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Inferences for New Weibull-Pareto Distribution Based on Progressively Type-II Censored Data

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Abstract: In this paper, the problem of estimation for the new Weibull-Pareto distribution based on progressive Type-II censored sample is studied. The maximum likelihood, Bayes and parametric bootstrap methods are used for estimating the three unknown parameters as well as some lifetime parameters reliability and hazard functions. Based on the asymptotic normality of maximum likelihood estimators we construct the approximate confidence intervals of the parameters. Futhermor, depending on the delta and parametric bootstrap methods we calculate the approximate confidence intervals (ACIs) of the reliability and hazard functions. Markov chain Monte Carlo (MCMC) technique is applied to computing the Bayes estimate and the credible intervals of the unknown parameters as well as reliability and hazard functions which are obtained under the assumptions of informative and non-informative priors based on the Gibbs within Metropolis-Hasting samplers procedure. The results of Bayes methed are obtained under squared error loss (SEL) function. Finally, Two examples used to a simulated data and a real life data sets have been presented for illustrative purposes.

Keywords: New Weibull-Pareto distribution (NWPD), Progressive Type-II censored samples, Parametric bootstrap, Bayesian estimation, MCMC technique.

1 Introduction

There are many situations in life testing and reliability experiments whose units are lost or removed from the experiment before the failure occures. However, in many situations, the removal of units prior to failure is pre-planned in order to provide saving in terms of time and cost associated with testing. There are many types of censored test, the most important and used censored schemes are Type-I and Type-II censoring. If an experimenter desires to remove surviving units at any point on the test. But using this type of censoring are not able him to removed units from the test at any other point than the final termination point of the life test. So these two traditional censoring schemes will not be of use to the experimenter. For this reason we consider a more general censoring scheme called progressive Type-II censoring. The progressively Type-II censored sample can be described as follows. Suppose that n independent units are put in the life test with continuous identical and independent distributed failure times $X_1, X_2, ..., X_n$ and censoring scheme $(R_1, R_2, ..., R_m)$. When the first failure X_1 occures, R_1 surviving units are withdrawn from the test at random. By the same when the second failure X_2 occures, R_2 surviving units are withdrawn from the test at random. Finally, when the m^{th} failure occures, all of the remaining surviving units are withdrawn from the test. The m ordered observed times is denoted by $X_{1:m:n}^{(R_1,...,R_m)}, X_{2:m:n}^{(R_1,...,R_m)}, ..., X_{m:m:n}^{(R_1,...,R_m)}$ are called progressively Type-II censored order statistics of size m taken from sample of size n with progressive censoring scheme $(R_1, R_2, ..., R_m)$. It is clear that $n = m + \sum_{i=1}^{m} R_i$. The special case when $R_1 = R_2 = \cdots = R_{m-1}$ imply $R_m = n - m$, then the progressive Type-II censoning sample reduce to the traditional Type-II censoring sample. Also when $R_1 = R_2 = \cdots = R_m = 0$ imply $m = n$, then the progressive Type-II censoring sample reduce to no censoring (ordinary order statistics). For more information on progressive censoring, we refer the reader to Balakrishnan and Aggarwala [3], Balakrishnan and Sandhu [4] and Balakrishnan [2]. Many authors have discussed inference under progressive Type-II censoning using different lifetime distributions, see for example, Musleh and Helu [11], Soliman et al. [12], Mahmoud et al. [10], Madi and Raqab [9] and EL-Sagheer [6].

In this paper we interested in the estimation of the parameters, reliability and hazard functions when sample is available progressive Type-II censoring scheme from the new Weibull-Pareto distribution. A NWPD is a generalization of the

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Weibull and the Pareto distributions as discussed in Suleman and Albert [13]. The probability density function (PDF), cumulative distribution function (CDF), reliability function $S(t)$ and hazard rate function $h(t)$ of the NWPD are given, respectively, by

$$f(x;\delta,\beta,\theta)=\frac{\beta\delta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1}\exp\left\{-\delta\left(\frac{x}{\theta}\right)^{\beta}\right\}, \tag{1}$$

$$F(x;\delta,\beta,\theta)=1-\exp\left\{-\delta\left(\frac{x}{\theta}\right)^{\beta}\right\}, \tag{2}$$

$$S(t)=\exp\left\{-\delta\left(\frac{t}{\theta}\right)^{\beta}\right\}, \tag{3}$$

and

$$h(t)=\frac{\beta\delta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1}, \tag{4}$$

where β is the shape and δ and θ are the scale parameters. From (1), it should be noted that the NWPD reduce to well-known distributions such as Weibull, Reyleigh, Exponential and Frechet distributions as follow:

(1)If $\delta=\theta=1$, then NWPD reduces to Weibull$(\beta,1)$.
(2)If $\delta=1$, then NWPD reduces to Weibull(β,θ).
(3)If $\delta=1/2$ and $\beta=2$, then NWPD reduces to Rayleigh(θ).
(4)If $\beta=\theta=1$, then NWPD reduces to Exponential distribution with mean equal $1/\delta$.
(5)If $\delta=1$ and $\beta=-\beta$ then NWPD reduces to Frechet distribution(β,θ).

It is clear that the shape of the hazard rate function $h(t)$ as in (4), depends on the parameter β and the following can be observed:

(i)If $\beta=1$, the failure rate is constant and given by $h(t)=\delta/\theta$. This makes the NWPD suitable for modeling systems or components with constant failure rate.
(ii)If $\beta>1$, the hazard is an increasing function of x, which makes the NWPD suitable for modeling components that wears faster with time.
(iii)If $\beta<1$, the hazard is a decreasing function of x, which makes the NWPD suitable for modeling components that wears slower with time.

Futhermor, the lifetime of the NWPD is able to model data with bathtube-shaped hazard rate, which is important feature engineering relaibility analysis. The NWPD is useful in modeling real life situation. The newly proposed distribution was used to model the exceedances of flood peaks (in m^3/s) of the Wheaton River near Carcross in Yukon Territory, Canada. More about this distribution, its properties and applications see Suleman and Albert [13].

In this paper, we investigate the estimation of the unknown parameters for the NWPD using the progressive Type-II censored sample. Based on the Newton–Raphson iteration method we obtain the MLEs of the parameters by solving the non-linear equations. The estimaion of some lifetime parameters such as reliability and hazard funtions are considered. The ACIs for the reliability and hazard funtions can be constructed by using delta and parametric bootstrap methods. In Bayesian study, we propose to discuss the Bayes estimate for the NWPD by using the MCMC techniques. Based on Metropolis algorithem within Gibbs sampler, the Bayes estimates and the credible intervals of the parameters as well as reliability and hazard functions are obtained. The Bayes estimates has been studied under SEL function. Two examples used a simulated data and a real-life data sets have been presented to illustrate all the methods of estimation developed here.

The rest of this paper is organized as follows: In Section 2 the MLEs of the unkown parameters, reliability and hazard functions are obtained. ACIs for the parameters, reliability and hazard functions are discussed in Section 3. In Section 4, we introduce two parametric bootstrap procedures to construct the confidence intervals for the unknown parameters, reliability and hazard functions. Section 5, Bayesian study is presented. Two examples one of them used a simulated data and the other used a real data sets have been analyzed in Section 6. Finaly, we conclude the paper in Section 7.

2 Maximum Likelihood Estimation

Maximum likelihood estimation (MLE) is a very popular technique used for estimating the parameters of continuous distributions. If the failure times of the units originally on test with progressive censoring scheme $(R_1,R_2,...,R_m)$ are from a continuous population with PDF (1) and CDF (2), then the joint probability density function of a progressively Type-II censored sample $\underline{x} = X_{1:m:n}^{(R_1,\ldots,R_m)}, X_{2:m:n}^{(R_1,\ldots,R_m)}, \ldots, X_{m:m:n}^{(R_1,\ldots,R_m)}$ of size m from a sample of size n is given (see Balakrishnan and Aggarwala [3]) by

$$f_{x_1,x_2,\ldots,x_m}(x_1,x_2,\ldots,x_m) = A\prod_{i=1}^{m} f(x_i)[1-F(x_i)]^{R_i}, \tag{5}$$

where x_i is used instead of $X_{i:m:n}^{(R_1,\ldots,R_m)}$, $R_i \geqslant 0$, $i = 1,2,\ldots,m$ and

$$A = n(n-1-R_1)(n-2-R_1-R_2)\ldots\left(n-\sum_{i=1}^{m-1}(R_i+1)\right). \tag{6}$$

From (1) and (2), the likelihood function can be weitten as

$$L(\underline{x};\delta,\beta,\theta) = A\beta^m\delta^m\theta^{(-m)}\left[\prod_{i=1}^{m}\left(\frac{x_i}{\theta}\right)^{\beta-1}\right]\exp\left\{-\delta\sum_{i=1}^{m}(R_i+1)\left(\frac{x_i}{\theta}\right)^{\beta}\right\}, \tag{7}$$

where A is defined in (6). Therefore without the additive constant, the log-likelihood function of the observed data $\ell(\underline{x};\delta,\beta,\theta) = \log L(\underline{x};\delta,\beta,\theta)$ can be written as

$$\ell(\underline{x};\delta,\beta,\theta) = m\log(\beta)+m\log(\delta)-m\log(\theta)+(\beta-1)\sum_{i=1}^{m}\log\left(\frac{x_i}{\theta}\right)-\delta\sum_{i=1}^{m}(R_i+1)\left(\frac{x_i}{\theta}\right)^{\beta}. \tag{8}$$

The corresponding likelihood equations are

$$\frac{\partial\ell(\underline{x};\delta,\beta,\theta)}{\partial\delta} = \frac{m}{\delta}-\sum_{i=1}^{m}(R_i+1)\left(\frac{x_i}{\theta}\right)^{\beta} = 0, \tag{9}$$

$$\frac{\partial\ell(\underline{x};\delta,\beta,\theta)}{\partial\beta} = \frac{m}{\beta}+\sum_{i=1}^{m}\log\left(\frac{x_i}{\theta}\right)-\delta\sum_{i=1}^{m}(R_i+1)\left(\frac{x_i}{\theta}\right)^{\beta}\log\left(\frac{x_i}{\theta}\right) = 0, \tag{10}$$

and

$$\frac{\partial\ell(\underline{x};\delta,\beta,\theta)}{\partial\theta} = \frac{-m\beta}{\theta}+\frac{\beta\delta}{\theta}\sum_{i=1}^{m}(R_i+1)(\frac{x_i}{\theta})^{\beta} = 0. \tag{11}$$

From (9), we get the MLE of δ as a function of the MLEs of β and θ as

$$\hat{\delta} = m\left[\sum_{i=1}^{m}(R_i+1)\left(\frac{x_i}{\hat{\theta}}\right)^{\hat{\beta}}\right]^{-1}. \tag{12}$$

Since Equations (10)–(12) do not have closed form solutions, the Newton–Raphson iteration method is used to obtain the estimates. The algorithm is described as follows:

1.Use the method of moments or any other methods to estimate the parameters δ, β and θ as starting point of iteration, denote the estimates as $(\delta_0,\beta_0,\theta_0)$ and set $k=0$.
2.Calculate $\left(\frac{\partial\ell}{\partial\delta},\frac{\partial\ell}{\partial\beta},\frac{\partial\ell}{\partial\theta}\right)_{(\delta_k,\beta_k,\theta_k)}$ and the observed Fisher Information matrix $I^{-1}(\delta,\beta,\theta)$, given in the next paragraph.
3.Update (δ,β,θ) as

$$(\delta_{k+1},\beta_{k+1},\theta_{k+1}) = (\delta_k,\beta_k,\theta_k)+\left(\frac{\partial\ell}{\partial\delta},\frac{\partial\ell}{\partial\beta},\frac{\partial\ell}{\partial\theta}\right)_{(\delta_k,\beta_k,\theta_k)}\times I^{-1}(\delta,\beta,\theta). \tag{13}$$

4.Set $k=k+1$ and then go back to Step 1.
5.Continue the iterative steps until $|(\delta_{k+1},\beta_{k+1},\theta_{k+1})-(\delta_k,\beta_k,\theta_k)|$ is smaller than a threshold value. The final estimates of (δ,β,θ) are the MLE of the parameters, denoted as $(\hat{\delta},\hat{\beta},\hat{\theta})$.

Moreover, using the invariance property of MLEs, the MLEs of $S(t)$ and $h(t)$ can be obtained after replacing δ, β and θ by $\hat{\delta}$, $\hat{\beta}$ and $\hat{\theta}$ as

$$\hat{S}(t) = \exp\left\{-\hat{\delta}(\frac{t}{\hat{\theta}})^{\hat{\beta}}\right\} \tag{14}$$

and

$$\hat{h}(t) = \frac{\hat{\beta}\hat{\delta}}{\hat{\theta}}(\frac{t}{\hat{\theta}})^{\hat{\beta}-1} \tag{15}$$

3 Approximate Confidence Intervals

From the log-likelihood function in (8), we have

$$\frac{\partial^2 \ell(\underline{x};\delta,\beta,\theta)}{\partial \delta^2} = \frac{-m}{\delta^2}, \tag{16}$$

$$\frac{\partial^2 \ell(\underline{x};\delta,\beta,\theta)}{\partial \delta \partial \beta} = \frac{\partial^2 \ell(\underline{x};\delta,\beta,\theta)}{\partial \beta \partial \delta} = -\sum_{i=1}^{m} (R_i+1)\left(\frac{x_i}{\theta}\right)^{\beta} \log\left(\frac{x_i}{\theta}\right), \tag{17}$$

$$\frac{\partial^2 \ell(\underline{x};\delta,\beta,\theta)}{\partial \delta \partial \theta} = \frac{\partial^2 \ell(\underline{x};\delta,\beta,\theta)}{\partial \theta \partial \delta} = \frac{\beta}{\theta}\sum_{i=1}^{m} (R_i+1)\left(\frac{x_i}{\theta}\right)^{\beta}, \tag{18}$$

$$\frac{\partial^2 \ell(\underline{x};\delta,\beta,\theta)}{\partial \beta^2} = \frac{-m}{\beta^2} - \delta\sum_{i=1}^{m} (R_i+1)\left(\frac{x_i}{\theta}\right)^{\beta} \left[\log\left(\frac{x_i}{\theta}\right)\right]^2, \tag{19}$$

$$\begin{aligned}\frac{\partial^2 \ell(\underline{x};\delta,\beta,\theta)}{\partial \beta \partial \theta} = \frac{\partial^2 \ell(\underline{x};\delta,\beta,\theta)}{\partial \theta \partial \beta} &= \frac{-m}{\theta} - \frac{\delta}{\theta}\sum_{i=1}^{m} (R_i+1)\left(\frac{x_i}{\theta}\right)^{\beta} \\ &- \frac{\beta\delta}{\theta}\sum_{i=1}^{m} (R_i+1)\left(\frac{x_i}{\theta}\right)^{\beta} \log\left(\frac{x_i}{\theta}\right),\end{aligned} \tag{20}$$

and

$$\frac{\partial^2 \ell(\underline{x};\delta,\beta,\theta)}{\partial \theta^2} = \frac{m\beta}{\theta^2} + \frac{\beta\delta(\beta+1)}{\theta^2}\sum_{i=1}^{m} (R_i+1)(\frac{x_i}{\theta})^{\beta}. \tag{21}$$

Now, we construct the ACIs of the parameters δ, β and θ based on the asymptotic normal distribution of the MLEs. So that we employ the asymptotic Fisher information matrix. The Fisher information matrix $\hat{I}(\delta,\beta,\theta)$ is given by taking expectation of minus (16)-(21), which can be written as

$$\hat{I}(\delta,\beta,\theta) = \begin{pmatrix} -\frac{\partial^2 \ell}{\partial \delta^2} & -\frac{\partial^2 \ell}{\partial \delta \partial \beta} & -\frac{\partial^2 \ell}{\partial \delta \partial \theta} \\ -\frac{\partial^2 \ell}{\partial \beta \partial \delta} & -\frac{\partial^2 \ell}{\partial \beta^2} & -\frac{\partial^2 \ell}{\partial \beta \partial \theta} \\ -\frac{\partial^2 \ell}{\partial \theta \partial \delta} & -\frac{\partial^2 \ell}{\partial \theta \partial \beta} & -\frac{\partial^2 \ell}{\partial \theta^2} \end{pmatrix}_{\downarrow(\delta,\beta,\theta)=(\hat{\delta},\hat{\beta},\hat{\theta})}. \tag{22}$$

Therefore, the asymptotic variance-covariance matrix of the MLEs is obtained by taking inverse of the elements on the observed Fisher information matrix and written by

$$\hat{I}^{-1}(\delta,\beta,\theta) = \begin{pmatrix} var(\delta) & cov(\delta,\beta) & cov(\delta,\theta) \\ cov(\beta,\delta) & var(\beta) & cov(\beta,\theta) \\ cov(\theta,\delta) & cov(\theta,\beta) & var(\theta) \end{pmatrix}_{\downarrow(\delta,\beta,\theta)=(\hat{\delta},\hat{\beta},\hat{\theta})}, \tag{23}$$

where $var(\hat{\delta}), var(\hat{\beta})$ and $var(\hat{\theta})$ are the elements of the main diagonal in variance-covariance matrix $\hat{I}^{-1}(\delta,\beta,\theta)$. Approximate confidence intervals for δ, β and θ can be given by to be multivariate normal with mean (δ,β,θ) and variance-covariance matrix $\hat{I}^{-1}(\delta,\beta,\theta)$. Thus, the $(1-\gamma)100\%$ ACIs for δ, β and θ can be obtained by

$$\left(\hat{\delta} \pm Z_{\gamma/2}\sqrt{var(\hat{\delta})}\right), \left(\hat{\beta} \pm Z_{\gamma/2}\sqrt{var(\hat{\beta})}\right) \text{ and } \left(\hat{\theta} \pm Z_{\gamma/2}\sqrt{var(\hat{\theta})}\right). \tag{24}$$

where $Z_{\gamma/2}$ is the percentile of the standard normal distribution with right-tail probability $\gamma/2$.

In order to find the approximate estimates of the variance of $S(t)$ and $h(t)$, we use the delta method discussed in Greene [7]. The delta method is a general approach for computing confidence intervals for functions of MLEs. It takes a function that is too complex for analytically computing the variance, creates a linear approximation of that function, and then computes the variance of the simpler linear function that can be used for large sample inference.

Let

$$\acute{B}_1 = \left(\frac{\partial S(t)}{\partial \delta}, \frac{\partial S(t)}{\partial \beta}, \frac{\partial S(t)}{\partial \theta}\right) \text{ and } \acute{B}_2 = \left(\frac{\partial h(t)}{\partial \delta}, \frac{\partial h(t)}{\partial \beta}, \frac{\partial h(t)}{\partial \theta}\right), \tag{25}$$

where

$$\frac{\partial S(t)}{\partial \delta} = -\left(\frac{t}{\theta}\right)^{\beta} \exp\left\{-\delta\left(\frac{t}{\theta}\right)^{\beta}\right\},$$
$$\frac{\partial S(t)}{\partial \beta} = -\delta\left(\frac{t}{\theta}\right)^{\beta} \exp\left\{-\delta\left(\frac{t}{\theta}\right)^{\beta}\right\}\log\left(\frac{t}{\theta}\right),$$
$$\frac{\partial S(t)}{\partial \theta} = \frac{\beta\delta}{\theta}\left(\frac{t}{\theta}\right)^{\beta} \exp\left\{-\delta\left(\frac{t}{\theta}\right)^{\beta}\right\}, \tag{26}$$

and

$$\frac{\partial h(t)}{\partial \delta} = \frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1},$$
$$\frac{\partial h(t)}{\partial \beta} = \frac{\delta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1} + \frac{\beta\delta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1}\log\left(\frac{t}{\theta}\right),$$
$$\frac{\partial h(t)}{\partial \theta} = -\frac{\beta^2\delta}{\theta^2}\left(\frac{t}{\theta}\right)^{\beta-1}, \tag{27}$$

Then, the approximate estimates of $var(\hat{S}(t))$ and $var(\hat{h}(t))$ can be given, respectively, by

$$var(\hat{S}) \simeq \left[\acute{B}_1 \hat{I}^{-1} B_1\right]_{(\delta,\beta,\theta)=(\hat{\delta},\hat{\beta},\hat{\theta})} \quad \text{and} \quad var(\hat{h}) \simeq \left[\acute{B}_1 \hat{I}^{-1} B_1\right]_{(\delta,\beta,\theta)=(\hat{\delta},\hat{\beta},\hat{\theta})}. \tag{28}$$

Thus,

$$\frac{(\hat{S}(t)-S(t))}{\sqrt{var(\hat{S})}} \sim N(0,1) \quad \text{and} \quad \frac{(\hat{h}(t)-h(t))}{\sqrt{var(\hat{h})}} \sim N(0,1), \tag{29}$$

asymptotically. These results yield the asymptotic $100(1-\gamma)\%$ confidence interval for $S(t)$ and $h(t)$ given by

$$\hat{S}(t) \pm Z_{\gamma/2}\sqrt{var(\hat{S})} \quad \text{and} \quad \hat{h}(t) \pm Z_{\gamma/2}\sqrt{var(\hat{h})}. \tag{30}$$

4 Bootstrap Confidence Intervals

The bootstrap is a resampling method for statistical inference. In common bootstraping is used for measurment of accuracy (as defined in trems bias, variance, confidence intervals and prediction error) and constructing hypothesis tests. In this section we construct two approixmate confidence intervals based on the parametric bootstrap percentile methods: (i) percentile bootstrap-p method (we call it BP) based on the idea of Efron[5] and (ii) bootstrap-t method (we call it BT) based on the idea of Hall [8]. The algorithms for estimating the confidence intervals using both methods are illustrated as follow

4.1 BP method

Algorithm 1.

1. Based on the original data $\underline{x} = X_{1:m:n}, X_{2:m:n}, \ldots, X_{m:m:n}$ compute the MLEs of δ, β and θ, say $\hat{\delta}$, $\hat{\beta}$ and $\hat{\theta}$. Then used their values to obtain the MLEs $\hat{S}(t)$ and $\hat{h}(t)$ from (14) and (15).
2. Use $\hat{\delta}$, $\hat{\beta}$ and $\hat{\theta}$ to generate the bootstrap sample $\underline{x}^* = X^*_{1:m:n}, X^*_{2:m:n}, \ldots, X^*_{m:m:n}$ with the same values of R_i and m, where $i = 1,2,\ldots,m$.
3. As in Step 1, obtain the MLEs based on $\underline{x}^*$ and compute the bootstrap estimates of $\delta, \beta, \theta, S(t)$ and $h(t)$, say $\hat{\delta}^*, \hat{\beta}^*, \hat{\theta}^*, \hat{S}^*(t)$ and $\hat{h}^*(t)$.
4. Repeat Steps 2-4 N times and obtain $\widehat{\varphi}^*_1, \widehat{\varphi}^*_2, \ldots, \widehat{\varphi}^*_N$ where $\widehat{\varphi}^*_j = \left(\widehat{\delta}^*_j, \widehat{\beta}^*_j, \widehat{\theta}^*_j, \widehat{S}^*_j, \widehat{h}^*_j\right)$.
5. Arrange all $\hat{\varphi}^*_j$, $j = 1,2,\ldots,N$ in an ascending order to obtain $\hat{\varphi}^*_1, \hat{\varphi}^*_2, \ldots, \hat{\varphi}^*_N$. Let $U_1(z) = P(\hat{\varphi}^* \le z)$ be the cumulative distribution function of $\hat{\varphi}^*$. Define $\hat{\varphi}_{boot-p} = U_1^{-1}(z)$ for given z. The approximate $(1-\gamma)100\%$ CIs of $\hat{\varphi}_{BP}$ is given by
$$[\hat{\varphi}_{BP}(\gamma/2), \hat{\varphi}_{BP}(1-\gamma/2)] \tag{31}$$

4.2 BT method

Algorithm 2.

1-4. The same as the BP algorithm 1**.**

5.Obtaining the asymptotic variance-covariance matrix $\hat{I}^{-1^*}(\hat{\delta}^*,\hat{\beta}^*,\hat{\theta}^*)$ and the variances of the reliability and hazard functions $var(\hat{S}^*)$ and $var(\hat{h}^*)$ by applying the asymptotic variance-covariance matrix and delta method.

6.Define the statistic T^* as

$$T^{*\varphi} = \sqrt{N}(\hat{\varphi}^* - \hat{\varphi}^*)/\sqrt{Var(\widehat{\varphi}^*)}$$

where $\varphi = \delta,\beta,\theta,S(t)$ and $h(t)$.

7.Repeat Steps 2-6 N times and obtain $T_1^{*\varphi},T_2^{*\varphi},...,T_N^{*\varphi}$.

8.Arrange $T_1^{*\varphi},T_2^{*\varphi},...,T_N^{*\varphi}$ in an ascending order to obtain $T_{(1)}^{*\varphi},T_{(2)}^{*\varphi},...,T_{(N)}^{*\varphi}$.

6.Let $U_2(z) = P(T^* \leq z)$ be the cumulative distribution function of T^* for given z, define

$$\hat{\varphi}_{BT} = \hat{\varphi} + N^{-1/2}\sqrt{var(\hat{\varphi}^*)}U_2^{-1}(z)$$

The approximate $(1-\gamma)100\%$ CIs of $\hat{\varphi}_{BT}$ is given by

$$[\hat{\varphi}_{BT}(\gamma/2),\hat{\varphi}_{BT}(1-\gamma/2)] \tag{32}$$

5 Bayesian Estimation

In this section we discuss how to obtain the Bayes estimates and the corresponding credible intervals of parameters δ, β, and β as well as the reliability $S(t)$ and hazard $h(t)$ functions of the NWPD. Let us consider the parameters δ, β and θ are independent and follow the gamma prior distributions, the prior density functions of δ, β and θ are given, respectively, by

$$\pi_1(\delta) \propto \delta^{\gamma_1-1}\exp\{-\eta_1\delta\}, \qquad \delta>0, \quad \gamma_1>0, \quad \eta_1>0, \tag{33}$$

$$\pi_2(\beta) \propto \delta^{\gamma_2-1}\exp\{-\eta_2\beta\}, \qquad \beta>0, \quad \gamma_2>0, \quad \eta_2>0, \tag{34}$$

and

$$\pi_3(\theta) \propto \delta^{\gamma_3-1}\exp\{-\eta_3\theta\}, \qquad \beta>0, \quad \gamma_3>0, \quad \eta_3>0, \tag{35}$$

where $\gamma_1,\eta_1,\gamma_2,\eta_2,\gamma_3$ and η_3 are selected to reflect the prior knowledge about δ, β and θ. Note that if $\gamma_1=\gamma_2=\gamma_3=\eta_1=\eta_2=\eta_3=0$, they the non-informative priors of δ, β and θ, we call it prior 0. Using the likelihood function given in (7), the joint posterior density function of δ, β and θ given $\underline{x}$ is thus

$$\begin{aligned}\pi^*(\delta,\beta,\theta|\underline{x}) &= \frac{L(\underline{x};\delta,\beta,\theta)\times\pi_1(\delta)\times\pi_2(\beta)\times\pi_3(\theta)}{\int_0^\infty\int_0^\infty\int_0^\infty L(\underline{x};\delta,\beta,\theta)\times\pi_1(\delta)\times\pi_2(\beta)\times\pi_3(\theta)\partial\delta\partial\beta\partial\theta} \\ &\propto \frac{\beta^{m+\gamma_2-1}\delta^{m+\gamma_1-1}}{\theta^{m+\gamma_3-1}}\left[\prod_{i=1}^m\left(\frac{x_i}{\theta}\right)^{\beta-1}\right] \\ &\quad\times\exp\left\{-\eta_2\beta-\eta_3\theta-\delta\left(\sum_{i=1}^m(R_i+1)\left(\frac{x_i}{\theta}\right)^\beta+\eta_1\right)\right\}\end{aligned} \tag{36}$$

It is clear that, the integral in (36) is so hard to evalaute analytically. To solve this problem we applied the MCMC technique to provide alternative method for parameter estimation. By using the MCMC technique we can obtain the Bayesian estimators of parameters δ, β and θ also approixmate the credible intervals. From (36), we can derive the conditional posterior distributions of δ, β and θ, respectively, as

$$\pi_1^*(\delta|\beta,\theta,\underline{x}) \propto \delta^{m+\gamma_1-1}\exp\left\{-\delta\left(\sum_{i=1}^m(R_i+1)\left(\frac{x_i}{\theta}\right)^\beta+\eta_1\right)\right\}, \tag{37}$$

$$\pi_2^*(\beta|\delta,\theta,\underline{x}) \propto \beta^{m+\gamma_2-1}\left[\prod_{i=1}^m\left(\frac{x_i}{\theta}\right)^{\beta-1}\right]\exp\left\{-\eta_2\beta-\delta\sum_{i=1}^m(R_i+1)\left(\frac{x_i}{\theta}\right)^\beta\right\}, \tag{38}$$

and

$$\pi_3^*(\theta|\delta,\beta,\underline{x}) \propto \frac{1}{\theta^{m+\gamma_3-1}} \left[\prod_{i=1}^{m} \left(\frac{x_i}{\theta} \right)^{\beta-1} \right] \exp\left\{ -\eta_3\theta - \delta \sum_{i=1}^{m} (R_i+1) \left(\frac{x_i}{\theta} \right)^{\beta} \right\}. \tag{39}$$

It can be seen that, as in (37) the full conditional posterior density is a gamma density with shape parameter $(m+\gamma_1)$ and scale parameter $\sum_{i=1}^{m}(R_i+1)(\frac{x_i}{\theta})^{\beta}+\eta_1$, therefor, it is easy to generate the samples of δ by using any gamma generating routine. However, in our case, we do not have well-known distridutions that the marginal posterior distributions of β and θ can have been analytically reduced and therefore the standard metheds is not possible use to direct sampling, but the plots of them show that they are similar to the normal distribution see Figures 1 and 2.

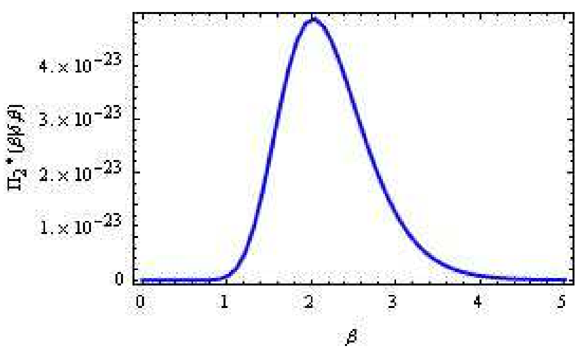

Fig. 1: Posterior density function $\pi_2^*(\beta|\delta,\theta,\underline{x})$ of β.

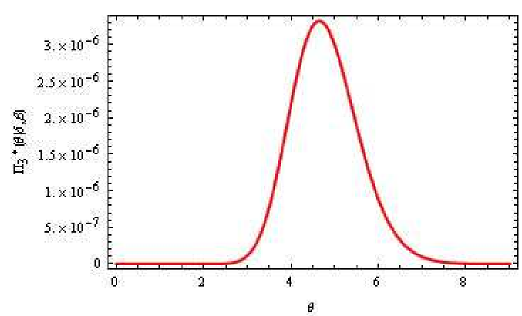

Fig. 2: Posterior density function $\pi_3^*(\theta|\delta,\beta,\underline{x})$ of θ.

To solve this problem we use a Metropolis-Hasting (M-H) sampling with the Gibbs sampling scheme by using the normal proposal distribution, as discussed in Tierney [14]. To sample from (38) and (39), we generate a proposal values of β and θ from a normal distributions $N(\beta^{(j-1)}, var(\beta))$ and $N(\theta^{(j-1)}, var(\theta))$ respectively, where $\beta^{(j-1)}$ and $\theta^{(j-1)}$ are the current values of β and θ and $var(\beta)$ and $var(\theta)$ are the varinaces of β and θ obtained from the variance-covariance matrix in (23). The hybrid algorithm M-H and Gibbs sampler works as follows:

1.Start with initial guess $(\delta^{(0)},\beta^{(0)},\theta^{(0)})$.
2.Set $j=1$.
3.Generate $\delta^{(j)}$ from Gamma$\left(m+\gamma_1, \sum_{i=1}^{m}(R_i+1)\left(\frac{x_i}{\theta}\right)^{\beta}+\eta_1\right)$.
4.Using M-H, generate $\beta^{(j)}$ and $\theta^{(j)}$ from $\pi_2^*(\beta|\delta,\theta,\underline{x})$ and $\pi_3^*(\theta|\delta,\beta,\underline{x})$ with normal proposal distribution $N(\beta^{(j-1)}, var(\beta))$ and $N(\theta^{(j-1)}, var(\theta))$.

(i)Generate a proposal β^* from $N(\beta^{(j-1)}, var(\beta))$ and θ^* from $N(\theta^{(j-1)}, var(\theta))$.
(ii)Evaluate the acceptance probabilities

$$\psi_\beta = \min\left[1, \frac{\pi_2^*(\beta^*|\delta^{(j)}, \theta^{(j-1)}, \underline{x})}{\pi_2^*(\beta^{(j-1)}|\delta^{(j)}, \theta^{(j-1)}, \underline{x})}\right] \text{ and } \psi_\theta = \min\left[1, \frac{\pi_3^*(\theta^*|\delta^{(j)}, \beta^{(j)}, \underline{x})}{\pi_3^*(\theta^{(j-1)}|\delta^{(j)}, \beta^{(j)}, \underline{x})}\right]$$

(iii)Generate a u_1 and u_2 from a Uniform$(0,1)$ distribution.
(iv)If $u_1 < \psi_\beta$ accept the proposal and set $\beta^{(j)} = \beta^*$, esle set $\beta^{(j)} = \beta^{(j-1)}$.
(iiv)If $u_2 < \psi_\theta$ accept the proposal and set $\theta^{(j)} = \theta^*$, esle set $\theta^{(j)} = \theta^{(j-1)}$.

5.Compute the reliability and hazard functions as

$$S^{(j)}(t) = \exp\left\{-\delta^{(j)}\left(\frac{t}{\theta^{(j)}}\right)^{\beta^{(j)}}\right\} \text{ and } h^{(j)}(t) = \frac{\beta^{(j)}\delta^{(j)}}{\theta^{(j)}}\left(\frac{t}{\theta^{(j)}}\right)^{\beta^{(j)}-1}.$$

6.Set $j = j+1$.
7.Repeat Steps 3-6 N times. In order to guarantee the convergence and to remove the affection of the selection of initial value, the first M simulated varieties are discarded. Then the selected sample are $\delta^{(i)}, \beta^{(i)}, \theta^{(i)}, S^{(i)}(t)$ and $h^{(i)}(t)$, $i = M+1, ..., N$, for sufficiently large N forms an approximate posterior sample which can be used to develop the Bayes estimate.
8.Based on SEL, the approixmate Bayes estimate of υ (where $\upsilon = \delta, \beta, \theta, S(t)$ and $h(t)$) under MCMC can be given as

$$\hat{\upsilon}_{MC} = \hat{E}(\upsilon|\underline{x}) = \frac{1}{N-M}\sum_{i=M+1}^{N} \upsilon^{(i)} \tag{40}$$

where M is burn-in and $\upsilon^{(i)} = \delta^{(i)}, \beta^{(i)}, \theta^{(i)}, S^{(i)}(t)$ and $h^{(i)}(t)$ respectively.
9.To compute the credible interval of $\upsilon^{(i)}$, order $\{\upsilon^{M+1}, \upsilon^{M+2}, ..., \upsilon^N\}$ as $\left\{\upsilon^{(1)}, \upsilon^{(2)}, ..., \upsilon^{(N)}\right\}$. Then the $100(1-\gamma)\%$ symmetric credible intervals of υ (where $\upsilon = \delta, \beta, \theta, S(t)$ and $h(t)$) can be given by

$$\left[\upsilon_{(N\ (\gamma/2))}, \quad \upsilon_{(N\ (1-\gamma/2))}\right] \tag{41}$$

6 Numerical Computations

To illustrate the computation of methods proposed in this paper, we discuss two different examples. The first example uses a simulated data set and the second uses a real life data set.

Example 1: (Simulated data set). By using the the algorithm described in Balakrishnan and Sandhu [2], we generate the progressive Type-II censored sample from NWPD with parameters $(\delta, \beta, \theta) = (19.5, 2.5, 5)$ of size $m = 20$, which's generated randomly of sample size $n = 30$ with censoring scheme $R = (1, 0, 1, 1, 0, 1, 1, 0, 0, 1, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0)$. The progressive Type-II censored sample is

0.1686	0.3892	0.4738	0.7142	0.7780	0.9456	1.0200	1.1951	1.2227	1.2483
1.2546	1.3254	1.3347	1.3846	1.4535	1.5786	1.6805	1.7143	2.0368	2.3460

1.*MLEs method:* Under the progressive Type-II censored sample and the method discussed in Section 2, we compute the MLEs of the parameters δ, β and θ, relaibility $S(t)$ and hazard $h(t)$ functions, the results become

$$\hat{\delta},\ \hat{\beta},\ \hat{\theta},\ \hat{S}(t=0.3),\ \hat{h}(t=0.3) = 20.6205,\ 2.4988,\ 5.2849,\ 0.9842,\ 0.1323.$$

Using the formula and the delta method as described in Section 3, we obtian the 95% ACIs of δ, β, θ, $S(t=0.3)$ and $h(t=0.3)$, the results are displayed in Table 1.

2.*Bootstrap methods:* Using the algorithms of the BP and BT methods described in Section 4, we present the mean of 1000 bootstrap (BP and BT) of the parameters, reliability and hazard functions are given, respectively, by

$$\hat{\delta}_{BP},\ \hat{\beta}_{BP},\ \hat{\theta}_{BP},\ \hat{S}_{BP}(t=0.3),\ \hat{h}_{BP}(t=0.3) = 20.3037,\ 2.6539,\ 5.2221,\ 0.9836,\ 0.1330.$$

and

$$\hat{\delta}_{BT},\ \hat{\beta}_{BT},\ \hat{\theta}_{BT},\ \hat{S}_{BT}(t=0.3),\ \hat{h}_{BT}(t=0.3) = 20.7372,\ 2.6333,\ 5.3055,\ 0.9837,\ 0.1329.$$

Also, the 95% BP and BT CIs are displayed in Table 1.

3.*MCMC method:* Under the Gibbs sampler within M-H algorithm as described in Section 5, we generate a Markov chain with 30 000 observations. Discarding the first 5000 values as 'burn-in' and taking every tenth variate as iid observations . We used the informative gamma priors for δ, β and θ, with the hyperparameters $\gamma_1 = \gamma_2 = \gamma_3 = 1.1$ and $\eta_1 = \eta_2 = \eta_3 = 0.015$. Based on the sample of size 25 000 the Bayes MCMC estimate of the parameters, reliability and hazard functions under SEL function are obtained as:

$$\hat{\delta}_{MC},\ \hat{\beta}_{MC},\ \hat{\theta}_{MC},\ \hat{S}_{MC}(t=0.3),\ \hat{h}_{MC}(t=0.3) = 21.3713,\ 2.5039,\ 5.2854,\ 0.9839,\ 0.1355.$$

Also, 95% CRI of δ, β, θ, $S(t)$ and $h(t)$ are displayed in Table 1.
Table 2, provides the MCMC results of the posterior mean, median, mode, standard deviation (S.D.) and skewness (Ske) of the parameters, reliability and hazard functions.

Table 1. 95% ACIs, BPCIs, BTCIs and CRIs of δ, β, θ, $S(t)$ and $h(t)$ for Example 1.

Parameters	ACIs		BPCIs	
	Interval	Length	Interval	Length
δ	[16.328, 24.913]	8.5850	[15.2637, 26.2276]	10.9639
β	[2.2235, 3.7740]	0.5506	[1.9961, 3.6433]	1.6472
θ	[4.8447, 8.7252]	3.8805	[2.0730, 6.8329]	4.7599
$S(t=0.3)$	[0.9763, 0.9921]	0.0158	[0.9547, 0.9981]	0.0433
$h(t=0.3)$	[0.0838, 0.1809]	0.0971	[0.0237, 0.3025]	0.2788
	BTCIs		CRIs	
	Interval	Length	Interval	Length
δ	[13.5819, 25.0266]	11.4447	[13.205, 26.413]	13.208
β	[2.0087, 3.7070]	1.6983	[2.1769, 3.5273]	1.3504
θ	[2.8589, 6.6199]	3.7610	[2.2851, 6.2858]	4.0007
$S(t=0.3)$	[0.9594, 0.9977]	0.0383	[0.9761, 0.9901]	0.0140
$h(t=0.3)$	[0.0286, 0.2958]	0.2672	[0.0831, 0.2015]	0.1184

Table 2. MCMC results for some posterior characteristics for Example 1.

Parameters	Mean	Median	Mode	SD	Ske
δ	21.3713	20.9926	20.2353	4.6734	0.4394
β	2.5039	2.5054	2.5084	0.0128	-0.3379
θ	5.2854	5.2854	5.2853	0.0002	0.4239
$S(t=0.3)$	0.9839	0.9842	0.9848	0.0036	-0.4698
$h(t=0.3)$	0.1355	0.1330	0.1279	0.0303	0.4730

Example 2: (Real life data). For illustrative purposes, considering the real data set of sample size 63 observed failure times. The data is represented the strength data measured in GPA, for single carbon fibers and impregnated 1000 carbon fiber tows, this data reported by Badar and Priest [1], as in Table 3. We have computed the Kolmogorov-Smirnov (KS) distance between the empirical and the fitted distribution functions. It is 0.16 and the associated p-value is 0.59. Since the p- value is quite high, we cannot reject the null hypothesis that the data is coming from the NWPD. Also, we plot both the empirical survival function (ESF) and the estimated survival functions in Figure 3 and we found that the NWPD fits the data very well.

Table 3. A real life data as in Badar and Priest.

0.101	0.332	0.403	0.428	0.457	0.550	0.561	0.596	0.597	0.645	0.654
0.674	0.718	0.722	0.725	0.732	0.775	0.814	0.816	0.818	0.824	0.859
0.875	0.938	0.940	1.056	1.117	1.128	1.137	1.137	1.177	1.196	1.230
1.325	1.339	1.345	1.420	1.423	1.435	1.443	1.464	1.472	1.494	1.532
1.546	1.577	1.608	1.635	1.693	1.701	1.737	1.754	1.762	1.828	2.052
2.071	2.086	2.171	2.224	2.227	2.425	2.595	3.22			

Accordingly the data set, which's discussed in Badar and Priest [14], we can generate the progressive Type-II censored sample of size $m = 25$ taken from sample size $n = 63$ with censoring scheme $R = (5,0,0,5,0,5,0,0,5,0,0,5,0,0,0,0,0,5,3,0,5,0,0,0,0)$. A progressively censored sample generated from the real data is

0.101	0.332	0.403	0.428	0.457	0.561	0.940	1.056	1.117	1.128	1.137	1.137
1.196	1.230	1.339	1.345	1.420	1.423	1.435	1.443	1.701	1.737	1.754	1.828
2.425											

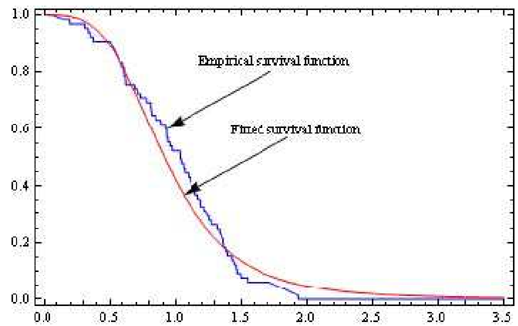

Fig. 3: The empirical and fitted survival functions for the data in Table 3.

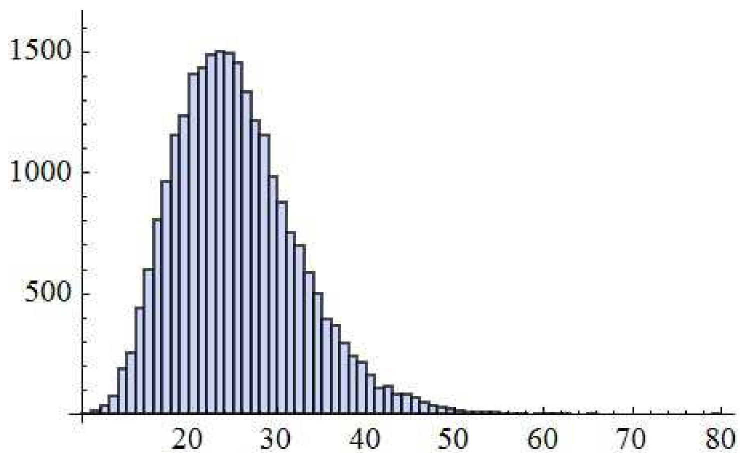

Fig. 4: Histogram and the kerney desnsity estimate of δ

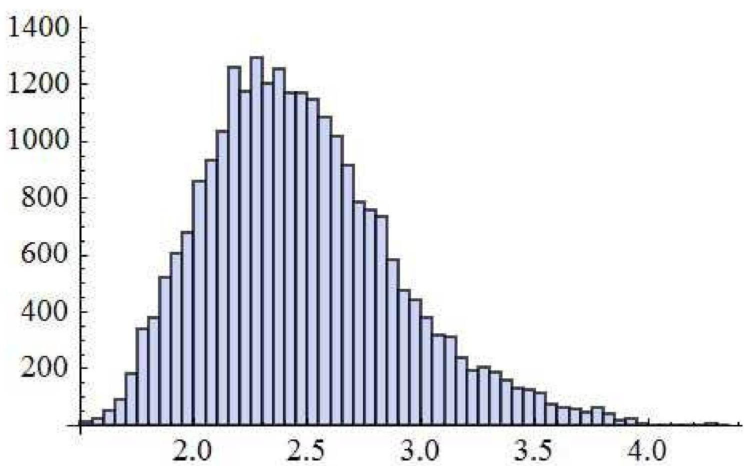

Fig. 5: Histogram and the kerney desnsity estimate of β

As the same in Example 1, we obtain the computations for different methods of estimations as follows:

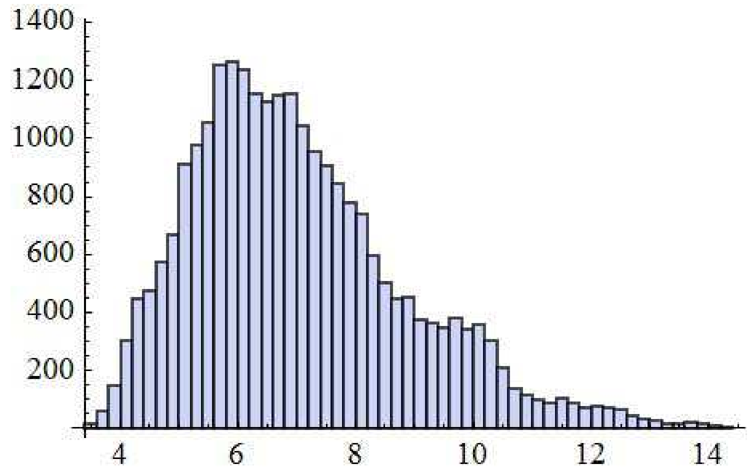

Fig. 6: Histogram and the kerney desnsity estimate of θ

1.*MLEs method:* Under the progressive Type-II censored sample and the method discussed in Section 2, we compute the MLEs of the parameters δ, β and θ, relaibility and hazard functions $S(t)$ and $h(t)$, the results become

$$\hat{\delta},\ \hat{\beta},\ \hat{\theta},\ \hat{S}(t=0.3),\ \hat{h}(t=0.3) = 25.6641,\ 2.4811,\ 6.5614,\ 0.5171,\ 1.0908.$$

Using the formula and the delta method as described in Section 3, we obtian the 95% ACIs of δ, β, θ, $S(t=0.3)$ and $h(t=0.3)$, the results are displayed in Table 4.

2.*Bootstrap methods:* Using the algorithms of the BP and BT methods described in Section 4, we present the mean of 1000 bootstrap (BP and BT) of the parameters, reliability and hazard functions are given, respectively, by

$$\hat{\delta}_{BP},\ \hat{\beta}_{BP},\ \hat{\theta}_{BP},\ \hat{S}_{BP}(t=0.3),\ \hat{h}_{BP}(t=0.3) = 24.2202,\ 2.6137,\ 6.2769,\ 0.5066,\ 1.1992.$$

and

$$\hat{\delta}_{BT}, \hat{\beta}_{BT}, \hat{\theta}_{BT}, \hat{S}_{BT}(t=0.3), \hat{h}_{BT}(t=0.3) = 26.0735,\ 2.6043,\ 6.6556,\ 0.5136,\ 1.1689.$$

Also, the 95% BP and BT CIs are displayed in Table 4.

3.*MCMC method:* Under the Gibbs sampler within M-H algorithm as described in Section 5, we generate a Markov chain with 30 000 observations. Discarding the first 5000 values as ‘burn-in’ and taking every tenth variate as iid observations . We used the non-informative gamma priors for δ, β and θ. Based on the sample of size 25 000 the Bayes MCMC estimate of the parameters, reliability and hazard functions under SEL function are obtained as

$$\hat{\delta}_{MC}, \hat{\beta}_{MC}, \hat{\theta}_{MC}, \hat{S}_{MC}(t=0.3), \hat{h}_{MC}(t=0.3) = 25.37,\ 2.4738,\ 7.0382,\ 0.517,\ 1.153.$$

Also, 95% CRI of δ, β, θ, $S(t)$ and $h(t)$ are displayed in Table 4.

Table 5, provides the MCMC results of the posterior mean, median, mode, standard deviation (S.D.) and skewness (Ske) of the parameters, reliability and hazard functions. Figures 4-8, display the histegram and the kerney desnsity estimate of the parameters δ, β and θ as well as reliability $S(t)$ and hazard $h(t)$ functions of Example 2.

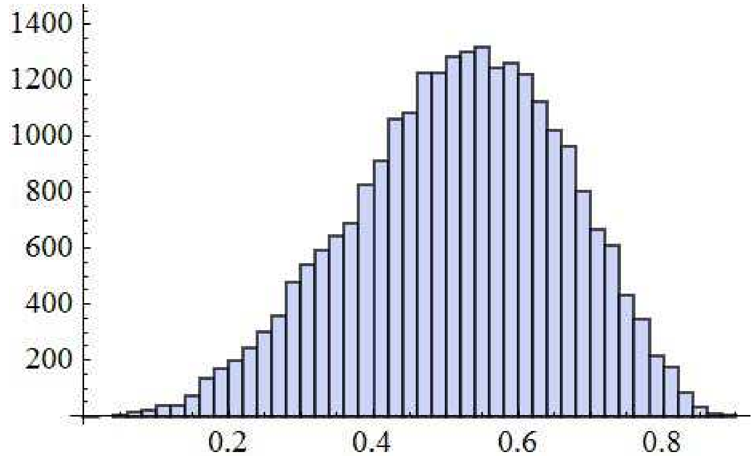

Fig. 7: Histogram and the kerney desnsity estimate of $S(t)$

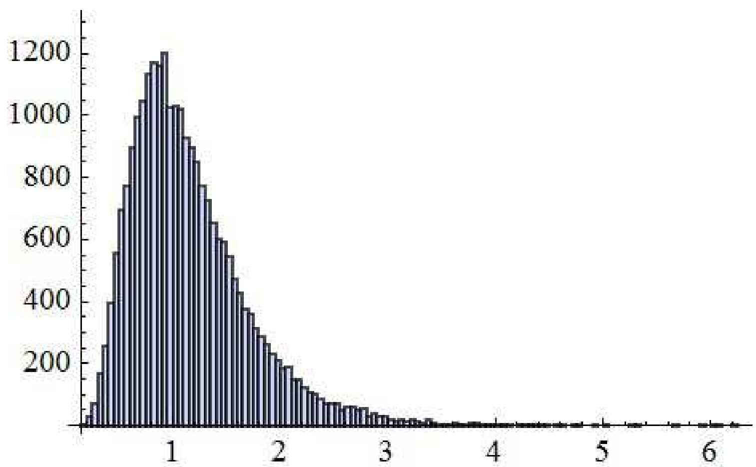

Fig. 8: Histogram and the kerney desnsity estimate of $h(t)$

Table 4. 95% ACIs, BPCIs, BTCIs and CRIs of δ, β, θ, $S(t)$ and $h(t)$ for Example 2.

Parameters	ACIs		BPCIs	
	Interval	Length	Interval	Length
δ	[12.564, 28.7643]	16.2003	[11.4538, 30.4408]	18.987
β	[1.2681, 4.694]	3.4259	[1.0461, 4.4412]	3.3951
θ	[3.242, 7.8809]	4.6389	[3.7784, 10.141]	6.3626
$S(t=0.3)$	[0.3922, 0.642]	0.2498	[0.3463, 0.6522]	0.3059
$h(t=0.3)$	[0.8116, 1.37]	0.5584	[0.7602, 2.1905]	1.4303
	BTCIs		CRIs	
	Interval	Length	Interval	Length
δ	[9.4578, 27.8403]	18.3825	[14.1179, 32.2699]	18.152
β	[2.01, 3.5974]	1.5874	[1.789, 3.462]	1.673
θ	[2.822, 8.4283]	5.6063	[4.2497, 10.4319]	6.1822
$S(t=0.3)$	[0.3483, 0.6421]	0.2938	[0.2132, 0.7726]	0.5594
$h(t=0.3)$	[0.7524, 1.9527]	1.2003	[0.4144, 2.5248]	2.1103

Table 5. MCMC results for some posterior characteristics for Example 2.

Parameters	Mean	Median	Mode	SD	Ske
δ	25.37	24.5926	23.0379	6.9857	0.7124
β	2.4738	2.4244	2.3256	0.4214	0.668
θ	7.0382	6.7401	6.1437	1.8361	0.7983
$S(t=0.3)$	0.517	0.5251	0.5412	0.1458	-0.2733
$h(t=0.3)$	1.153	1.0457	0.8313	0.5485	1.4352

7 conclusion

In this paper, we have studied different methods such as maximum likelihood, parameteric bootstrap and Bayes estimate to obtain the estimation value for parameters having the new- Weibull-Pareto distribution and their reliability and hazard functions based on progressive Type-II censored sample. Futhermore, the paper has explained how to construct the approximate confidence intervals for the unknown parameters by using the asymptotic normality of maximum likelihood estimation as well as the reliability and hazard functions depending on the delta and parametric bootstrap metheds. It is clear that, after studying the Bayesian estimate the posterior distribution equations of the unknown parameters is complicated and so hard to reduce analytically to well-known forms. For this reason we have applied the MCMC techinque to compute the Bayes estimates also construct the crideble intervals. The Bayes estimates have been obtained under SEL function. For illustrative purposes, we have applied two numerical example using a simulated data and a real life data sets as well.

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