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Theory of System of Nonlinear Fractional Differential Equations

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Abstract: This paper is devoted to investigate the basic results such as existence, uniqueness and continuous dependence of solutions of system of fractional differential equations involving the Caputo fractional derivative. Validity and convergence of Picard's successive approximations for the solutions of the system of fractional differential equations have been proved and obtained bound for the error.

Keywords: Fractional differential equations, Caputo fractional derivative, existence and uniqueness, successive approximations, fixed point theorem.

1 Introduction

The general theory of fractional calculus and fractional differential equations have deeply been given in excellent monographs, for instance, by Miller and Ross [1], Podlubny [2], Kilbas et al. [3], Lakshmikantham et al. [4] and Diethelm [5]. Several papers have been devoted to studying existence and qualitative properties for various kinds of fractional differential equations, see [6,7,8,9,10,11,12,13,14,15,16,17] and some of the references cited therein. The system of fractional differential equations with fractional derivative defined in the Caputo sense have been investigated in the interesting papers by Zhou [14] and Daftardar-Gejji et al. [15,16].

Motivated by [9,15,17], we consider a more general problem for the system of fractional differential equations (SFDEs) of the form:

$$a_0\frac{d\overline{u}(t)}{dt}+a_1\,{}^cD_t^{\alpha}\overline{u}(t)=\overline{f}(t,\overline{u}),\ \overline{u}(0)=\overline{u}_0,\ 0<\alpha<1,$$

where $a_0\neq 0$ and a_1 be any real numbers, ${}^cD_t^{\alpha}$ denotes Caputo fractional derivative of order α with base limit 0 and $\overline{f}:\mathfrak{D}\,(\subset\mathbb{R}\times\mathbb{R}^n)\to\mathbb{R}^n$. Dividing by a_0 we obtain the SFDEs of the form:

$$\frac{d\overline{u}(t)}{dt}+a\,{}^cD_t^{\alpha}\overline{u}(t)=\overline{f}(t,\overline{u}),\ \overline{u}(0)=\overline{u}_0,\ 0<\alpha<1, \tag{1}$$

where a is some real number.

The main objective of this paper is to obtain existence results, uniqueness and continuous dependence of solutions on initial conditions for SFDEs (1). Further, we prove validity and convergence of Picards successive approximations to the solutions of SFDEs (1.1) and obtain bound for the error. Our analysis is based on the techniques used in [9,15,17,18].

The important aspect of this paper is that with very few restrictions on f we have obtained various existence results and different properties of solutions. This paper extends few results of [9,15,17] and for $a=0$ in (1) it includes the study of system of nonlinear ordinary differential equations.

This paper is organized as follows. Section 2 contains definitions and basic results of the fractional calculus. In Section 3, equivalent system of fractional Volterra integral equation is obtained and proved theorems on the existence

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and uniqueness of a solution to the problem (1). In Section 4, continuous dependence of solutions on initial condition is proved. Section 5 deals with convergence of Picard's successive approximations to the solutions of SFDEs and obtained bound for the error. In Section 6, an illustrative example is provided in support of the results obtained. The conclusions are provided in Section 7.

2 Preliminaries

In this section we recall definitions and few basic results of the fractional calculus from [2,3,5,19].

Definition 1.*The Riemann–Liouville fractional integral of order $\mu \geq 0$ $(\mu \in \mathbb{R})$ with base limit 0 of the function $g \in C[0,+\infty)$ is defined as*

$$I_t^{\mu} g(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1} g(s)\,ds,\ t>0 \text{ if } \mu>0,$$
$$I_t^0 g(t) = g(t),$$

where $\Gamma(\cdot)$ is the Euler's Gamma function.
Note that $I_t\, g(t) = \int_0^t g(s)\,ds$ and $\Gamma(\mu+1) = \mu\Gamma(\mu)$.

Definition 2.*Let $n-1<\mu\leq n\,(n\in\mathbb{N})$. Then the Caputo fractional derivative of order μ with base limit 0 of the function $g \in C^n[0,+\infty)$ is defined as*

$${}^cD_t^{\mu} g(t) = I_t^{n-\mu}\left\{\frac{d^n}{dt^n} g(t)\right\} = \frac{1}{\Gamma(n-\mu)} \int_0^t (t-s)^{n-\mu-1} g^{(n)}(s)\,ds,\ t>0 \text{ if } n-1<\mu<n,$$
$${}^cD_t^{\mu} g(t) = g^{(n)}(t) \text{ if } \mu = n.$$

Lemma 1.*Let $g \in C^n[0,+\infty)$ and $n-1<\mu\leq n$ $(n\in\mathbb{N})$. Then*

$$I_t^{\mu}\left({}^cD_t^{\mu} g(t)\right) = g(t) - \sum_{k=0}^{n-1} \frac{g^{(k)}(0^+)}{k!} t^k.$$

Note that:

(i)${}^cD_t^{\mu} g(t) = I_t^{1-\mu}\left\{\frac{d}{dt} g(t)\right\},\ \mu \in (0,1]$.
(ii)$I_t^{\mu}\left({}^cD_t^{\mu} g(t)\right) = g(t) - g(0),\ \mu \in (0,1]$.
(iii)${}^cD_t^{\mu} K = 0$, where $\mu>0$ and K is a constant.

Following fixed point theorems are used to establish the existence and uniqueness results.

Theorem 1.*[[20],(Schauder)] Let X be a Banach space, $D \subset X$ a nonempty convex bounded closed set and let $T: D \to D$ be a completely continuous operator. Then T has at least one fixed point.*

Theorem 2.*[9] Let V be a nonempty closed subset of a Banach space E, and let $\alpha_n \geq 0$, $n \in \mathbb{N}\cup\{0\}$ be a sequence such that $\sum_{n=0}^{\infty} \alpha_n$ converges. Moreover, let the mapping $A: V \to V$ satisfy the inequality*

$$\|A^n u - A^n v\| \leq \alpha_n \|u-v\|$$

for every $n \in \mathbb{N}$ and every $u,v \in V$. Then, A has a uniquely defined fixed point u^*. Furthermore, for each $u_0 \in V$ the sequence $\{A^n u_0\}_{n=1}^{\infty}$ converges to this fixed point u^*.

3 Existence and Uniqueness Results

In the following lemma we obtain an equivalent system of fractional Volterra integral equation to the initial value problem for SFDEs (1).

Lemma 2. *If $\bar{f} = (f_1, ..., f_n)$ is continuous then the initial value problem for SFDEs (1) is equivalent to the system of fractional Volterra integral equation.*

$$\bar{u}(t) = \left(1 + \frac{a}{\Gamma(2-\alpha)} t^{1-\alpha}\right) \bar{u}_0 - \frac{a}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \bar{u}(s) ds + \int_0^t \bar{f}(s, \bar{u}(s)) ds. \tag{2}$$

Proof. Let $\bar{u}(t)$ is the solution of SFDEs (1). Integrating (1), we obtain

$$\bar{u}(t) - \bar{u}(0) + a I_t {}^c D_t^\alpha \bar{u}(t) = I_t \bar{f}(t, \bar{u}(t)).$$

Note that, $I_t {}^c D_t^\alpha \bar{u}(t) = I_t \left(I_t^{\,1-\alpha} \frac{d}{dt} \bar{u}(t)\right) = I_t^{1-\alpha} \left(I_t \frac{d}{dt} \bar{u}(t)\right) = I_t^{1-\alpha} (\bar{u}(t) - \bar{u}(0))$ above equation reduces to

$$\bar{u}(t) - \bar{u}_0 + \frac{a}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \bar{u}(s) ds - \frac{a}{\Gamma(2-\alpha)} t^{1-\alpha} \bar{u}_0 = \int_0^t \bar{f}(s, \bar{u}(s)) ds.$$

This gives

$$\bar{u}(t) = \left(1 + \frac{a}{\Gamma(2-\alpha)} t^{1-\alpha}\right) \bar{u}_0 - \frac{a}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \bar{u}(s) ds + \int_0^t \bar{f}(s, \bar{u}(s)) ds,$$

which is (2).
Conversely, let $\bar{u}(t)$ satisfies equation system of fractional Volterra integral equation (2). Differentiating (2) we obtain

$$\frac{d\bar{u}(t)}{dt} = \frac{a}{\Gamma(1-\alpha)} t^{-\alpha} \bar{u}_0 - \frac{a}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} \bar{u}(s) ds + \bar{f}(t, \bar{u}(t)). \tag{3}$$

Integrating by parts and then differentiating, we have

$$\begin{aligned} \frac{a}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} \bar{u}(s) ds &= \frac{a}{\Gamma(1-\alpha)} \frac{d}{dt} \left\{ \frac{t^{1-\alpha}}{1-\alpha} \bar{u}(0) + \int_0^t \frac{(t-s)^{1-\alpha}}{1-\alpha} \left(\frac{d\bar{u}(s)}{ds}\right) ds \right\} \\ &= \frac{a}{\Gamma(1-\alpha)} \left\{ t^{-\alpha} \bar{u}_0 + \int_0^t (t-s)^{-\alpha} \left(\frac{d\bar{u}(s)}{ds}\right) ds \right\} \\ &= \frac{a}{\Gamma(1-\alpha)} t^{-\alpha} \bar{u}_0 + a I_t^{1-\alpha} \frac{d\bar{u}(t)}{dt} \\ &= \frac{a}{\Gamma(1-\alpha)} t^{-\alpha} \bar{u}_0 + a {}^c D_t^\mu \bar{u}(t). \end{aligned} \tag{4}$$

Using (4) in (3) we obtain

$$\frac{d\bar{u}(t)}{dt} + a {}^c D_t^\alpha \bar{u}(t) = \bar{f}(t, \bar{u}(t)).$$

Further, from (2) we have $\bar{u}(0) = \bar{u}_0$. This proves $\bar{u}(t)$ satisfies the IVP (1). We have proved that Volterra integral equation (2) is equivalent to SFDEs (1).

Theorem 3. *(Existence): Let $f_j : \mathfrak{D} \to \mathbb{R}$ $(j = 1, ..., n)$ be continuous where*

$$\mathfrak{D} = [0, T] \times \prod_{j=1}^n [u_j(0) - b_j, u_j(0) + b_j], \; T > 0, \; b_j > 0,$$

and let $\bar{f} = (f_1, ..., f_n)$. Then, the SFDEs (1) has a solution $\bar{u}(t) : [0, \chi] \to \mathbb{R}^n$ where

$$\chi = \min \left\{ T, \left[\frac{b\Gamma(2-\alpha)}{2|a|(2\|\bar{u}_0\|_\infty + b)}\right]^{\frac{1}{1-\alpha}}, \frac{b}{2\|f\|_\infty} \right\} \text{ and } b = \min\{b_1, ..., b_n\}.$$

Proof. Define the operator $A(\bar{u}) = (A_1 \bar{u}, A_2 \bar{u}, ..., A_n \bar{u})$ in which for each j $(j = 1, 2, ..., n)$,

$$A_j(\bar{u}(t)) = \left(1 + \frac{a}{\Gamma(2-\alpha)} t^{1-\alpha}\right) u_j(0) - \frac{a}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} u_j(s) ds + \int_0^t f_j(s, \bar{u}(s)) ds. \tag{5}$$

With this operator the Volterra integral equation (2) can be written as $\bar{u}(t) = A\bar{u}(t)$. We prove that A has a fixed point. Set $V = \{\bar{u} \in \mathscr{B} : \bar{u}(0) = \bar{u}_0, |u_j(t) - u_j(0)| < b, 1 \le j \le n\}$, where $\mathscr{B} = C[0,\chi]^n$. Since $\bar{u}_0 \in V$, V is nonempty. One can verify that V is closed, bounded and convex subset of $\mathscr{B}$. Firstly we prove that A maps the set V into itself. For any $\bar{u} \in V$ and $t \in [0,\chi]$ from the definition of operator A, we have

$$A\bar{u}(t) - \bar{u}(0) = \frac{a}{\Gamma(2-\alpha)} t^{1-\alpha} \bar{u}(0) - \frac{a}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \{[\bar{u}(s) - \bar{u}(0)] + \bar{u}(0)\} ds + \int_0^t \bar{f}(s,\bar{u}(s)) ds. \tag{6}$$

Therefore,

$$\begin{aligned}
\|A\bar{u}(t) - \bar{u}(0)\|_\infty &\le \frac{|a|\chi^{1-\alpha}}{\Gamma(2-\alpha)} \|\bar{u}_0\|_\infty + \frac{|a|(b + \|\bar{u}_0\|_\infty)}{\Gamma(1-\alpha)} \int_0^\chi (\chi - s)^{-\alpha} ds + \int_0^\chi \sup_{\substack{0\le\tau\le s\\ 1\le j\le n}} \left| f_j(\tau,\bar{u}(\tau)) \right| ds \\
&\le \frac{|a|\chi^{1-\alpha}}{\Gamma(2-\alpha)} \|\bar{u}_0\|_\infty + \frac{|a|(b + \|\bar{u}_0\|_\infty)}{\Gamma(1-\alpha)} \frac{\chi^{1-\alpha}}{1-\alpha} + \chi \|\bar{f}\|_\infty \\
&= \frac{|a|(2\|\bar{u}_0\|_\infty + b)}{\Gamma(2-\alpha)} \chi^{1-\alpha} + \chi \|\bar{f}\|_\infty \\
&\le \frac{|a|(2\|\bar{u}_0\|_\infty + b)}{\Gamma(2-\alpha)} \frac{b\Gamma(2-\alpha)}{2|a|(2\|\bar{u}_0\|_\infty + b)} + \frac{b}{2\|f_0\|_\infty} \|\bar{f}\|_\infty \\
&= b.
\end{aligned}$$

Further, for any $\bar{u} \in V$ and $0 \le \tau_1 \le \tau_2 \le \chi$, we have

$$\begin{aligned}
&\|A\bar{u}(\tau_1) - A\bar{u}(\tau_2)\|_\infty \\
&\le \frac{|a|\|\bar{u}_0\|_\infty}{\Gamma(2-\alpha)} \left|\tau_1^{1-\alpha} - \tau_2^{1-\alpha}\right| - \frac{|a|}{\Gamma(1-\alpha)} \int_0^{\tau_1} (\tau_1 - s)^{-\alpha} \{\|\bar{u}(s) - \bar{u}_0\|_\infty + \|\bar{u}_0\|_\infty\} ds \\
&\quad + \frac{|a|}{\Gamma(1-\alpha)} \int_0^{\tau_2} (\tau_2 - s)^{-\alpha} \{\|\bar{u}(s) - \bar{u}_0\|_\infty + \|\bar{u}_0\|_\infty\} ds + \left\| \int_0^{\tau_1} \bar{f}(s,\bar{u}(s)) ds - \int_0^{\tau_2} \bar{f}(s,\bar{u}(s)) ds \right\|_\infty \\
&\le \frac{|a|\|\bar{u}_0\|_\infty}{\Gamma(2-\alpha)} \left|\tau_1^{1-\alpha} - \tau_2^{1-\alpha}\right| - \frac{|a|(b + \|\bar{u}_0\|_\infty)}{\Gamma(1-\alpha)} \left(\int_0^{\tau_1} (\tau_1 - s)^{-\alpha} ds - \int_0^{\tau_2} (\tau_2 - s)^{-\alpha} ds \right) + |\tau_1 - \tau_2| \|\bar{f}\|_\infty \\
&\le \frac{|a|\|\bar{u}_0\|_\infty}{\Gamma(2-\alpha)} \left|\tau_1^{1-\alpha} - \tau_2^{1-\alpha}\right| + \frac{|a|(b + \|\bar{u}_0\|_\infty)}{\Gamma(2-\alpha)} \left|\tau_1^{1-\alpha} - \tau_2^{1-\alpha}\right| + \|\bar{f}\|_\infty |\tau_1 - \tau_2| \\
&= \frac{|a|(b + 2\|\bar{u}_0\|_\infty)}{\Gamma(2-\alpha)} \left|\tau_1^{1-\alpha} - \tau_2^{1-\alpha}\right| + \|\bar{f}\|_\infty |\tau_1 - \tau_2|.
\end{aligned}$$

The above inequality implies that $A\bar{u}$ is continuous.

We have proved that for any $\bar{u} \in V$ we have $A\bar{u} \in \mathscr{B}$, $A\bar{u}(0) = \bar{u}_0$ and $\|A\bar{u}(t) - \bar{u}(0)\|_\infty \le b$. Therefore $A\bar{u} \in V$, whenever $\bar{u} \in V$. This proves A maps the set V into itself.

Next we decompose the operator A as $A = F + G$, where F and G are defined on V by

$$\begin{aligned}
F\bar{u}(t) &= -\frac{a}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \bar{u}(s) ds, \\
G\bar{u}(t) &= \left(1 + \frac{a}{\Gamma(2-\alpha)} t^{1-\alpha}\right) \bar{u}(0) + \int_0^t \bar{f}(s,\bar{u}(s)) ds.
\end{aligned}$$

Let $\{\bar{u}_n\} \subseteq V$ be a sequence such that $\bar{u}_n \to \bar{u}$ in V. Then we have

$$\|F\bar{u}_n(t) - F\bar{u}(t)\|_\infty = \left\| \frac{a}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} [\bar{u}_n(s) - \bar{u}(s)] ds \right\|_\infty \le \frac{|a|\|\bar{u}_n - \bar{u}\|_\infty}{\Gamma(1-\alpha)} \int_0^\chi (\chi - s)^{-\alpha} ds.$$

Therefore

$$\|F\bar{u}_n - F\bar{u}\|_\infty \le \frac{|a|\chi^{1-\alpha}}{\Gamma(2-\alpha)} \|\bar{u}_n - \bar{u}\|_\infty \to 0.$$

Thus $F\bar{u}_n \to F\bar{u}$ whenever $\bar{u}_n \to \bar{u}$ in V. This implies F is continuous.

Let $\varepsilon > 0$ be given. Since each f_j is continuous on compact set $\mathfrak{D}$, it is uniformly continuous on $\mathfrak{D}$ and hence there exists $\delta > 0$ such that

$$\bar{u},\bar{v} \in V \text{ and } \|\bar{u}-\bar{v}\|_\infty < \delta \Rightarrow \left|f_j(t,\bar{u}) - f_j(t,\bar{v})\right| < \frac{\varepsilon}{\chi}.$$

Thus for $\bar{u},\bar{v} \in V$ and $\|\bar{u}-\bar{v}\|_\infty < \delta$ we have

$$\begin{aligned}
\|G\bar{u}(t) - G\bar{v}(t)\|_\infty &= \left\| \int_0^t \left[\bar{f}(s,\bar{u}(s)) - \bar{f}(s,\bar{v}(s))\right] ds \right\|_\infty \\
&= \sup_{1\le j\le n} \left| \int_0^t \left[f_j(s,\bar{u}(s)) - f_j(s,\bar{v}(s))\right] ds \right| \\
&\le \int_0^\chi \sup_{\substack{0\le\tau\le s \\ 1\le j\le n}} \left|f_j(\tau,\bar{u}(\tau)) - f_j(\tau,\bar{v}(\tau))\right| ds \\
&< \frac{\varepsilon}{\chi}\chi = \varepsilon.
\end{aligned}$$

This proves G is continuous. We have proved that $A = F + G$ is continuous.

Fix any $\bar{v} \in A(V) = \{A\bar{u} : \bar{u} \in V\}$, then for all $t \in [0,\chi]$,

$$\begin{aligned}
\|\bar{v}(t)\|_\infty &= \|A\bar{u}(t)\|_\infty \\
&\le \left(1 + \frac{|a|}{\Gamma(2-\alpha)}\chi^{1-\alpha}\right)\|\bar{u}_0\|_\infty + \left\| \frac{a}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha}(\bar{u}(s) - \bar{u}(0) + \bar{u}(0))ds \right\|_\infty \\
&\quad + \left\| \int_0^t \bar{f}(s,\bar{u}(s))ds \right\|_\infty \\
&\le \left(1 + \frac{|a|}{\Gamma(2-\alpha)}\chi^{1-\alpha}\right)\|\bar{u}_0\|_\infty + \frac{|a|(b + \|\bar{u}_0\|_\infty)}{\Gamma(1-\alpha)} \int_0^\chi (\chi - s)^{-\alpha} ds \\
&\quad + \sup_{1\le j\le n} \left| \int_0^t f_j(s,\bar{u}(s))ds \right| \\
&\le \left(1 + \frac{|a|}{\Gamma(2-\alpha)}\chi^{1-\alpha}\right)\|\bar{u}_0\|_\infty + \frac{|a|(b + \|\bar{u}_0\|_\infty)}{\Gamma(2-\alpha)}\chi^{1-\alpha} + \int_0^\chi \sup_{\substack{0\le\tau\le s \\ 1\le j\le n}} \left|f_j(\tau,\bar{u}(\tau))\right| ds \\
&\le \|\bar{u}_0\|_\infty + \frac{|a|(2\|\bar{u}_0\|_\infty + b)}{\Gamma(2-\alpha)}\chi^{1-\alpha} + \chi\|\bar{f}\|_\infty \\
&\le \|\bar{u}_0\|_\infty + \frac{|a|(2\|\bar{u}_0\|_\infty + b)}{\Gamma(2-\alpha)} \frac{b\Gamma(2-\alpha)}{2|a|(2\|\bar{u}_0\|_\infty + b)} + \frac{b}{2\|f_0\|_\infty}\|\bar{f}\|_\infty \\
&= \|\bar{u}_0\|_\infty + b.
\end{aligned}$$

It remains to prove that $A(V)$ is equicontinuous. For any $\bar{u} \in V$ and any $\tau_l, \tau_2 \in [0,\chi]$ with $0 \le \tau_l \le \tau_2 \le \chi$ we have already proved that

$$\|A\bar{u}(\tau_1) - A\bar{u}(\tau_2)\|_\infty \le \frac{|a|(b + 2\|\bar{u}_0\|_\infty)}{\Gamma(2-\alpha)}\left|\tau_1^{1-\alpha} - \tau_2^{1-\alpha}\right| + \|\bar{f}\|_\infty|\tau_1 - \tau_2|.$$

Noting that right hand side of above inequality is independent of $\bar{u}$ and $|\tau_1 - \tau_2| \to 0$ implies $||A\bar{u}(\tau_1) - A\bar{u}(\tau_2)||_\infty \to 0$. This proves $A(V)$ is equicontinuous.

In the view of Ascoli-Arzela theorem [21], $A(V)$ is relatively compact. This completes the proof of $A : V \to V$ is completely continuous operator. Hence by the Theorem 1 A has fixed point $\bar{u} : [0,\chi] \to \mathbb{R}^n$. This fixed point is then desired solution of SFDEs (1).

In the next theorem, assuming $\bar{f}$ is Lipschitz and using the generalisation of Banach's fixed point theorem we prove another existence result, which guarantee the uniqueness of solution also.

Theorem 4.*(Uniqueness): Let $\mathfrak{D}$, χ and b are as defined in the Theorem 3.2. Assume that $f_j : \mathfrak{D} \to \mathbb{R}$ $(j = 1,...,n)$ be continuous and $\bar{f} = (f_1,..., f_n)$ satisfies the Lipschitz condition,*

$$\|\bar{f}(t,\bar{u}) - \bar{f}(t,\bar{v})\|_\infty \le L\|\bar{u}-\bar{v}\|_\infty.$$

If $\gamma := \frac{|a|\chi^{1-\alpha}}{\Gamma(2-\alpha)} + L\chi < 1$ then the SFDEs (1) has unique solution $\bar{u}(t) : [0,\chi] \to \mathbb{R}^n$.

*Proof.*We use the same operator A defined in (2). In the Theorem 3.2 it is already proved that A is continuous operator and it maps the nonempty, closed and convex set

$$V = \left\{\bar{u} \in \mathscr{B} : \bar{u}(0) = \bar{u}_0,\, |u_j(t) - u_j(0)| < b, 1 \le j \le n\right\}$$

into itself. By method of induction we shall prove that for every $n \in \mathbb{N}\cup\{0\}$ and every $\bar{u},\bar{v} \in V$, A satisfies the condition

$$\|A^n\bar{u} - A^n\bar{v}\|_\infty \le \gamma^n\|\bar{u}-\bar{v}\|_\infty. \tag{7}$$

The inequality (7) is trivial for $n = 0$. Firstly, we prove that it is hold for $n = 1$. By using definition of operator A and Lipschitz condition on $\bar{f}$, we obtain

$$\begin{aligned}
&\|A\bar{u}(t) - A\bar{v}(t)\|_\infty \\
&\le \left\|\frac{a}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}(\bar{u}(s) - \bar{v}(s))ds\right\|_\infty + \left\|\int_0^t \bar{f}(s,\bar{u}(s))ds - \int_0^t \bar{f}(s,\bar{v}(s))ds\right\|_\infty \\
&\le \frac{|a|}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha} \sup_{0\le\tau\le s}\|\bar{u}(\tau) - \bar{v}(\tau)\|_\infty ds + \int_0^t \sup_{\substack{0\le\tau\le s\\ 1\le j\le n}}\left|f_j(\tau,\bar{u}(\tau)) - f_j(\tau,\bar{v}(\tau))\right| ds \\
&\le \frac{|a|}{\Gamma(1-\alpha)}\|\bar{u}-\bar{v}\|_\infty \int_0^\chi (\chi-s)^{-\alpha}ds + L\int_0^t \sup_{0\le\tau\le s}\|\bar{u}(\tau) - \bar{v}(\tau)\|_\infty ds \\
&\le \left(\frac{|a|\chi^{1-\alpha}}{\Gamma(2-\alpha)} + L\chi\right)\|\bar{u}-\bar{v}\|_\infty \\
&= \gamma\|\bar{u}-\bar{v}\|_\infty.
\end{aligned}$$

Let us assume (7) is true for $n = m-1$. We prove it is hold for $n = m$. Again using definition of operator A and the Lipschitz condition on $\bar{f}$, we have

$$\begin{aligned}
&\|A^m\bar{u}(t) - A^m\bar{v}(t)\|_\infty \\
&= \|A(A^{m-1}\bar{u}(t)) - A(A^{m-1}\bar{v}(t))\|_\infty \\
&\le \left\|\frac{a}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}(A^{m-1}\bar{u}(s) - A^{m-1}\bar{v}(s))ds\right\|_\infty + \left\|\int_0^t \bar{f}(s,\bar{u}(s))ds - \int_0^t \bar{f}(s,\bar{v}(s))ds\right\|_\infty \\
&\le \frac{|a|}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha} \sup_{0\le\tau\le s}\|A^{m-1}\bar{u}(\tau) - A^{m-1}\bar{v}(\tau)\|_\infty ds \\
&\quad + \int_0^t \sup_{\substack{0\le\tau\le s\\ 1\le j\le n}}\left|f_j(\tau,A^{m-1}\bar{u}(\tau)) - f_j(t,A^{m-1}\bar{v}(\tau))\right| ds \\
&\le \frac{|a|}{\Gamma(1-\alpha)}\left(\frac{|a|\chi^{1-\alpha}}{\Gamma(2-\alpha)} + L\chi\right)^{m-1}\|\bar{u}-\bar{v}\|_\infty \int_0^\chi (\chi-s)^{-\alpha}ds \\
&\quad + L\int_0^t \sup_{0\le\tau\le s}\|A^{m-1}\bar{u}(\tau) - A^{m-1}\bar{v}(\tau)\|_\infty ds \\
&\le \left(\frac{|a|\chi^{1-\alpha}}{\Gamma(2-\alpha)} + L\chi\right)^{m-1}\frac{|a|\chi^{1-\alpha}}{\Gamma(2-\alpha)}\|\bar{u}-\bar{v}\|_\infty + L\chi\left(\frac{|a|\chi^{1-\alpha}}{\Gamma(2-\alpha)} + L\chi\right)^{m-1}\|\bar{u}-\bar{v}\|_\infty \\
&= \left(\frac{|a|\chi^{1-\alpha}}{\Gamma(2-\alpha)} + L\chi\right)^{m-1}\left(\frac{|a|\chi^{1-\alpha}}{\Gamma(2-\alpha)} + L\chi\right)\|\bar{u}-\bar{v}\|_\infty \\
&= \gamma^m\|\bar{u}-\bar{v}\|_\infty.
\end{aligned}$$

By induction the proof of inequality (7) is completed.

Let $\alpha_n = \gamma^n,\ n \in \mathbb{N} \cup \{0\}$. Since $0 < \gamma < 1,\ \sum_{n=0}^{\infty} \alpha_n = \sum_{n=0}^{\infty} \gamma^n = \frac{1}{1-\gamma}$. Further, form inequality 7

$$\|A^n \bar{u} - A^n \bar{v}\|_\infty \le \alpha_n \|\bar{u} - \bar{v}\|_\infty,\ \bar{u}, \bar{v} \in V,\ n \in \mathbb{N} \cup \{0\}.$$

Therefore by the Theorem 2 we conclude that A has a unique fixed point. This fixed point is the solution of SFDEs (1).

4 Continuous Dependence of Solution

Theorem 5. *Assume that $\bar{f} = (f_1, ..., f_n)$ satisfies the Lipschitz condition,*

$$\|\bar{f}(t,\bar{u}) - \bar{f}(t,\bar{v})\|_\infty \le L\|\bar{u} - \bar{v}\|_\infty,$$

where $f_j : (j = 1, ..., n)$ is as in Theorem 3.2. Let $\bar{u}(t)$ and $\bar{v}(t)$ are the solutions of SFDEs

$$\frac{d\bar{u}(t)}{dt} + a\, {}^cD_t^\alpha \bar{u}(t) = \bar{f}(t,\bar{u}),\ \ t \in [0,T], 0 < \alpha < 1, \tag{8}$$

corresponding the initial conditions $\bar{u}(0) = \bar{u}_0$ and $\bar{v}(0) = \bar{v}_0$. If $\beta := \frac{|a|T^{1-\alpha}}{\Gamma(2-\alpha)} + LT < 1$ then

$$\|\bar{u}(t) - \bar{v}(t)\|_\infty \le \left\{ \left(1 + \frac{|a|t^{1-\alpha}}{\Gamma(2-\alpha)}\right) + \frac{\beta}{1-\beta} \right\} \|\bar{u}_0 - \bar{v}_0\|_\infty, t \in [0,T]. \tag{9}$$

Proof. Let $\bar{u}(t)$ and $\bar{v}(t)$ be the solutions of SFDEs (8) corresponding to the initial conditions $\bar{u}(0) = \bar{u}_0$ and $\bar{v}(0) = \bar{v}_0$. Consider the sequences $\{A^n \bar{u}_0\}$ and $\{A^n \bar{v}_0\}$ defined respectively by

$$\begin{aligned}
(A^0 \bar{u}_0)(t) &= \left(1 + \frac{a}{\Gamma(2-\alpha)} t^{1-\alpha}\right) \bar{u}_0 \\
(A^n \bar{u}_0)(t) &= \left(1 + \frac{a}{\Gamma(2-\alpha)} t^{1-\alpha}\right) \bar{u}_0 - \frac{a}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} (A^{n-1}\bar{u}_0)(s) ds \\
&\quad + \int_0^t \bar{f}(s, (A^{n-1}\bar{u}_0)(s)) ds,\ (n = 1,2,3,...),
\end{aligned}$$

and

$$\begin{aligned}
(A^0 \bar{v}_0)(t) &= \left(1 + \frac{a}{\Gamma(2-\alpha)} t^{1-\alpha}\right) \bar{v}_0 \\
(A^n \bar{v}_0)(t) &= \left(1 + \frac{a}{\Gamma(2-\alpha)} t^{1-\alpha}\right) \bar{v}_0 - \frac{a}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} (A^{n-1}\bar{v}_0)(s) ds \\
&\quad + \int_0^t \bar{f}(s, (A^{n-1}\bar{v}_0)(s)) ds,\ (n = 1,2,3,...).
\end{aligned}$$

Then

$$\begin{aligned}
\|(A^n \bar{u}_0)(t) - A^n \bar{v}_0(t)\|_\infty &\le \left(1 + \frac{|a|}{\Gamma(2-\alpha)} t^{1-\alpha}\right) \|\bar{u}_0 - \bar{v}_0\|_\infty \\
&\quad + \frac{|a|}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \left\|(A^{n-1}\bar{u}_0)(s) - (A^{n-1}\bar{v}_0)(s))\right\|_\infty ds \\
&\quad + \left\| \int_0^t \bar{f}(s, (A^{n-1}\bar{u}_0)(s))) ds - \int_0^t \bar{f}(s, (A^{n-1}\bar{v}_0)(s)) ds \right\|_\infty.
\end{aligned}$$

Proceeding as in the proof of Theorem 4, we obtain

$$\begin{aligned}
\|(A^n \bar{u}_0)(t) - A^n \bar{v}_0(t)\|_\infty &\le \left(1 + \frac{|a|}{\Gamma(2-\alpha)} t^{1-\alpha}\right) \|\bar{u}_0 - \bar{v}_0\|_\infty + \left(\frac{|a|T^{1-\alpha}}{\Gamma(2-\alpha)} + LT\right)^n \|\bar{u}_0 - \bar{v}_0\|_\infty \\
&= \left\{ \left(1 + \frac{|a|}{\Gamma(2-\alpha)} t^{1-\alpha}\right) + \beta^n \right\} \|\bar{u}_0 - \bar{v}_0\|_\infty.
\end{aligned}$$

From above inequality it follows that

$$\|(A^n\bar{u}_0)(t)-A^n\bar{v}_0(t)\|_\infty \leq \left\{\left(1+\frac{|a|}{\Gamma(2-\alpha)}t^{1-\alpha}\right)+\sum_{k=1}^{n}\beta^n\right\}\|\bar{u}_0-\bar{v}_0\|_\infty, t\in[0,T].$$

Taking limit as $n\to\infty$ in above inequality and using the Theorem 2, we obtain

$$\|\bar{u}(t)-\bar{v}(t)\|_\infty \leq \left\{\left(1+\frac{|a|}{\Gamma(2-\alpha)}t^{1-\alpha}\right)+\sum_{k=1}^{\infty}\beta^n\right\}\|\bar{u}_0-\bar{v}_0\|_\infty, t\in[0,T].$$

Since $0<\beta<1$,

$$\|\bar{u}(t)-\bar{v}(t)\|_\infty \leq \left\{\left(1+\frac{|a|}{\Gamma(2-\alpha)}t^{1-\alpha}\right)+\frac{\beta}{1-\beta}\right\}\|\bar{u}_0-\bar{v}_0\|_\infty, t\in[0,T].$$

5 Successive Approximation (Picard)

Define the successive approximations to the solution of initial value problem (1) by

$$\bar{u}_0(t)=\left(1+\frac{a}{\Gamma(2-\alpha)}t^{1-\alpha}\right)\bar{u}_0, \tag{10}$$

$$\bar{u}_{n+1}(t)=\left(1+\frac{a}{\Gamma(2-\alpha)}t^{1-\alpha}\right)\bar{u}_0-\frac{a}{\Gamma(1-\alpha)}\int_0^t(t-s)^{-\alpha}\bar{u}_n(s)ds$$
$$+\int_0^t\bar{f}(s,\bar{u}_n(s))ds,\ (n=0,1,2,\ldots). \tag{11}$$

Theorem 6. *Let $\bar{f}$ and χ are as defined in the Theorem 3.2. Then, the successive approximations given by (10)-(11) are valid on $[0,\chi]$, and satisfy*

$$\|\bar{u}_n(t)-\bar{u}_0\|_\infty\leq b,\ \forall n\in\mathbb{N}\cup\{0\},\ t\in[0,\chi]. \tag{12}$$

Proof. Obviously $\bar{u}_0(t)$ is defined on $[0,\chi]$, and from (10), we have

$$\|\bar{u}_0(t)-\bar{u}_0\|_\infty\leq\frac{\chi^{1-\alpha}}{\Gamma(2-\alpha)}\|\bar{u}_0\|_\infty\leq\frac{b\Gamma(2-\alpha)}{2(2\|\bar{u}_0\|_\infty+b)}\frac{\|\bar{u}_0\|_\infty}{\Gamma(2-\alpha)}\leq b,$$

which is (12) for $n=0$.

Assume, for any $n=m\in\mathbb{N}$, $\bar{u}_m(t)$ is defined on $[0,\chi]$ and satisfies (12). Therefore $\bar{u}_{m+1}(t)$ is defined on $[0,\chi]$ and from (11) we have

$$\begin{aligned}
&\|\bar{u}_{m+1}(t)-\bar{u}(0)\|_\infty\\
&\leq\frac{|a|\chi^{1-\alpha}}{\Gamma(2-\alpha)}\|\bar{u}_0\|_\infty+\frac{|a|}{\Gamma(1-\alpha)}\left\|\int_0^t(t-s)^{-\alpha}\left([\bar{u}_m(s)-\bar{u}(0)]+\bar{u}(0)\right)ds\right\|_\infty+\left\|\int_0^t\bar{f}(s,\bar{u}_m(s))ds\right\|_\infty\\
&\leq\frac{|a|\chi^{1-\alpha}}{\Gamma(2-\alpha)}\|\bar{u}_0\|_\infty+\frac{|a|(b+\|\bar{u}_0\|_\infty)}{\Gamma(1-\alpha)}\int_0^\chi(\chi-s)^{-\alpha}ds+\int_0^\chi\sup_{\substack{0\leq\tau\leq s\\1\leq j\leq n}}\left|f_j(\tau,\bar{u}_m(\tau))\right|ds\\
&\leq\frac{|a|\chi^{1-\alpha}}{\Gamma(2-\alpha)}\|\bar{u}_0\|_\infty+\frac{|a|(b+\|\bar{u}_0\|_\infty)}{\Gamma(2-\alpha)}\chi^{1-\alpha}+\chi\|\bar{f}_0\|_\infty\\
&=\frac{|a|(2\|\bar{u}_0\|_\infty+b)}{\Gamma(2-\alpha)}\chi^{1-\alpha}+\chi\|\bar{f}\|_\infty\\
&\leq\frac{|a|(2\|\bar{u}_0\|_\infty+b)}{\Gamma(2-\alpha)}\frac{b\Gamma(2-\alpha)}{2|a|(2\|\bar{u}_0\|_\infty+b)}+\frac{b}{2\|\bar{f}_0\|_\infty}\|\bar{f}\|_\infty\\
&=b.
\end{aligned}\tag{13}$$

Thus $\bar{u}_{m+1}(t)$ satisfies (12). By induction proof of the theorem is completed.

In the next theorem it is proved that the successive approximations $\{\bar{u}_n(t)\}$ to the solutions of IVP (1) defined by (10)-(11) converges uniformly to a function $\bar{u}(t)$ on $[0,\chi]$, and this limit function $\bar{u}(t)$ is a solution of SFDEs (1).

Theorem 7. *Let $\mathfrak{D}$, χ and b are as defined in the Theorem 3.2. Assume that $f_j : \mathfrak{D} \to \mathbb{R}$ $(j = 1,...,n)$ be continuous and $\bar{f} = (f_1,..., f_n)$ satisfies the Lipschitz condition,*

$$\|\bar{f}(t,\bar{u}) - \bar{f}(t,\bar{v})\|_\infty \le L\|\bar{u}-\bar{v}\|_\infty.$$

If $\gamma := \frac{|a|\chi^{1-\alpha}}{\Gamma(2-\alpha)} + L\chi < 1$ then the successive approximations $\{u_n(t)\}_{n=0}^{\infty}$ to the solution of initial value problem for system of FDEs (1) defined by (10)-(11) converges on the interval $[0,\chi]$ to a solution $\bar{u}(t)$ of SFDEs (1).

Proof. For any $t \in [0,\chi]$, we can write $\bar{u}_n(t) = \bar{u}_0(t) + \sum_{k=1}^{n}(\bar{u}_k(t) - \bar{u}_{k-1}(t))$, which is a partial sum of the series $\bar{u}_0(t) + \sum_{k=1}^{\infty}(\bar{u}_k(t) - \bar{u}_{k-1}(t))$. Now from (10) and (11) we have

$$\begin{aligned}
\|\bar{u}_1(t) - \bar{u}_0(t)\|_\infty &= \left\| -\frac{a}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha}\bar{u}_0(s)ds + \int_0^t \bar{f}(s,\bar{u}(s))ds \right\|_\infty \\
&\le \left(1 + \frac{|a|}{\Gamma(2-\alpha)}\chi^{1-\alpha}\right) \|\bar{u}_0\|_\infty \frac{|a|}{\Gamma(1-\alpha)} \int_0^\chi (\chi - s)^{-\alpha} ds + \chi \|\bar{f}\|_\infty \\
&= \left(1 + \frac{|a|}{\Gamma(2-\alpha)}\chi^{1-\alpha}\right) \frac{|a|}{\Gamma(2-\alpha)}\chi^{1-\alpha}\|\bar{u}_0\|_\infty + \chi\|\bar{f}\|_\infty.
\end{aligned}$$

Again, from (10) and (11),

$$\begin{aligned}
&\|\bar{u}_2(t) - \bar{u}_1(t)\|_\infty \\
&\le \frac{|a|}{\Gamma(1-\alpha)} \left\| \int_0^t (t-s)^{-\alpha}(\bar{u}_1(s) - \bar{u}_0(s))ds \right\|_\infty + \left\| \int_0^t [\bar{f}(s,\bar{u}_1(s)) - \bar{f}(s,\bar{u}_0(s))]ds \right\|_\infty \\
&\le \frac{|a|}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \sup_{0\le\tau\le s} \|\bar{u}_1(\tau) - \bar{u}_0(\tau)\|_\infty ds + L\int_0^t \sup_{0\le\tau\le s} \|\bar{u}_1(\tau) - \bar{u}_0(\tau)\|_\infty ds \\
&\le \frac{|a|}{\Gamma(1-\alpha)} \left\{ \left(1 + \frac{|a|}{\Gamma(2-\alpha)}\chi^{1-\alpha}\right) \frac{|a|}{\Gamma(2-\alpha)}\chi^{1-\alpha}\|\bar{u}_0\|_\infty + \chi\|\bar{f}\|_\infty \right\} \int_0^\chi (\chi-s)^{-\alpha}ds \\
&\quad + L\chi \left\{ \left(1 + \frac{|a|}{\Gamma(2-\alpha)}\chi^{1-\alpha}\right) \frac{|a|}{\Gamma(2-\alpha)}\chi^{1-\alpha}\|\bar{u}_0\|_\infty + \chi\|\bar{f}\|_\infty \right\} \\
&= \left\{ \left(1 + \frac{|a|}{\Gamma(2-\alpha)}\chi^{1-\alpha}\right) \frac{|a|}{\Gamma(2-\alpha)}\chi^{1-\alpha}\|\bar{u}_0\|_\infty + \chi\|\bar{f}\|_\infty \right\} \left(\frac{|a|}{\Gamma(2-\alpha)} \frac{\chi^{1-\alpha}}{1-\alpha} + L\chi \right) \\
&= \left\{ \left(1 + \frac{|a|}{\Gamma(2-\alpha)}\chi^{1-\alpha}\right) \frac{|a|}{\Gamma(2-\alpha)}\chi^{1-\alpha}\|\bar{u}_0\|_\infty + \chi\|\bar{f}\|_\infty \right\} \left(\frac{|a|\chi^{1-\alpha}}{\Gamma(2-\alpha)} + L\chi \right) \\
&= \lambda\gamma,
\end{aligned}$$

where, $\lambda := \left(1 + \frac{|a|}{\Gamma(2-\alpha)}\chi^{1-\alpha}\right) \frac{|a|}{\Gamma(2-\alpha)}\chi^{1-\alpha}\|\bar{u}_0\|_\infty + \chi\|\bar{f}\|_\infty$ and $\gamma := \frac{|a|\chi^{1-\alpha}}{\Gamma(2-\alpha)} + L\chi$.

We shall prove by induction that

$$\|\bar{u}_n(t) - \bar{u}_{n-1}(t)\|_\infty \le \lambda\gamma^{n-1}. \tag{14}$$

Already it is proved for $n=1,2$. Now assume (14) form $n=m\in\mathbb{N}$. By definition of $\bar{u}_{m+1}(t)$ and $\bar{u}_m(t)$ we obtain

$$\begin{aligned}
&\|\bar{u}_{m+1}(t)-\bar{u}_m(t)\|_\infty\\
&\le \frac{|a|}{\Gamma(1-\alpha)}\left\|\int_0^t (t-s)^{-\alpha}(\bar{u}_m(s)-\bar{u}_{m-1}(s))ds\right\|_\infty+\left\|\int_0^t [\bar{f}(s,\bar{u}_m(s))-\bar{f}(s,\bar{u}_{m-1}(s))]ds\right\|_\infty\\
&\le \frac{|a|}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\sup_{0\le\tau\le s}\|\bar{u}_m(\tau)-\bar{u}_{m-1}(\tau)\|_\infty ds+L\int_0^t \sup_{0\le\tau\le s}\|\bar{u}_m(\tau)-\bar{u}_{m-1}(\tau)\|_\infty ds\\
&\le \frac{|a|}{\Gamma(1-\alpha)}\lambda\gamma^{m-1}\int_0^\chi (\chi-s)^{-\alpha}ds+\lambda\gamma^{m-1}L\chi\\
&=\lambda\gamma^{m-1}\left(\frac{|a|\chi^{1-\alpha}}{\Gamma(2-\alpha)}+L\chi\right)\\
&=\lambda\gamma^m.
\end{aligned}$$

Above inequality is just (14) for $n=m+1$. We have proved that (14) is true far all $n\in\mathbb{N}$.

Since $0<\gamma<1$, $\sum_{m=1}^{\infty}\lambda\gamma^m=\frac{\lambda\gamma}{1-\gamma}$ and therefore from the inequality (14) it follows that $\bar{u}_0(t)+\sum_{k=1}^{\infty}(\bar{u}_k(t)-\bar{u}_{k-1}(t))$, $t\in[0,\chi]$ converges with respect to the norm $\|\cdot\|_\infty$. This implies the sequence $\{\bar{u}_n(t)\}_{n=1}^{\infty}$ converges uniformly to a continuous function $\bar{u}(t)$, $t\in[0,\chi]$. Thus taking limit as $n\to\infty$ on both sides of (10), we obtain

$$\bar{u}(t)=\left(1+\frac{a}{\Gamma(2-\alpha)}t^{1-\alpha}\right)\bar{u}(0)-\frac{}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\bar{u}(s)ds+\int_0^t \bar{f}(s,\bar{u}(s))ds.$$

This proves $\bar{u}(t)$, $t\in[0,\chi]$ is a solution of the integral equation equivalent to IVP (1).

Corollary 1.*Let $\bar{u}_n(t)$ be the n^{th} successive approximation to the solution $\bar{u}(t)$ of the initial value problem* (1)*. Then the error $\bar{u}(t)-\bar{u}_n(t)$ satisfies the estimate*

$$\|\bar{u}(t)-\bar{u}_n(t)\|_\infty\le\frac{\lambda}{1-\gamma}\gamma^{n+1}, \tag{15}$$

where, $\gamma:=\frac{|a|\chi^{1-\alpha}}{\Gamma(2-\alpha)}+L\chi<1$ *and* $\lambda:=\left(1+\frac{|a|}{\Gamma(2-\alpha)}\chi^{1-\alpha}\right)\frac{|a|}{\Gamma(2-\alpha)}\chi^{1-\alpha}\|\bar{u}_0\|_\infty+\chi\|\bar{f}\|_\infty.$

*Proof.*As discussed in above theorem for any $t\in[0,\chi]$, we have

$$\bar{u}(t)=\bar{u}_0(t)+\sum_{k=1}^{\infty}(\bar{u}_k(t)-\bar{u}_{k-1}(t))$$

and

$$\bar{u}_n(t)=\bar{u}_0(t)+\sum_{k=1}^{n}(\bar{u}_k(t)-\bar{u}_{k-1}(t)).$$

Using the inequality (14), we obtain

$$\|\bar{u}(t)-\bar{u}_n(t)\|_\infty=\left\|\sum_{k=1}^{\infty}(\bar{u}_k(t)-\bar{u}_{k-1}(t))\right\|_\infty\le\sum_{k=n+1}^{\infty}\lambda\gamma^k=\lambda\frac{\gamma^{n+1}}{1-\gamma}.$$

Remark 5.1: Since $0<\gamma<1$, from above inequality it follows $\bar{u}_n(t)\to\bar{u}(t)$ as $n\to\infty$ i.e. the sequence $\{\bar{u}_n(t)\}$ of successive approximation converges to the solution $\bar{u}(t)$ of system of FDE (1).

6 Examples

Consider the following system of FDEs:

$$\frac{du_1}{dt}+\frac{\Gamma(\frac{7}{3})}{2}\,{}^cD_t^{\frac{2}{3}}u_1(t)=1+2t+t^{\frac{4}{3}}+u_1^2-u_2,\ u_1(0)=0, \tag{16}$$

$$\frac{du_2}{dt}+\frac{\Gamma(\frac{7}{3})}{2}\,{}^cD_t^{\frac{2}{3}}u_2(t)=4tu_1+12\frac{\Gamma(\frac{7}{3})}{\Gamma(\frac{13}{3})}t^{\frac{10}{3}},\ u_2(0)=1. \tag{17}$$

Let $T > 0, b_j > 0$ $(j = 1,2)$ and consider the functions $f_j : \mathfrak{D} \to \mathbb{R}$ $(j = 1,2)$ defined by

$$f_1(t,\bar{u}) = 1 + 2t + t^{\frac{4}{3}} + u_1^2 - u_2,$$
$$f_2(t,\bar{u}) = 4tu_1 + 12\frac{\Gamma(\frac{7}{3})}{\Gamma(\frac{13}{3})}t^{\frac{10}{3}},$$

where $\mathfrak{D} = [0,T] \times [-b_1,b_1] \times [1-b_2, 1+b_2]$ and $\bar{u}(t) = (u_1(t),u_2(t))$.

Let $\bar{f}(t,\bar{u}) = \left(\bar{f_1}(t,\bar{u}), \bar{f_2}(t,\bar{u})\right) = \left(1 + 2t + t^{\frac{4}{3}} + u_1^2 - u_2, 4tu_1 + 12\frac{\Gamma(\frac{7}{3})}{\Gamma(\frac{13}{3})}t^{\frac{10}{3}}\right)$. Then above SFDEs can be written in the form

$$\frac{d\bar{u}(t)}{dt} + \frac{\Gamma(\frac{7}{3})}{2}\, {}^cD_t^{\frac{2}{3}}\bar{u}(t) = \bar{f}(t,\bar{u}),\ \bar{u}(0) = (0,1),\ t \in [0,T].$$

Note that $\bar{f} : \mathfrak{D} \to \mathbb{R}^2$ is clearly continuous. We prove that $\bar{f}$ satisfies a Lipschitz condition with respect to the second variable.

For any $(t,\bar{u}),(t,\bar{v}) \in \mathfrak{D}$, we have

$$\bar{f}(t,\bar{u}) - \bar{f}(t,\bar{v}) = \left(u_1^2 - v_1^2 - u_2 + y_2,\ 4t(u_1 - v_1)\right).$$

Therefore,

$$\left\|\bar{f}(t,\bar{u}) - \bar{f}(t,\bar{v})\right\|_\infty = \sup_{0\le t\le T}\left\{\sup\{|u_1^2 - v_1^2 - u_2 + y_2|, |4t(u_1 - v_1)|\}\right\}.$$

Note that

$$\begin{aligned} |u_1^2 - v_1^2 - u_2 + y_2| &\le |u_1^2 - v_1^2| + |u_2 - y_2| \\ &\le |u_1 + v_1||u_1 - v_1| + |u_2 - y_2| \\ &\le (|u_1| + |v_1|)|u_1 - v_1| + |u_2 - y_2| \\ &\le 2b_1|u_1 - v_1| + |u_2 - y_2| \\ &\le (1 + 2b_1)\{|u_1 - v_1| + |u_2 - y_2|\} \\ &\le 2(1 + 2b_1)\max\{|u_1 - v_1|,\ |u_2 - y_2|\} \\ &= 2(1 + 2b_1)\|\bar{u} - \bar{v}\|_\infty. \end{aligned}$$

and

$$4t|u_1 - v_1| \le 4t\max\{|u_1 - v_1|,\ |u_2 - y_2|\} = 4t\|\bar{u} - \bar{v}\|_\infty.$$

Hence

$$\left\|\bar{f}(t,\bar{u}) - \bar{f}(t,\bar{v})\right\|_\infty = \sup_{0\le t\le T}\{\sup\{2(1 + 2b_1)\|\bar{u} - \bar{v}\|_\infty,\ 4t\|\bar{u} - \bar{v}\|_\infty\}\} \le L\|\bar{u} - \bar{v}\|_\infty.$$

This proves $\bar{f}$ satisfies a Lipschitz condition on $\mathfrak{D}$ with Lipschitz constant $L = \max\{2(1+2b_1), 4T\}$. Further

$$\left\|\bar{f}\right\|_\infty = \sup_{\substack{0\le t\le T \\ 1\le j\le 2}}\left|f_j(t,\bar{u}(t))\right| = \max\left\{2T + T^{\frac{4}{3}} + b_1^2 - b_2,\ 4Tb_1 + 12\frac{\Gamma(\frac{7}{3})}{\Gamma(\frac{13}{3})}T^{\frac{10}{3}}\right\}.$$

All the assumptions of the Theorems 3 and 4 are satisfied and hence the SFDEs (16)- (17) has a unique solution $\bar{u} : [0,\chi] \to \mathbb{R}^2$ where

$$\chi = \min\left\{T, \left[\frac{b\Gamma(\frac{4}{3})}{\Gamma(\frac{7}{3})(2+b)}\right]^3, \frac{b}{2\|f\|_\infty}\right\} \text{ and } b = \min\{b_1, b_2\}.$$

One can verify that $\bar{u}(t) = (t^2,\ t^4 + 1)$ is the unique solution of the SFDEs (16)- (17).

7 Conclusions

With boundedness and Lipschitz condition on $\bar{f}$ we have proved various results such as existence, uniqueness, continuous dependence of solutions and convergence of Picards successive approximations related to the system of nonlinear ordinary differential equations (1).

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