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Global Convergence of Successive Approximations for Partial Fractional Differential Equations and Inclusions

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Abstract: This paper deals with the global convergence of successive approximations as well as the uniqueness of solutions for some classes of partial functional differential equations and inclusions involving the Caputo fractional derivative. We show a theorem on the global convergence of successive approximations to the unique solution of our problems.

Keywords: Fractional differential equation, Fractional differential inclusion, left-sided mixed Riemann-Liouville integral, Caputo fractional order derivative, Darboux problem, generalized solution, global convergence, successive approximations.

1 Introduction

Fractional calculus is a powerful tool in applied mathematics to investigate several problems from various fields of science and engineering, with many break-through results which can be seen in physics, finance, hydrology, biophysics, thermodynamics, control theory, statistical mechanics, astrophysics, and bioengineering [1,2,3,4,5]. There has been a significant development in ordinary and partial fractional differential equations in recent years. We recommend the reader to check the monographs of Abbas *et al.* [6,7], Kilbas *et al.* [8], Miller and Ross [9], Zhou [10], the papers of Abbas *et al.* [11,12,13,14], Vityuk *et al.* [15,16,17,18], and the references therein.

Convergence of successive approximations for ordinary functional differential equations as well as for integral functional equations is a well established property. It has been studied by De Blasi and Myjak [19], Chen [20], Faina [21], Shin [22], and the references therein. Człapiński [23] got the global convergence of successive approximations as well as the uniqueness of solutions for the Darboux problem

$$\begin{cases} D_{xy}z(x,y) = f(x,y,z_{(x,y)}); & \text{if } (x,y) \in J := [0,a]\times[0,b], \\ z(x,y) = \Phi(x,y); & \text{if } (x,y) \in E_0 := (-\infty,a]\times(-\infty,b]\backslash(0,a]\times(0,b], \end{cases} \tag{1}$$

where $f : J\times\mathscr{B} \to \mathbb{R}$ and $\Phi : E_0 \to \mathbb{R}$ are given functions, and $\mathscr{B}$ is a phase space. In [24], Abbas *et al.* presented some global convergence of successive approximations of the following partial Hadamard integral equation

$$u(x,y) = \mu(x,y) + \int_1^x\int_1^y \left(\log\frac{x}{s}\right)^{r_1-1}\left(\log\frac{y}{t}\right)^{r_2-1}\frac{f(s,t,u(s,t))}{st\Gamma(r_1)\Gamma(r_2)}dtds; \text{ if } (x,y)\in J, \tag{2}$$

where $J := [1,a]\times[1,b],\ a,b > 1,\ r_1,r_2 > 0,\ \mu : J \to \mathbb{R},\ f : J\times\mathbb{R} \to \mathbb{R}$ are given continuous functions, and $\Gamma(\cdot)$ is the (Euler's) Gamma function defined by

$$\Gamma(\zeta) = \int_0^\infty t^{\zeta-1}e^{-t}dt;\ \zeta > 0.$$

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Also in [24], the authors discussed the global convergence of successive approximations for the fractional partial Hadamard integral inclusion

$$u(x,y)-\mu(x,y)\in({}^{H}I^{r}_{\sigma}F)(x,y,u(x,y));\ (x,y)\in J, \tag{3}$$

where $\sigma=(1,1)$, $F:J\times\mathbb{R}\to\mathscr{P}(\mathbb{R})$ is a compact valued multi-valued map, ${}^{H}I^{r}_{\sigma}F$ is the definite Hadamard integral for the set-valued function F of order $r=(r_1,r_2)\in(0,\infty)\times(0,\infty)$, and $\mu:J\to\mathbb{R}$ is a given continuous function, and $\mathscr{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$.

Motivated by the above papers, in the present article, we discuss the global convergence of successive approximations for the fractional partial differential equation

$$^{c}D^{r}_{\theta}u(x,y)=f(x,y,u(x,y));\ \text{ if } (x,y)\in J, \tag{4}$$

with the initial conditions

$$\begin{cases} u(x,0)=\varphi(x);\ x\in[0,a],\\ u(0,y)=\psi(y);\ y\in[0,b],\\ \varphi(0)=\psi(0), \end{cases} \tag{5}$$

where $a,b>0,\ \theta=(0,0),\ {}^{c}D^{r}_{\theta}$ is the fractional Caputo derivative of order $r=(r_1,r_2)\in(0,1]\times(0,1],\ f:J\times E\to E$ is a given functions, E is a (real or complex) Banach space, and $\varphi:[0,a]\to E,\ \psi:[0,b]\to E$ are given absolutely continuous functions.

Next, we discuss the global convergence of successive approximations for the fractional partial differential inclusion

$$^{c}D^{r}_{\theta}u(x,y)\in F(x,y,u(x,y));\ \text{ if } (x,y)\in J, \tag{6}$$

with the initial conditions (5), where $F:J\times E\to\mathscr{P}(E)$ is a compact valued multi-valued map, $\mathscr{P}(E)$ is the family of all nonempty subsets of the Banach space E.

This paper initiates the convergence of successive approximations for fractional differential equations and inclusions. The paper is organized as follows. In Section 2 some preliminary results are introduced. The main result is presented in Section 3, and two examples are presented in the last section.

2 Preliminaries

Denote $L^1(J)$ the space of Bochner-integrable functions $u:J\to E$ with the norm

$$\|u\|_{L^1}=\int_0^a\int_0^b\|u(x,y)\|_E\,dydx,$$

where $\|.\|_E$ denotes a norm on E.
$AC(J)$ denotes the space of absolutely continuous functions from J into E, and $\mathscr{C}:=C(J)$ is the Banach space of all continuous functions from J into E with the norm $\|.\|_\infty$, namely

$$\|u\|_\infty=\sup_{(x,y)\in J}\|u(x,y)\|_E.$$

Definition 1. *The function $f:J\times E\to E$ is said to be L^1-Carathéodory if*

(i)$(x,y)\longmapsto f(x,y,u)$ is measurable for each $u\in E$;
(ii)$u\longmapsto f(x,y,u)$ is continuous for almost all $(x,y)\in J$;
(iii)there exists a real positive function $\delta\in L^1(J)$ such that

$$\|f(x,y,u)\|_E\le\delta(x,y);\ \textit{for all } u\in E \textit{ and almost all } (x,y)\in J.$$

Let (X,d) be a metric space. We use the following notations:

$$\mathscr{P}_{bd}(X)=\{Y\in\mathscr{P}(X):Y\ \textit{bounded}\},\ \mathscr{P}_{cl}(X)=\{Y\in\mathscr{P}(X):Y\ \textit{closed}\},$$

$$\mathscr{P}_{cp}(X) = \{Y \in \mathscr{P}(X) : Y \ compact\}, \ \ and \ \ \mathscr{P}_{cv}(X) = \{Y \in \mathscr{P}(X) : Y \ convex\}.$$

A multivalued map $G : X \to \mathscr{P}(X)$ has *convex (closed) values* if $G(x)$ is convex (closed) for all $x \in X$. We say that G is *bounded* on bounded sets if $G(B)$ is bounded in X for each bounded set B of X, i.e.,

$$\sup_{x\in B}\{\sup\{\ \|u\| : \ u \in G(x)\}\} < \infty.$$

G is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood N_0 of x_0 such that $G(N_0) \subseteq N$. Finally, we say that G has a *fixed point* if there exists $x \in X$ such that $x \in G(x)$.

For each $u \in \mathscr{C}$ let the set $S_{F\circ u}$ known as *the set of selectors* from F defined by

$$S_{F\circ u} = \{v \in L^1(J) : v(x,y) \in F(x,y,u(x,y)))\ ,\ a.e.\ x,y \in J\}.$$

For more details on multivalued maps we refer to the books of Deimling [25] and Górniewicz [26].

Consider $H_d : \mathscr{P}(X) \times \mathscr{P}(X) \longrightarrow \mathbb{R}_+ \cup \{\infty\}$, given by

$$H_d(\mathscr{A},\mathscr{B}) = \max\left\{ \sup_{a\in\mathscr{A}} d(a,\mathscr{B})\ ,\ \sup_{b\in\mathscr{B}} d(\mathscr{A},b) \right\},$$

where $d(\mathscr{A},b) = \inf_{a\in\mathscr{A}} d(a,b)$, $d(a,\mathscr{B}) = \inf_{b\in\mathscr{B}} d(a,b)$. Then $(\mathscr{P}_{bd,cl}(X),H_d)$ is a metric space and $(\mathscr{P}_{cl}(X),H_d)$ is a generalized (complete) metric space (see [27]).

Definition 2. *A multivalued map $F : J \times E \to \mathscr{P}(E)$ is said to be Carathéodory if*

(i)$(x,y) \longmapsto F(x,y,u)$ is measurable for each $u \in E$;
(ii)$u \longmapsto F(x,y,u)$ is upper semicontinuous for almost all $(x,y) \in J$.

F is said to be L^1-Carathéodory if $(i),(ii)$ and the following condition holds;

(iii)for each $c > 0$, there exists $\sigma_c \in L^1(J,\mathbb{R}_+)$ such that

$$\begin{aligned}\|F(x,y,u)\|_{\mathscr{P}} &= \sup\{\|f\| : f \in F(x,y,u)\} \\ &\le \sigma_c(x,y) \ \ \text{for all } \|u\| \le c \text{ and for } a.e.\ (x,y) \in J.\end{aligned}$$

Now, we introduce notations, definitions and a preliminary Lemma concerning to partial fractional calculus theory.

Definition 3. *[15] Let $r_1,r_2 \in (0,\infty)$ and $r = (r_1,r_2)$. For $u \in L^1(J)$, the expression*

$$(I^r_\theta u)(x,y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}u(s,t)dtds,$$

is called the left-sided mixed Riemann-Liouville integral of order r, where $\Gamma(.)$ is the (Euler's) Gamma function defined by $\Gamma(\xi) = \int_0^\infty t^{\xi-1}e^{-t}dt;\ \xi > 0$.

In particular,

$$(I^\theta_\theta u)(x,y) = u(x,y),\ (I^{(1,1)}_\theta f)(x,y) = \int_0^x \int_0^y u(s,t)dtds; \text{ for almost all } (x,y) \in J.$$

For instance, $I^r_\theta f$ exists for all $r_1,r_2 \in (0,\infty)$, when $f \in L^1(J)$. Note also that when $u \in \mathscr{C}$, then $(I^r_\theta f) \in \mathscr{C}$, moreover

$$(I^r_\theta u)(x,0) = (I^r_\theta u)(0,y) = 0;\ x \in [0,a],\ y \in [0,b].$$

Example 1. Let $\lambda,\omega \in (0,\infty)$ and $r = (r_1,r_2) \in (0,\infty) \times (0,\infty)$, then

$$I^r_\theta \frac{x^\lambda y^\omega}{\Gamma(1+\lambda)\Gamma(1+\omega)} = \frac{x^{\lambda+r_1}y^{\omega+r_2}}{\Gamma(1+\lambda+r_1)\Gamma(1+\omega+r_2)};\ \text{ for almost all } (x,y) \in J.$$

By $1-r$ we mean $(1-r_1,1-r_2)\in[0,1)\times[0,1)$. Denote by $D^2_{xy}:=\frac{\partial^2}{\partial x\partial y}$, the mixed second order partial derivative.

Definition 4. *[6,18] Let $r\in(0,1]\times(0,1]$ and $u\in L^1(J)$. The Caputo fractional-order derivative of order r of u is defined by the expression*

$$^cD^r_\theta u(x,y)=(I^{1-r}_\theta D^2_{xy}u)(x,y)=\frac{1}{\Gamma(1-r_1)\Gamma(1-r_2)}\int_0^x\int_0^y\frac{D^2_{st}u(s,t)}{(x-s)^{r_1}(y-t)^{r_2}}dtds.$$

The case $r=(1,1)$ is included and we have

$$(^cD^{(1,1)}_\theta u)(x,y)=(D^2_{xy}u)(x,y);\text{ for almost all }(x,y)\in J.$$

Example 2. Let $\lambda,\omega\in(0,\infty)$ and $r=(r_1,r_2)\in(0,1]\times(0,1]$, then

$$^cD^r_\theta\frac{x^\lambda y^\omega}{\Gamma(1+\lambda)\Gamma(1+\omega)}=\frac{x^{\lambda-r_1}y^{\omega-r_2}}{\Gamma(1+\lambda-r_1)\Gamma(1+\omega-r_2)};\text{ for almost all }(x,y)\in J.$$

In the sequel, we need the following Lemmas:

Lemma 1. *[11] Let $r_1,r_2\in(0,1]$ and $\mu(x,y)=\varphi(x)+\psi(y)-\varphi(0)$. A function $u\in\mathscr{C}$ is a solution of the fractional integral equation*

$$u(x,y)=\mu(x,y)+\int_0^x\int_0^y\frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}f(s,t,u(s,t))dtds,\tag{7}$$

if and only if u is a solution of the problem (4)-(5).

Lemma 2. *[14] Let $r_1,r_2\in(0,1]$ and $\mu(x,y)=\varphi(x)+\psi(y)-\varphi(0)$. A function $u\in\mathscr{C}$ is a solution of the fractional integral equation*

$$u(x,y)=\mu(x,y)+\int_0^x\int_0^y\frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}f(s,t)dtds,\tag{8}$$

where $f\in S_{F\circ u}$, if and only if u is a solution of the inclusion (6) with the initial conditions (5).

3 Successive Approximations and Uniqueness Results

In this section, we present the main result for the global convergence of successive approximations to a unique solution of our problems.

Definition 5. *A generalized solution of the problem (4)-(5) is an absolutely continuous function satisfying the fractional integral equation (7) almost everywhere on J.*

Define the successive approximations of the problem (4)-(5) as follows:

$$u^{(0)}(x,y)=\mu(x,y);\ (x,y)\in J,$$

$$u^{(n+1)}(x,y)=\mu(x,y)+\int_0^x\int_0^y\frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}f(s,t,u^{(n)}(s,t))dtds;\ (x,y)\in J.$$

Set $J_\sigma:=[0,\sigma a]\times[0,\sigma b]$; for any $\sigma\in[0,1]$. Let us introduce the following hypotheses.

(H_1) The function $f:J\times E\to E$ is L^1-Carathéodory,

(H_2) There exist a constant $\rho>0$ and a Carathéodory function $w:J\times[0,2\rho]\to[0,\infty)$ such that $w(x,y,.)$ is nondecreasing for almost all $(x,y)\in J$, and the inequality

$$\|f(x,y,u)-f(x,y,\overline{u})\|_E\le w(x,y,\|u-\overline{u}\|_E)\tag{9}$$

holds for all $(x,y)\in J$ and $u,\overline{u}\in E$ such that $\|u-\overline{u}\|_E\le 2\rho$,

$(H_3) v \equiv 0$ is the only function in $\mathscr{C}(J_\lambda, [0,2\rho])$ satisfying the integral inequality

$$v(x,y) \leq \int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} w(s,t,v(s,t)) dt ds, \tag{10}$$

with $\sigma \leq \lambda \leq 1$.

Theorem 1. *Assume that the hypotheses $(H_1)-(H_3)$ are satisfied. Then the successive approximations $u^{(n)}$; $n \in \mathbb{N}$ are well defined and converge to the unique solution of the problem (4)-(5) uniformly on J.*

Proof. From (H_1), the successive approximations are well defined. Furthermore, the sequences $\{u^{(n)}(x,y);\ n \in \mathbb{N}\}$ is equi-continuous on J. Indeed, for each $(x_1,y_1),\ (x_2,y_2) \in J$ with $x_1 < x_2$ and $y_1 < y_2$, and for all $(x,y) \in J$, we have

$$\begin{aligned}
&\|u^{(n)}(x_2,y_2) - u^{(n)}(x_1,y_1)\|_E \leq \|\mu(x_1,y_1) - \mu(x_2,y_2)\|_E \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{x_1} \int_0^{y_1} [(x_2-s)^{r_1-1}(y_2-t)^{r_2-1} - (x_1-s)^{r_1-1}(y_1-t)^{r_2-1}] \\
&\quad \times \|f(s,t,u^{(n-1)}(s,t))\|_E \, dt\, ds \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x_2} \int_{y_1}^{y_2} (x_2-s)^{r_1-1}(y_2-t)^{r_2-1} \|f(s,t,u^{(n-1)}(s,t))\|_E \, dt\, ds \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{x_1} \int_{y_1}^{y_2} (x_2-s)^{r_1-1}(y_2-t)^{r_2-1} \|f(s,t,u^{(n-1)}(s,t))\|_E \, dt\, ds \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x_2} \int_0^{y_1} (x_2-s)^{r_1-1}(y_2-t)^{r_2-1} \|f(s,t,u^{(n-1)}(s,t))\|_E \, dt\, ds,
\end{aligned}$$

then, from hypothesis (iii) of Definition 1, we obtain

$$\begin{aligned}
&\|u^{(n)}(x_2,y_2) - u^{(n)}(x_1,y_1)\|_E \leq \|\mu(x_1,y_1) - \mu(x_2,y_2)\|_E \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{x_1} \int_0^{y_1} [(x_2-s)^{r_1-1}(y_2-t)^{r_2-1} - (x_1-s)^{r_1-1}(y_1-t)^{r_2-1}] \\
&\quad \times \delta(s,t) \, dt\, ds \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x_2} \int_{y_1}^{y_2} (x_2-s)^{r_1-1}(y_2-t)^{r_2-1} \delta(s,t) \, dt\, ds \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{x_1} \int_{y_1}^{y_2} (x_2-s)^{r_1-1}(y_2-t)^{r_2-1} \delta(s,t) \, dt\, ds \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x_2} \int_0^{y_1} (x_2-s)^{r_1-1}(y_2-t)^{r_2-1} \delta(s,t) \, dt\, ds \\
&\longrightarrow 0,\ as\ x_1 \to x_2\ \ and\ \ y_1 \to y_2.
\end{aligned}$$

Let

$$\tau := \sup\{\sigma \in [0,1] : \{u^n(x,y)\} \text{ converges uniformly on } J_\sigma\}.$$

If $\tau = 1$, then we have the global convergence of successive approximations. Suppose that $\tau < 1$, then the sequence $\{u^{(n)}(x,y)\}$ converges uniformly on J_τ. Since this sequence is equi-continuous, then it converges uniformly to a continuous function $\tilde{u}(x,y)$. If we prove that there exists $\lambda \in (\tau, 1]$ such that $\{u^n(x,y)\}$ converges uniformly on J_λ, this will yield a contradiction.

Put $u(x,y) = \tilde{u}(x,y)$; for $(x,y) \in J_\tau$. From (H_2), there exist a constant $\rho > 0$ and a Carathéodory function $w : J \times [0,2\rho] \to [0,\infty)$ satisfying inequality (9). Also, there exist $\lambda \in [\tau,1]$ and $n_0 \in \mathbb{N}$, such that, for all $(x,y) \in J_\lambda$ and $n,m > n_0$, we have

$$\|u^{(n)}(x,y) - u^{(m)}(x,y)\|_E \leq 2\rho.$$

For any $(x,y) \in J_\lambda$, put

$$v^{(n,m)}(x,y) = \|u^{(n)}(x,y) - u^{(m)}(x,y)\|_E,\ and\ \ v^{(k)}(x,y) = \sup_{n,m \geq k} v^{(n,m)}(x,y).$$

Since the sequence $v^{(k)}(x,y)$ is non-increasing, it is convergent to a function $v(x,y)$ for each $(x,y)\in J_\lambda$. From the equicontinuity of $\{v^{(k)}(x,y)\}$ it follows that $\lim_{k\to\infty} v^{(k)}(x,y)=v(x,y)$ uniformly on J_λ. Furthermore, for $(x,y)\in J_\lambda$ and $n,m\geq k$, we have

$$\begin{aligned} v^{(n,m)}(x,y) &= \|u^{(n)}(x,y)-u^{(m)}(x,y)\|_E \\ &\leq \sup_{(s,t)\in[0,x]\times[0,y]}\|u^{(n)}(s,t)-u^{(m)}(s,t)\|_E \\ &\leq \int_0^x\int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \\ &\times\|f(s,t,u^{(n-1)}(s,t))-f(s,t,u^{(m-1)}(s,t))\|_E dtds \\ &\leq \int_0^{\lambda a}\int_0^{\lambda b} \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \\ &\times\|f(s,t,u^{(n-1)}(s,t))-f(s,t,u^{(m-1)}(s,t))\|_E dtds. \end{aligned}$$

Thus, by (9) we get

$$\begin{aligned} v^{(n,m)}(x,y) \leq &\int_0^{\lambda a}\int_0^{\lambda b} \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \\ &\times w(s,t,\|u^{(n-1)}(s,t)-u^{(m-1)}(s,t)\|_E)dtds \\ =&\int_0^{\lambda a}\int_0^{\lambda b} \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}w(s,t,v^{(n-1,m-1)}(s,t))dtds. \end{aligned}$$

Hence

$$v^{(k)}(x,y) \leq \int_0^{\lambda a}\int_0^{\lambda b} \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}w(s,t,v^{(k-1)}(s,t))dtds.$$

By the Lebesgue dominated convergence theorem we get

$$v(x,y) \leq \int_0^{\lambda a}\int_0^{\lambda b} \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}w(s,t,v(s,t))dtds.$$

Then, by the Carathéodory condition (iii), and (H_3) we get $v\equiv 0$ on J_λ, which yields that $\lim_{k\to\infty} v^{(k)}(x,y)=0$ uniformly on J_λ. Thus $\{u^{(k)}(x,y)\}_{k=1}^\infty$ is a Cauchy sequence on J_λ. Consequently $\{u^{(k)}(x,y)\}_{k=1}^\infty$ is uniformly convergent on J_λ which yields the contradiction.

Thus $\{u^{(k)}(x,y)\}_{k=1}^\infty$ converges uniformly on J to a continuous function $u^*(x,y)$. By the Carathéodory condition (iii) and the Lebesgue dominated convergence theorem, we get

$$\lim_{k\to\infty}\int_0^x\int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}f(s,t,u^{(k)}(s,t)dtds$$

$$=\int_0^x\int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}f(s,t,u^*(s,t)dtds,$$

for each $(x,y)\in J$. This yields that u^* is a solution of the problem (4)-(5).

Finally, we show the uniqueness of solutions of the problem (4)-(5). Let u_1 and u_2 be two solutions of (7). As above, put

$$\tau := \sup\{\sigma\in[0,1] : u_1(x,y)=u_2(x,y) \ for\ (x,y)\in J_\sigma\},$$

and suppose that $\tau<1$. There exist a constant $\rho>0$ and a comparison function $w: J_\tau\times[0,2\rho]\to[0,\infty)$ satisfying inequality (9). We choose $\lambda\in(\sigma,1)$ such that

$$\|u_1(x,y)-u_2(x,y)\|_E\leq 2\rho;\ for\ (x,y)\in J_\lambda.$$

Then for all $(x,y) \in J_\lambda$ we obtain

$$\begin{aligned}\|u_1(x,y)-u_2(x,y)\|_E &\leq \int_0^{\tau a}\int_0^{\tau b}\frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}\\ &\times \|f(s,t,u_1(s,t))-f(s,t,u_2(s,t))\|_E dtds\\ &\leq \int_0^{\tau a}\int_0^{\tau b}\frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}\\ &\times w(s,t,\|u_1(s,t)-u_2(s,t)\|_E)dtds.\end{aligned}$$

Again, by the Carathéodory condition (iii), and (H_3) we get $u_1-u_2\equiv 0$ on J_λ. This gives $u_1=u_2$ on J_λ, which yields a contradiction. Consequently, $\tau=1$ and the solution of the problem (4)-(5) is unique on J.

Now, we present the main result for the global convergence of successive approximations to a unique solution of the problem (6)-(5).

Definition 6. *A function $u\in\mathscr{C}$ is a generalized solution of the problem (6)-(5), if u is an absolutely continuous function, and there exists $f\in S_{F\circ u}$ such that u satisfies (8) almost everywhere on J.*

Define the successive approximations of the problem (6)-(5) as follows:

$$u^{(0)}(x,y)=\mu(x,y);\ (x,y)\in J,$$

$$u^{(n+1)}(x,y)=\mu(x,y)+\int_0^x\int_0^y\frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}f_n(s,t)dtds;\ (x,y)\in J,$$

where $f_n\in S_{F,u^{(n)}}$ with $\|f_n\|=\|F(x,y,u_n)\|_{\mathscr{P}}$.

Set $J_\sigma:=[0,\sigma a]\times[0,\sigma b]$; for any $\sigma\in[0,1]$. Let us introduce the following hypotheses.

(H_1')The multifunction $F:J\times E\to\mathscr{P}(E)$ is L^1-Carathéodory,

(H_2')There exist a constant $\rho>0$ and a Carathéodory function $w:J\times[0,\rho]\to[0,\infty)$ such that $w(x,y,.)$ is nondecreasing for almost all $(x,y)\in J$, and the inequality

$$H_d(F(x,y,u),F(x,y,\overline{u}))\leq w(x,y,\|u-\overline{u}\|_E) \tag{11}$$

holds for all $(x,y)\in J$ and $u,\overline{u}\in E$ such that $\|u-\overline{u}\|_E\leq\rho$,

$(H_3')v\equiv 0$ is the only function in $\mathscr{C}(J_\lambda,[0,\rho])$ satisfying the integral inequality

$$v(x,y)\leq\int_0^x\int_0^y\frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}w(s,t,v(s,t))dtds, \tag{12}$$

with $\sigma\leq\lambda\leq 1$.

Theorem 2. *Assume that the hypotheses $(H_1')-(H_3')$ are satisfied. Then the successive approximations $u^{(n)}$; $n\in\mathbb{N}$ are well defined and converge to the unique solution of the problem (6)-(5) uniformly on J.*

Proof. From (H_1'), the successive approximations are well defined. Furthermore, the sequences $\{u^{(n)}(x,y);\ n\in\mathbb{N}\}$ is equi-continuous on J. Let

$$\tau:=\sup\{\sigma\in[0,1]:\{u^n(x,y)\}\text{ converges uniformly on }J_\sigma\}.$$

If $\tau=1$, then we have the global convergence of successive approximations. Suppose that $\tau<1$, then the sequence $\{u^{(n)}(x,y)\}$ converges uniformly on J_τ. Since this sequence is equi-continuous, then it converges uniformly to a continuous function $\tilde{u}(x,y)$. If we prove that there exists $\lambda\in(\tau,1]$ such that $\{u^n(x,y)\}$ converges uniformly on J_λ, this will yield a contradiction.

Put $u(x,y)=\tilde{u}(x,y)$; for $(x,y)\in J_\tau$. From (H_2'), there exist a constant $\rho>0$ and a Carathéodory function $w:J\times[0,\rho]\to[0,\infty)$ satisfying inequality (11). Also, there exist $\lambda\in[\tau,1]$ and $n_0\in\mathbb{N}$, such that, for all $(x,y)\in J_\lambda$ and $n,m>n_0$, we have

$$\|u^{(n)}(x,y)-u^{(m)}(x,y)\|_E\leq\rho.$$

For any $(x,y)\in J_\lambda$, put

$$v^{(n,m)}(x,y)=\|u^{(n)}(x,y)-u^{(m)}(x,y)\|_E,\ and\ \ v^{(k)}(x,y)=\sup_{n,m\geq k}v^{(n,m)}(x,y).$$

Since the sequence $v^{(k)}(x,y)$ is non-increasing, it is convergent to a function $v(x,y)$ for each $(x,y)\in J_\lambda$. From the equi-continuity of $\{v^{(k)}(x,y)\}$ it follows that $\lim_{k\to\infty}v^{(k)}(x,y)=v(x,y)$ uniformly on J_λ. Furthermore, for $(x,y)\in J_\lambda$ and $n,m\geq k$, there exist $f_{n-1}\in S_{F\circ u_{n-1}}$ and $f_{m-1}\in S_{F\circ u_{m-1}}$ with $\|f_{n-1}\|=\|F(x,y,u_{n-1})\|_{\mathscr{P}}$ and $\|f_{m-1}\|=\|F(x,y,u_{m-1})\|_{\mathscr{P}}$, such that

$$\begin{aligned}v^{(n,m)}(x,y)&=\|u^{(n)}(x,y)-u^{(m)}(x,y)\|_E\\&\leq\sup_{(s,t)\in[0,x]\times[0,y]}\|u^{(n)}(s,t)-u^{(m)}(s,t)\|_E\\&\leq\int_0^x\int_0^y\frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}\\&\times\|f_{n-1}(s,t)-f_{m-1}(s,t)\|_E\,dtds\\&\leq\int_0^{\lambda a}\int_0^{\lambda b}\frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}\\&\times H_d(F(s,t,u_{n-1})),F(s,t,u_{m-1})))dtds.\end{aligned}$$

Thus, by (11) we get

$$\begin{aligned}v^{(n,m)}(x,y)&\leq\int_0^{\lambda a}\int_0^{\lambda b}\frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}\\&\times w(s,t,\|u^{(n-1)}(s,t)-u^{(m-1)}(s,t)\|_E)dtds\\&=\int_0^{\lambda a}\int_0^{\lambda b}\frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}w(s,t,v^{(n-1,m-1)}(s,t))dtds.\end{aligned}$$

Hence

$$v^{(k)}(x,y)\leq\int_0^{\lambda a}\int_0^{\lambda b}\frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}w(s,t,v^{(k-1)}(s,t))dtds.$$

By the Lebesgue dominated convergence theorem we get

$$v(x,y)\leq\int_0^{\lambda a}\int_0^{\lambda b}\frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}w(s,t,v(s,t))dtds.$$

Then, by the Carathéodory condition (iii), and (H_3') we get $v\equiv 0$ on J_λ, which yields that $\lim_{k\to\infty}v^{(k)}(x,y)=0$ uniformly on J_λ. Thus $\{u^{(k)}(x,y)\}_{k=1}^{\infty}$ is a Cauchy sequence on J_λ. Consequently $\{u^{(k)}(x,y)\}_{k=1}^{\infty}$ is uniformly convergent on J_λ which yields the contradiction.

Thus $\{u^{(k)}(x,y)\}_{k=1}^{\infty}$ converges uniformly on J to a continuous function $u^*(x,y)$. By the Carathéodory condition (iii) and the Lebesgue dominated convergence theorem, for each $(x,y)\in J$ we get

$$\begin{aligned}&\lim_{k\to\infty}\int_0^x\int_0^y\frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}f_k(s,t)dtds\\&=\int_0^x\int_0^y\frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}f_*(s,t)dtds,\end{aligned}$$

where $f_k\in S_{F\circ u_k}$ and $f_*\in S_{F\circ u_*}$ with $\|f_k\|=\|F(x,y,u_k)\|_{\mathscr{P}}$ and $\|f_*\|=\|F(x,y,u_*)\|_{\mathscr{P}}$. This yields that u^* is a solution of the problem (6)-(5).

Finally, we show the uniqueness of solutions of the problem (6)-(5). Let u_1 and u_2 be two solutions of (8). As above, put

$$\tau:=\sup\{\sigma\in[0,1]:u_1(x,y)=u_2(x,y)\ for\ (x,y)\in J_\sigma\},$$

and suppose that $\tau < 1$. There exist a constant $\rho > 0$ and a comparison function $w : J_\tau \times [0,\rho] \to [0,\infty)$ satisfying inequality (10). We choose $\lambda \in (\sigma, 1)$ such that

$$\|u_1(x,y) - u_2(x,y)\|_E \leq \rho;\ for\ (x,y) \in J_\lambda.$$

Then for all $(x,y) \in J_\lambda$ we obtain

$$\begin{aligned}\|u_1(x,y) - u_2(x,y)\|_E &\leq \int_0^{\tau a}\int_0^{\tau b} \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \\ &\times H_d(F(s,t,u_1(s,t)), F(s,t,u_2(s,t)))dtds \\ &\leq \int_0^{\tau a}\int_0^{\tau b} \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \\ &\times w(s,t,\|u_1(s,t) - u_2(s,t)\|_E)dtds.\end{aligned}$$

Again, by the Carathéodory condition (iii), and (H_3') we get $u_1 - u_2 \equiv 0$ on J_λ. This gives $u_1 = u_2$ on J_λ, which yields a contradiction. Consequently, $\tau = 1$ and the solution of the problem (6)-(5) is unique on J.

4 Examples

Let

$$E = l^1 = \left\{ w = (w_1, w_2, \ldots, w_p, \ldots) : \sum_{p=1}^{\infty} |w_p| < \infty \right\},$$

be the Banach space with the norm

$$\|w\|_E = \sum_{p=1}^{\infty} |w_p|.$$

Example 1. Consider the following partial hyperbolic functional differential equation of the form

$$({}^cD_\theta^r u_p)(x,y) = \frac{xye^{x+y-3}}{1+|u_p(x,y)|};\ (x,y) \in [0,1]\times[0,1];\ p \in \mathbb{N}^*, \tag{13}$$

with the initial conditions

$$\begin{cases} u(x,0) = (1+x^2, 0, \ldots, 0, \ldots);\ x \in [0,1], \\ u(0,y) = (e^y, 0, \ldots, 0, \ldots);\ y \in [0,1], \end{cases} \tag{14}$$

where $(r_1, r_2) \in (0,1] \times (0,1]$,

$$u = (u_1, u_2, \ldots, u_p, \ldots),\quad {}^cD_\theta^r u = ({}^cD_\theta^r u_1, {}^cD_\theta^r u_2, \ldots, {}^cD_\theta^r u_p, \ldots),\quad f = (f_1, f_2, \ldots, f_p, \ldots).$$

For each $p \in \mathbb{N}^*$, set

$$f_p(x,y,u(x,y)) = \frac{xye^{x+y-3}}{1+|u_p(x,y)|};\ (x,y) \in [0,1]\times[0,1].$$

For each $u,\ \overline{u} \in E,\ p \in \mathbb{N}^*$ and $(x,y) \in [0,1]\times[0,1]$ we have

$$|f_p(x,y,u) - f_p(x,y,\overline{u})| \leq xye^{x+y}|u_p - \overline{u}_p|.$$

Thus, for each $u, \overline{u} \in E$ and $(x,y) \in [0,1]\times[0,1]$, we get

$$\begin{aligned}&\|f(x,y,u(x,y)) - f(x,y,\overline{u}(x,y))\|_E \\ &= \sum_{p=1}^{\infty} |f_p(x,y,u(x,y)) - f_p(x,y,\overline{u}(x,y))| \\ &\leq xye^{x+y} \sum_{p=1}^{\infty} |u_p - \overline{u}_p| \\ &= xye^{x+y}\|u - \overline{u}\|_E.\end{aligned}$$

This means that condition (9) holds with any $(x,y)\in[0,1]\times[0,1],\ \rho>0$ and a comparison function $w:[0,1]\times[0,1]\times[0,\rho]\to[0,\infty)$ given by

$$w(x,y,v)=xye^{x+y}v.$$

We see that w satisfies the Carathéodory conditions with $\delta:[0,1]\times[0,1]\to[0,\infty)$ given by $\delta(x,y)=\rho xye^{x+y}$.

The integral equation (10) in our case takes the form

$$v(x,y)\leq\int_0^x\int_0^y\frac{st(x-s)^{r_1-1}(y-t)^{r_2-1}e^{s+t}}{\Gamma(r_1)\Gamma(r_2)}v(s,t)dtds. \tag{15}$$

Since w is nondecreasing with respect to v, then integral inequality (18) has only the zero solution. Consequently, Theorem 1 implies that the successive approximations $u^{(n)};\ n\in\mathbb{N}$, defined by

$$u^{(0)}(x,y)=(x^2+e^y,0,\ldots,0,\ldots);\ (x,y)\in[0,1]\times[0,1],$$

$$u^{(n+1)}(x,y)=u^{(0)}(x,y)$$
$$+\int_0^x\int_0^y\frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}f(s,t,u^{(n)}(s,t))dtds;\ (x,y)\in[0,1]\times[0,1],$$

converge to a unique solution of the problem (13)-(14) uniformly on $[0,1]\times[0,1]$.

Example 2. Consider now the following partial functional differential inclusion of the form

$$({}^cD^r_\theta u_p)(x,y)\in\left[\frac{xye^{x+y-3}}{(1+2^p)(1+|u_p(x,y)|)},\frac{xye^{x+y-3}}{2^p(1+|u_p(x,y)|)}\right];\ (x,y)\in[0,1]\times[0,1];\ p\in\mathbb{N}^*, \tag{16}$$

with the initial conditions

$$\begin{cases}u(x,0)=(1+x^2,0,\ldots,0,\ldots);\ x\in[0,1],\\ u(0,y)=(e^y,0,\ldots,0,\ldots);\ y\in[0,1],\end{cases} \tag{17}$$

where $(r_1,r_2)\in(0,1]\times(0,1]$,

$$u=(u_1,u_2,\ldots,u_p,\ldots),\ \ {}^cD^r_\theta u=({}^cD^r_\theta u_1,{}^cD^r_\theta u_2,\ldots,{}^cD^r_\theta u_p,\ldots),\ \ F=(F_1,F_2,\ldots,F_p,\ldots).$$

For each $p\in\mathbb{N}^*$, set

$$F_p(x,y,u(x,y))=\left[\frac{xye^{x+y-3}}{(1+2^p)(1+|u_p(x,y)|)},\frac{xye^{x+y-3}}{2^p(1+|u_p(x,y)|)}\right];\ (x,y)\in[0,1]\times[0,1];\ p\in\mathbb{N}^*.$$

For each $u,\ \overline{u}\in E,\ (x,y)\in[0,1]\times[0,1]$ and $p\in\mathbb{N}^*$, we have

$$H_d(F_p(t,x,u_p)-F_p(t,x,\overline{u}_p))\leq xye^{x+y-3}|u-\overline{u}|.$$

Thus,

$$\begin{aligned}H_d(F(x,y,u(x,y)),F(x,y,\overline{u}(x,y)))&=\sum_{p=1}^{\infty}H_d(F_p(x,y,u_p(x,y)),F_p(x,y,\overline{u}_p(x,y))|\\&\leq xye^{x+y-3}\sum_{p=1}^{\infty}|u_p-\overline{u}_p|\\&=xye^{x+y-3}\|u-\overline{u}\|_E.\end{aligned}$$

This means that condition (10) holds with any $(x,y)\in[0,1]\times[0,1],\ \rho>0$ and a comparison function $w:[0,1]\times[0,1]\times[0,\rho]\to[0,\infty)$ given by

$$w(x,y,v)=xye^{x+y}v.$$

We see that w satisfies the Carathéodory conditions with $\delta:[0,1]\times[0,1]\to[0,\infty)$ given by $\delta(x,y)=\rho xye^{x+y}$.

The integral inequality (18) in our case takes the form

$$v(x,y) \leq \int_0^x \int_0^y \frac{st(x-s)^{r_1-1}(y-t)^{r_2-1}e^{s+t}}{\Gamma(r_1)\Gamma(r_2)} v(s,t)dtds. \tag{18}$$

Since w is nondecreasing with respect to v, then integral inequality (18) has only the zero solution. Defined the successive approximations $u^{(n)}$; $n \in \mathbb{N}$ by

$$u^{(0)}(x,y) = (x^2 + e^y, 0, \ldots, 0, \ldots); \ (x,y) \in [0,1] \times [0,1],$$

$$u^{(n+1)}(x,y) = u^{(0)}(x,y)$$

$$+ \int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f^{(n)}(s,t)dtds; \ (x,y) \in [0,1] \times [0,1],$$

where

$$f^{(n)}(x,y) = (f_1^{(n)}(x,y), f_2^{(n)}(x,y), \ldots, f_p^{(n)}(x,y), \ldots) \in S_{F,u^{(n)}},$$

$$f_p^{(n)}(x,y) = \frac{xy}{2^p} e^{x+y-3}; \ p \in \mathbb{N}^*,$$

and

$$\|f^{(n)}\| = \|F(x,y,u^{(n)})\|_{\mathscr{P}} = e^{-1}.$$

Consequently, Theorem 2 implies that the successive approximations $u^{(n)}$; $n \in \mathbb{N}$, converge to a unique solution of the problem (16)-(17) uniformly on $[0,1] \times [0,1]$.

5 Conclusion

In the present work, the global convergence of successive approximations to the unique solution of some classes of partial functional differential equations and inclusions involving the Caputo fractional derivative was studied. A theorem on the global convergence of successive approximations to the unique solution of our problems is obtained.

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