$$

As the same way as in [4], by using the Young's inequality and the assumption (E_k), we also get (E_{k+j}) :

for any $i \in [0, k+j-1]$, $\sup_{t\in(0,T]} ||f(t,\cdot)||_{H^{it}} \le C_3^j \cdot C_0^{i+1}(i!)^s$.

This completes the proof of Theorem 1.1 by induction as in [4].

3. Appendix

This section is devoted to the proofs of Lemma 2.4 and Lemma 2.5.

Proof of Lemma 2.4. Since $\xi^{\pm} = \frac{\xi \pm |\xi|\sigma}{2}$, by using Lemma 2.2, it follows from Lemma 2.6 of [4] that

$$\begin{aligned} I_0(\tau) = &\int_{\mathbb{R}^3} < |\xi| >^r \overline{\hat{f}(\xi)} \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} [h(v,\xi^+) - h(v,\xi)]\hat{\mu}(\xi^+) \\ &bf(v)e^{-iv\cdot\xi^-} d\sigma dv d\xi \\ &+ \int_{\mathbb{R}^3} < |\xi| >^r \overline{\hat{f}(\xi)} \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} h(v,\xi)[\hat{\mu}(\xi^+) - \hat{\mu}(\xi)] \\ &bf(v)e^{-iv\cdot\xi^-} d\sigma dv d\xi \\ &+ \int_{\mathbb{R}^3} < |\xi| >^r \overline{\hat{f}(\xi)} \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} h(v,\xi)\hat{\mu}(\xi) \\ &bf(v)[e^{-iv\cdot\xi^-} - e^0] d\sigma dv d\xi \\ &= I_{01} + I_{02} + I_{03}. \end{aligned}$$

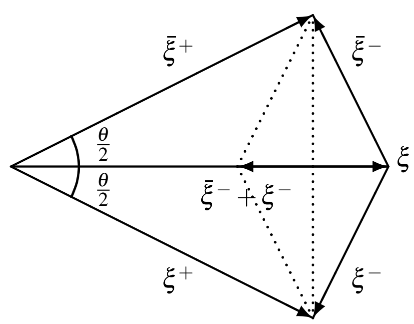
Here,

$$|I_{02}| \le C||f||_{L^1_{1+\gamma}}||f||_{L^1}(r+3)!$$

To estimate I_{01} and I_{03}, we use the symmetry of the integral variables. Notice that the unit vector σ denotes the rotation of the unit vector $\frac{\xi}{|\xi|}$ by an angle θ, i.e., $\sigma = R_\theta(\frac{\xi}{|\xi|})$. Corresponding, the inverse rotation $\bar{\sigma} = R_{-\theta}(\frac{\xi}{|\xi|})$ and moreover, suppose that $\bar{\xi}^{\pm} = \frac{\xi \pm |\xi|\bar{\sigma}}{2}$, then by a simple calculation, we get (cf. Figure 1 as follows)

$$\begin{cases} |\xi^+| = |\bar{\xi}^+| = |\xi|\cos\frac{\theta}{2}, \\ |\xi^-| = |\bar{\xi}^-| = |\xi|\sin\frac{\theta}{2}, \\ |\xi^+ + \bar{\xi}^+ - 2\xi| = |\xi^- + \bar{\xi}^-| = 2|\xi|\sin^2\frac{\theta}{2}. \end{cases} \quad (3.1)$$

Figure 1:



Therefore,

$$
\begin{aligned}
I_{03} &= \int_{\mathbb{R}^3} <|\xi|>^r \overline{\hat{f}(\xi)} \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} h(v,\xi)\hat{\mu}(\xi) bf(v) \\
&\quad (e^{-iv\cdot\xi^-} - e^0) d\sigma dv d\xi \\
&= -\int_{\mathbb{R}^6}\int_{\mathbb{S}^2} <|\xi|>^r \overline{\hat{f}(\xi)} h(v,\xi)\hat{\mu}(\xi) bf(v) \\
&\quad \Big(\sin^2\frac{v\cdot\xi^-}{2} + \sin^2\frac{v\cdot\bar{\xi}^-}{2}\Big) d\sigma dv d\xi \\
&\quad - i\int_{\mathbb{R}^6}\int_{\mathbb{S}^2} <|\xi|>^r \overline{\hat{f}(\xi)} h(v,\xi)\hat{\mu}(\xi) bf(v) \\
&\quad \sin\frac{v\cdot(\xi^- + \bar{\xi}^-)}{2}\cos\frac{v\cdot(\xi^- - \bar{\xi}^-)}{2} d\sigma dv d\xi \\
&= I_{031} + I_{032}.
\end{aligned}
$$

Here we have used the fact that $d\sigma = d\bar{\sigma}$. Since in this case $1 < \nu < 2$, $\int_{\mathbb{S}^2} b\theta^2 d\sigma < \infty$, by using Lemma 2.3 and (3.1), we get

$$
\begin{aligned}
|I_{031}| &\le C\int_{\mathbb{R}^6}\int_{\mathbb{S}^2} <|\xi|>^r |\overline{\hat{f}(\xi)}||h(v,\xi)|\hat{\mu}(\xi) bf(v)|v|^2 \\
&\quad |\xi^-|^2 d\sigma dv d\xi \\
&\le C'\int_{\mathbb{R}^6} <|\xi|>^{r+\gamma+2} |\overline{\hat{f}(\xi)}|\hat{\mu}(\xi) f(v) <|v|>^{\gamma+2} \\
&\quad dv d\xi \\
&\le C''||f||_{L^1_{2+\gamma}}||f||_{L^1}(r+3)!
\end{aligned}
$$

and

$$
\begin{aligned}
|I_{032}| &\le C\int_{\mathbb{R}^6}\int_{\mathbb{S}^2} <|\xi|>^r |\overline{\hat{f}(\xi)}||h(v,\xi)|\hat{\mu}(\xi) bf(v)|v||\xi| \\
&\quad \theta^2 d\sigma dv d\xi \\
&\le C'\int_{\mathbb{R}^6} <|\xi|>^{r+\gamma+1} |\overline{\hat{f}(\xi)}|\hat{\mu}(\xi) f(v) <|v|>^{\gamma+1} \\
&\quad dv d\xi \\
&\le C''||f||_{L^1_{1+\gamma}}||f||_{L^1}(r+3)!,
\end{aligned}
$$

which implies

$$|I_{03}| \le C||f||_{L^1_{2+\gamma}}||f||_{L^1}(r+3)!.$$

To prove I_{01}, we first note that

$$
\begin{aligned}
& h(v,\xi^+) - h(v,\xi) \\
&= \int_0^1 (\xi^+ - \xi)\cdot[\nabla_\xi h(v,\xi_\tau) - \nabla_\xi h(v,\xi)] d\tau \\
&\quad + (\xi^+ - \xi)\cdot\nabla_\xi h(v,\xi) \\
&= \int_0^1 (\xi^+ - \xi)\cdot\{\int_0^1 (\xi_\tau - \xi)\cdot\nabla_\xi^2 h\big(v,\xi + \eta(\xi_\tau - \xi)\big) \\
&\quad d\eta\} d\tau + (\xi^+ - \xi)\cdot\nabla_\xi h(v,\xi)
\end{aligned}
$$

where $\xi_\tau = \xi + \tau(\xi^+ - \xi)$. Therefore,

$$
\begin{aligned}
I_{01} &= \int_{\mathbb{R}^6} <|\xi|>^r \overline{\hat{f}(\xi)} \int_{\mathbb{S}^2} \hat{\mu}(\xi^+) bf(v) e^{-iv\cdot\xi^-} \int_0^1 (\xi^+ - \xi) \\
&\quad \cdot\{\int_0^1 (\xi_\tau - \xi)\cdot\nabla_\xi^2 h\big(v,\xi + \eta(\xi_\tau - \xi)\big) d\eta\} d\tau d\sigma dv d\xi \\
&\quad + \int_{\mathbb{R}^6} <|\xi|>^r \overline{\hat{f}(\xi)} \int_{\mathbb{S}^2} \hat{\mu}(\xi^+) bf(v)(e^{-iv\cdot\xi^-} - 1) \\
&\quad \cdot(\xi^+ - \xi)\nabla_\xi h(v,\xi) d\sigma dv d\xi \\
&\quad + \int_{\mathbb{R}^6} <|\xi|>^r \overline{\hat{f}(\xi)} \int_{\mathbb{S}^2} \hat{\mu}(\xi^+) bf(v)(\xi^+ - \xi) \\
&\quad \cdot\nabla_\xi h(v,\xi) d\sigma dv d\xi \\
&= I_{011} + I_{012} + I_{013}.
\end{aligned}
$$

By Lemma 2.3,

$$
\begin{aligned}
& |\int_0^1 (\xi^+ - \xi)\cdot\{\int_0^1 (\xi_\tau - \xi)\cdot\nabla_\xi^2 h\big(v,\xi + \eta(\xi_\tau - \xi)\big) d\eta\} d\tau| \\
&\le \int_0^1\int_0^1 |(\xi^+ - \xi)|\cdot|(\xi_\tau - \xi)|\cdot|\nabla_\xi^2 h\big(v,\xi + \eta(\xi_\tau - \xi)\big)| d\eta d\tau \\
&\le C <|v|>^\gamma <|\xi|>^{\gamma+2} \theta^2
\end{aligned}
$$

which gives

$$|I_{011}| \le C||f||_{L^1_\gamma}||f||_{L^1}(r+3)!$$

and

$$|I_{012}| \le C||f||_{L^1_{\gamma+1}}||f||_{L^1}(r+3)!.$$

Finally, we estimate I_{013}, recall $d\sigma = d\bar{\sigma}$ and notice that $\hat{\mu}(\xi^+) = \hat{\mu}(\bar{\xi}^+)$, we have

$$
\begin{aligned}
I_{013} &= \int_{\mathbb{R}^6} <|\xi|>^r \overline{\hat{f}(\xi)} \int_{\mathbb{S}^2} \hat{\mu}(\xi^+) bf(v)(\xi^+ - \xi) \\
&\quad \cdot\nabla_\xi h(v,\xi) d\sigma dv d\xi \\
&= \int_{\mathbb{R}^6} <|\xi|>^r \overline{\hat{f}(\xi)} \int_{\mathbb{S}^2} \hat{\mu}(\bar{\xi}^+) bf(v)(\bar{\xi}^+ - \xi) \\
&\quad \cdot\nabla_\xi h(v,\xi) d\bar{\sigma} dv d\xi \\
&= \frac{1}{2}\int_{\mathbb{R}^6} <|\xi|>^r \overline{\hat{f}(\xi)} \int_{\mathbb{S}^2} \hat{\mu}(\xi^+) bf(v)(\xi^+ + \bar{\xi}^+ - 2\xi) \\
&\quad \cdot\nabla_\xi h(v,\xi) d\sigma dv d\xi.
\end{aligned}
$$

Together with (3.1) and Lemma 2.3, we get

$$|I_{013}| \le C||f||_{L^1_\gamma}||f||_{L^1}(r+3)!.$$

The above estimates complete the proof of Lemma 2.4. □

Proof of Lemma 2.5. setting $v'_\tau = v' + \tau(v - v')$, we have

$$I_3 = \int_0^1 \int_{\mathbb{R}^6\times\mathbb{S}^2} b\mu(v_*)M_k f(v')(v-v')\big(\nabla_v[M_k,\Phi^*]f(v'_\tau) - \nabla_v[M_k,\Phi^*]f(v')\big)d\sigma dv dv_* d\tau + \int_{\mathbb{R}^6\times\mathbb{S}^2} b\mu(v_*)M_k f(v')(v-v') \cdot \big(\nabla_v[M_k,\Phi^*]f(v')\big)d\sigma dv dv_* = \int_0^1 I_{32}(\tau)d\tau + I_{31}. \tag{3.2}$$

It follows from [1, P. 468] that

$$I_{31} = 0. \tag{3.3}$$

By using the formulas (5.1)-(5.3) in [4], we can decompose $I_{32}(\tau)$ as follows:

$$\begin{aligned} I_{32}(\tau) &= \int_{\mathbb{R}^6\times\mathbb{S}^2} b\mu(v_*)M_k f(v')(v-v') \cdot [\nabla_v\Gamma_1 f(v'_\tau) - \nabla_v\Gamma_1 f(v')]d\sigma dv dv_* \\ &+ \sum_{j=2}^{k+7}\int_{\mathbb{R}^6\times\mathbb{S}^2} b\mu(v_*)M_k f(v')(v-v') \cdot [\nabla_v\Gamma_j f(v'_\tau) - \nabla_v\Gamma_j f(v')]d\sigma dv dv_* \\ &+ \int_{\mathbb{R}^6\times\mathbb{S}^2} b\mu(v_*)M_k f(v')(v-v') \cdot (\nabla_v\Gamma_{k+8} f'_\tau - \nabla_v\Gamma_{k+8} f')d\sigma dv dv_* \\ &= I_{321}(\tau) + I_{322}(\tau) + I_{323}(\tau). \end{aligned} \tag{3.4}$$

Since

$$\nabla_v^2\Gamma_j f(v) = \frac{(-i)^j}{j!}\{M_k^{(j)}\nabla_v^2 f(v)\cdot\partial_v^j\Phi^*(v) + 2M_k^{(j)}\nabla_v f(v)\cdot\partial_v^{j+1}\Phi^*(v) + M_k^{(j)} f(v)\cdot\partial_v^{j+2}\Phi^*(v)\}$$

and

$$\begin{aligned} &\nabla_v\Gamma_{k+8} f'_\tau - \nabla_v\Gamma_{k+8} f' \\ &= \frac{(-i)^{k+8}}{(k+8)!}\Big\{\int_{\mathbb{R}^6}[e^{i(v'_\tau-y)\cdot\xi} - e^{i(v'-y)\cdot\xi}]\cdot(i\xi) \times \partial_\xi^{k+8}M_k(\xi)d\xi f(y)\partial_v^{k+8}\Phi^*(c'_\tau)dy \\ &+ \int_{\mathbb{R}^6} e^{i(v'-y)\cdot\xi}\cdot(i\xi)\partial_\xi^{k+8}M_k(\xi)d\xi f(y) \times [\partial_v^{k+8}\Phi^*(c'_\tau) - \partial_v^{k+8}\Phi^*(c')]dy \\ &+ \int_{\mathbb{R}^6}[e^{i(v'_\tau-y)\cdot\xi} - e^{i(v'-y)\cdot\xi}]\partial_\xi^{k+8}M_k(\xi)d\xi f(y) \times \partial_v^{k+9}\Phi^*(c'_\tau)dy \\ &+ \int_{\mathbb{R}^6} e^{i(v'-y)\cdot\xi}\partial_\xi^{k+8}M_k(\xi)d\xi f(y) \times [\partial_v^{k+9}\Phi^*(c'_\tau) - \partial_v^{k+9}\Phi^*(c')]dy. \end{aligned}$$

For any $j \ge 2$, putting $v''_\eta = \eta v'_\tau + (1-\eta)v'$, $\eta \in [0,1]$. Using the facts that

$$\begin{cases} |v - v'| = O(1)|v'_\tau - v'| \le C\cdot <|v_*|><|v'|> \theta; \\ |\frac{dv}{dv'}| = O(1)|\frac{dv}{dv''_\eta}| \le C, \end{cases}$$

we conclude

$$\begin{aligned} &\int_{\mathbb{R}^6\times\mathbb{S}^2} b\mu(v_*)M_k f(v')(v-v')\cdot[\nabla_v\Gamma_j f(v'_\tau) - \nabla_v\Gamma_j f(v')] d\sigma dv dv_* \\ &\le \int_0^1\int_{\mathbb{R}^6\times\mathbb{S}^2} b\mu(v_*)|M_k f(v')|\cdot|v-v'|\cdot|v'_\tau - v'| \cdot|\nabla_v^2\Gamma_j f(v''_\eta)|d\sigma dv dv_* d\eta \\ &\le C\int_0^1\int_{\mathbb{R}^6\times\mathbb{S}^2} b\theta^2\mu(v_*) <|v_*|>^2 \big(|M_k f(v')|^2 + |\nabla_v^2\Gamma_j f(v''_\eta)|^2\big)d\sigma dv dv_* d\eta \\ &\le C\big(||M_k f||_{L^2}^2 + ||\nabla_v^2\Gamma_j f||_{L^2}^2\big). \end{aligned}$$

Use the same way as in the proof of Proposition 3.1 of [4], by Lemma 2.1, Remark 2.1, it follows that

$$|I_{322}(\tau)| \le C_{17}\{[C_0^{k+1}(k!)^s]^2 + k^4||M_k f||_{L^2}^2\}. \tag{3.5}$$

Similarly, by using the fact that $|c'_\tau - c'| \le C|v'_\tau - v'|$, we obtain

$$|I_{323}(\tau)| \le C_{18}[C_0^{k+1}(k!)^s]^2. \tag{3.6}$$

Hence, it remains to estimate $I_{321}(\tau)$. To do this, we decompose $I_{321}(\tau)$ as below:

$$I_{321}(\tau) = (-i)\cdot\{J_1(\tau) + J_2(\tau) + J_3(\tau) + J_4(\tau)\}, \tag{3.7}$$

where

$$J_1(\tau) = \int_{\mathbb{R}^6\times\mathbb{S}^2} b\mu(v_*)M_k f(v')(v-v')\cdot M_k^{(1)} f(v'_\tau) [\partial_v^2\Phi^*(v'_\tau) - \partial_v^2\Phi^*(v')]d\sigma dv dv_*,$$

$$J_2(\tau) = \int_{\mathbb{R}^6\times\mathbb{S}^2} b\mu(v_*)M_k f(v')(v-v') \cdot[M_k^{(1)} f(v'_\tau) - M_k^{(1)} f(v')]\partial_v^2\Phi^*(v')d\sigma dv dv_*$$

and

$$J_3(\tau) = \int_{\mathbb{R}^6\times\mathbb{S}^2} b\mu(v_*)M_k f(v')(v-v')\cdot M_k^{(1)}\nabla_v f(v'_\tau) [\partial_v^1\Phi^*(v'_\tau) - \partial_v^1\Phi^*(v')]d\sigma dv dv_*;$$

$$J_4(\tau) = \int_{\mathbb{R}^6\times\mathbb{S}^2} b\mu(v_*)M_k f(v')(v-v')\cdot\partial_v^1\Phi^*(v') [M_k^{(1)}\nabla_v f(v'_\tau) - M_k^{(1)}\nabla_v f(v')]d\sigma dv dv_*.$$

Taking the same analysis as in (3.5), we also conclude that

$$|J_n(\tau)| \le C\{[C_0^{k+1}(k!)^s]^2 + k^2||M_k f||_{L^2}^2\}, \tag{3.8}$$

for $n = 1,2,3$. In order to estimate $J_4(\tau)$, we shall take the Littlewood-Paley partition of unity $\{\psi_j(\xi)\}$ such as

$$\sum_{j=0}^{+\infty}\psi_j(\xi)\equiv 1,\ \psi_j(\xi)=\psi(2^{-j}\xi)\text{ for } j\geq 1.$$

Here, $0\leq\psi_0,\psi\in C_0^{+\infty}(\mathbb{R}^3)$. Therefore,

$$J_4(\tau)=\sum_{j=0}^{+\infty}\int_{\mathbb{R}^6\times\mathbb{S}^2} b\mu(v_*)M_kf(v')(v-v')\cdot\partial_v^1\Phi^*(v')\,[\tilde{\psi}_j(v'_\tau)-\tilde{\psi}_j(v')]d\sigma dv dv_* \tag{3.9}$$

where $\tilde{\psi}_j(v)=\psi_j(D_v)M_k^{(1)}(D_v)\nabla_v f(v)$. Fix a number $0<\varepsilon<1-\frac{\nu}{2}$, put

$$\Omega_j=\Omega_j(v,v_*)=\Big\{\sigma\in\mathbb{S}^2;\frac{v-v_*}{|v-v_*|}\cdot\sigma\geq\frac{1-2^{1-j(2+2\varepsilon-\nu)/(2-\nu)}}{<|v-v_*|>^2}\Big\}.$$

It follows from [1, P. 470] that

$$\int_{\Omega_j}b\theta^2d\sigma\leq C\cdot 2^{j(\frac{\nu}{2}-\varepsilon-1)}<|v-v_*|>^{\nu-2}\leq C\cdot 2^{j(\frac{\nu}{2}-\varepsilon-1)} \tag{3.10}$$

and

$$\int_{\Omega_j^c}b\theta d\sigma\leq C\cdot 2^{j(\frac{\nu}{2}-\varepsilon)}<|v-v_*|>^{\nu-1}. \tag{3.11}$$

Therefore,

$$\begin{aligned}&\int_{\mathbb{R}^6\times\Omega_j}b\mu(v_*)M_kf(v')(v-v')\cdot\partial_v^1\Phi^*(v')\,[\tilde{\psi}_j(v'_\tau)-\tilde{\psi}_j(v')]d\sigma dv dv_*\\ &\leq C\int_0^1\int_{\mathbb{R}^6\times\Omega_j}b\theta^2\mu(v_*)<|v_*|>^2\times\{2^{\frac{(2-\nu)j}{2}}|M_kf(v)|^2+2^{\frac{(\nu-2)j}{2}}|\nabla_v\tilde{\psi}_j(v)|^2\}d\sigma dv dv_* d\eta\\ &\leq C\{2^{-j\varepsilon}||M_kf(v)||_{L^2}^2+2^{j(\nu-\varepsilon-2)}||\nabla_v\tilde{\psi}_j(v)||_{L^2}^2\}\\ &\leq C\{2^{-j\varepsilon}||M_kf(v)||_{L^2}^2+k^2||\psi_jM_kf||^2_{H^{(\nu-\varepsilon)/2}}\}.\end{aligned} \tag{3.12}$$

Here we use the fact that

$$2^{\frac{j(\nu-\varepsilon-2)}{2}}||\nabla_v\tilde{\psi}_j(v)||_{L^2}\leq Ck||\psi_jM_kf||_{H^{(\nu-\varepsilon)/2}}.$$

On the other hand, by (3.11), we also have

$$\begin{aligned}&\int_{\mathbb{R}^6\times\Omega_j^c}b\mu(v_*)M_kf(v')(v-v')\cdot\partial_v^1\Phi^*(v')[\tilde{\psi}_j(v'_\tau)-\tilde{\psi}_j(v')]d\sigma dv dv_*\\ &\leq C\cdot 2^{-j\varepsilon}||M_kf(v)||_{L^2}^2+C\cdot 2^{j(\nu-\varepsilon)}||\tilde{\psi}_j(v)||^2_{L^2}\\ &\leq C\cdot 2^{-j\varepsilon}||M_kf(v)||_{L^2}^2+C\cdot k^2||\psi_jM_kf||^2_{H^{(\nu-\varepsilon)/2}}.\end{aligned}$$

This, together with (3.12) and the fact that $\sum_{j=0}^{+\infty}|\psi_j|^2<C<+\infty$, gives

$$|J_4(\tau)|\leq C\{||M_kf||^2_{L^2}+k^2||M_kf||^2_{H^{(\nu-\varepsilon)/2}}\} \tag{3.13}$$

It follows from (3.2)-(3.13) that

$$|I_3|\leq C\{[C_0^{k+1}(k!)^s]^2+k^4||M_kf||^2_{H^{(\nu-\varepsilon)/2}}\}$$

This completes the proof of Lemma 2.5. □

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