Journal of Analysis & Number Theory
An International Journal

http://dx.doi.org/10.18576/jant/050112

Weak Convergence Theorems of Explicit Iteration Process with Errors and Applications in Optimization

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Received: 7 Jul. 2016, Revised: 21 Sep. 2016, Accepted: 23 Sep. 2016
Published online: 1 Jan. 2017

Abstract: This paper deals with a new explicit metric projection iteration with errors scheme of finding a common fixed point of infinite families of nonlinear mappings in a Hilbert space and we prove weak convergence theorems for finding common fixed points of this families of firmly nonexpansive mappings. Also, we apply our results to prove the convergence of some iterative algorithms with error terms for solving variational inclusion problems, equilibrium problems and split feasibility problems in Hilbert spaces.

Keywords: explicit iteration process, firmly nonexpansive mapping; fixed point; Hilbert space. *AMS Mathematics subject Classification.* 47H09,47H10,47H20.

1 Introduction

Many problems in physics, optimization, image processing and economics can be recast in terms of a fixed point problem of nonlinear mappings in Hilbert space [[1],[2], [3], [4], [5], [6]]. A lot of this studies consider this mappings as nonexpansive which defined as: let H be a real Hilbert space and K be a nonempty closed convex subset of H. Then, a mapping R of K into H is called nonexpansive if $\|Rx-Ry\| \leq \|x-y\|$ for all $x,y \in K$, and R is called firmly nonexpansive if

$$||Rx-Ry||^2 + ||(Id-R)x-(Id-R)y||^2 \leq ||x-y||^2 \quad (1)$$

for all $x,y \in K$, where $Id : K \to K$ denote the identity operator. We have known that every firmly nonexpansive mapping is nonexpansive mapping. In 1953, Mann [7] consider the following iteration scheme

$$\begin{cases} x_1 \in K \text{ is chosen arbitrarily} \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)Rx_n, \forall\, n \in N \end{cases}$$

where $\{\alpha_n\}$ is a sequence in [0,1]. In 1967, Halpern [8] study the strong convergence of the following iteration: Fix a point $u \in K$

$$\begin{cases} x_1 \in K \text{ is chosen arbitrarily} \\ x_{n+1} = \alpha_n u + (1-\alpha_n)Rx_n, \forall\, n \in N \end{cases}$$

here $\{\alpha_n\}$ is a real sequence in the interval [0,1]. In 1996, Bauschke [9] defined another iteration process by using a finite family of nonexpansive mappings in Hilbert space as follow: Let $\{R_1, R_2, \ldots.., R_r : r \in N\}$ be a finite set of r nonexpansive self mappings of K such that: $F := \cap_{i=1}^{r} F(R_i) \neq \emptyset$. Define $\{x_n\}$ as follows: Fix $u \in K$ and $\{\alpha_n\}$ be a real sequence in [0,1]

$$\begin{cases} x_1 \in K \text{ is chosen arbitrarily} \\ x_{n+1} = \alpha_n u + (1-\alpha_n)R_n x_n, \forall\, n \in N, \end{cases}$$

where $R_k = R_{k \bmod r}$ (Here the mod r function take value in $\{1,2,\ldots.,r\}$). Bauschke Succeed to find a common fixed point of this iteration.

In 2001, Xu and Ori [10] gave the following implicit iteration:

$$\begin{cases} x_1 \in K \text{ is chosen arbitrarily} \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)R_n x_{n+1}, \forall\, n \in N \end{cases}$$

where $\{\alpha_n\}$ be a real sequence in [0,1], and $R_k = R_{k \bmod r}$, and proved the convergence of this iteration to a common fixed point. In 2005, Kimura et al. [11], studied the convergence of an iterative scheme to a common fixed point of a finite family of nonexpansive mappings in Banach space.
The problem of finding a common fixed point of families

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of nonlinear mappings has been investigated by many researchers; see, for instance, ([12]-[17]).

Recently, Chuang and Takahashi [18] defined the new Mann's type iteration process by metric projection from H to K and gave weak convergence theorems for finding a common fixed point of a sequence of firmly nonexpansive mappings in a Hilbert space. They introduced a new iterative sequence for finding a common fixed point of the families of nonlinear mappings in a Hilbert space as follows: Let $\{R_n\}$ be a sequence of firmly nonexpansive mappings from H to K and $\{x_n\}$ be a sequence in K defined by

$$\begin{cases} x_1 \in K \text{ is chosen arbitrarily} \\ x_{n+1} := P_K(\alpha_n x_n + (1-\alpha_n)R_n x_n), \forall\, n \in N \end{cases}$$

Where P_K is the metric projection from H onto K, $\{R_n\}$ satisfies the resolvent property and $\{\alpha_n\}$ be a sequence in $(0,2)$. Also, they proved that the sequence $\{x_n\}$ converges weakly to a common fixed point of $\{R_n\}$.

In this paper we prove that: If $\{x_n\}$ be a sequence defined by:

$$\begin{cases} x_1 \in K \text{ is chosen arbitrarily} \\ x_{n+1} := P_K(\alpha_n x_n + (1-\alpha_n)R_n x_n + e_n), \forall\, n \in N \end{cases} \quad (2)$$

where, $\{\alpha_n\}$ be a sequence in (0,1) which satisfies the following condition: $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, and $\{e_n\}$ be a bounded sequence in K which satisfies that: $\sum_{n=1}^{\infty} \|e_n\| < \infty$. Then $x_n \rightharpoonup \bar{x}$, where $\bar{x} \in \cap_{n=1}^{\infty} F(R_n)$. We extended our result to study the convergence of iterative process with errors of another type of nonlinear mapping in H under other certain conditions. Also, we apply our results to prove the convergence of some algorithms with error analysis for solving variational inclusion problems, equilibrium problems and split feasibility problems in Hilbert spaces.

2 Preliminaries

Let H be a real Hilbert space. The inner product and the induced norm on H are denoted by $< .,. >$ and $\| \,.\, \|$ respectively. Throughout this paper, we denote by N the set of positive integers and strongly (respectively weak) convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ (respectively $x_n \rightharpoonup x$). Denote by $F(R)$ the set of fixed points of R (i.e., $F(R) = \{x \in K : Rx = x\}$).

Lemma 2.1 ([19,20]). Let $R : K \to H$. Then the following statements are equivalent:
(i) R is firmly nonexpansive,
(ii) $Id - R$ is firmly nonexpansive,
(iii) $2R - Id$ is nonexpansive,
(iv) $||Rx - Ry||^2 \le \langle x-y, Rx-Ry\rangle$ $(\forall x,y \in K)$,
(v) $0 \le \langle Rx - Ry, (Id-R)x - (Id-R)y\rangle$ $(\forall x,y \in K)$.

Lemma 2.2 ([21]). Let H be a real Hilbert space then the following equations hold:
(i) $||x-y||^2 = ||x||^2 - ||y||^2 - 2\langle x-y, y\rangle$, for all $x,y \in H$,
(ii) $||tx + (1-t)y||^2 = t||x||^2 + (1-t)||y||^2 - t(1-t)||x-y||^2$, for all $t \in [0,1]$ and $x,y \in H$,
(iii) $2\langle x-y, u-v\rangle = ||x-v||^2 + ||y-u||^2 - ||x-u||^2 - ||y-v||^2$ for all $x,y,u,v \in H$.

Definition 2.1 [22]. A linear subspace M of a normed space X is called proximinal (resp. Chebyshev) if for each $x \in X$, the best approximations to X from M,

$$P_M := \{y \in M : ||x-y|| = \inf_{m\in M} ||x-m||\}, \quad (3)$$

is nonempty (resp. a singleton). It will know that for each element of a Hilbert space H there exist Chebyshev convex subset.

Definition 2.2 [23]. The mapping $P_K : H \to K$ which is defined by $P_K x = z_x$ for $x \in H$ such that:

$$||z_x - x|| \le ||y - x||, \quad (4)$$

for all $y \in K$, is called the metric projection of H onto K. We have known that P_K is firmly nonexpansive, therefore P_K is nonexpansive.

Lemma 2.3 [24]. Let K be a nonempty, closed and convex subset of a Hilbert space H. Let P_K be the metric projection from H onto K. Then $\langle x - P_K x, P_K x - y\rangle \ge 0, \forall\, x \in H, y \in K$.

Definition 2.3 [25]. Let K be nonempty subset of a Hilbert space H. Let $\{R_n\}$ be a sequence of mappings from K into itself. We say that $\{R_n\}$ Satisfies AKTT-condition if:

$$\sum_{n=1}^{\infty} \sup_{x\in B} ||R_{n+1}x - R_n x|| < \infty, \quad (5)$$

for each nonempty and bounded subset B of K.

Lemma 2.4 [25]. Let K be a nonempty and closed subset of a Hilbert space H and let $\{R_n\}$ be a sequence of mappings from K into itself which satisfies AKTT-condition. Then, for each $x \in K, \{R_n x\}$ converges strongly to a point in K. Furthermore, define a mapping $R : K \to K$ by $Rx := \lim_{n\to\infty} R_n x,\ x \in K$. Then, for each bounded subset B of K,

$$\lim_{n\to\infty} \sup\{||Rz - R_n z|| : z \in B\} = 0. \quad (6)$$

Definition 2.4 [18]. Let K be a nonempty, closed and convex subset of a Hilbert space H. Let $\{R_n\}$ be a firmly nonexpansive mapping from K into H. Then we say that $\{R_n\}$ satisfies the resolvent property if there exist a nonexpansive mapping $R : K \to H$ and two natural number n_0 and k such that: $||x - Rx|| \le k||x - R_n x||$ for all $x \in K$ and $n \in N$ with $n \ge n_0$ and $F(R) = \cap_{n=1}^{\infty} F(R_n)$.

Definition 2.5. Let K be a nonempty, closed and convex subset of a Hilbert space H. A family $\Gamma := \{T(s) : 0 \le s \le \infty\}$ of mappings from K into itself is called a one-parameter nonexpansive semigroup of K if it satifies the following conditions: for all $x,y \in K$ and $s,t \ge 0$
(i) $T(0)x := x$,

(ii) $T(s+t) = T(s)T(t)$,
(iii)$||T(s)x - T(s)y|| \leq ||x-y||$,
(iv) for each $x \in K, s \to T(s)x$ is continuous.

Lemma 2.5 [26]. Let K be a nonempty, closed and convex subset of a Hilbert space H and let $R : K \to K$ be a firmly nonexpansive mapping with $F(R) \neq \emptyset$. Then $\langle x - Rx, Rx - z\rangle \geq 0$ for all $x \in K$ and $z \in F(R)$.

Lemma 2.6 [24]. Let K be a nonempty, closed and convex subset of a Hilbert space H. Let R be a nonexpansive mapping of K into itself and let $\{x_n\}$ be a sequence in K. If $x_n \rightharpoonup w$ and $\lim_{n\to\infty} ||x_n - Rx_n|| = 0$, then $Rw = w$.

Definition 2.6 [27]. A space X is said to satisfy Opial's condition if for each sequence $\{x_n\}$ in X which $x_n \rightharpoonup x$, we have $\forall\, y \in X, y \neq x$ the following:
(i) $\liminf_{n\to\infty} ||x_n - x|| < \liminf_{n\to\infty} ||x_n - y||$,
(ii) $\limsup_{n\to\infty} ||x_n - x|| < \limsup_{n\to\infty} ||x_n - y||$.

Lemma 2.7 [27]. A Hilbert space has Opial's property.

Definition 2.7. Let $A : H \to 2^H$ be a set valued mapping. The sets $domA = \{x \in H : Ax \neq \emptyset\}$ and $grA = \{(x,u) \in H^2 : u \in Ax\}$ are the domain and the graph of A, respectively. A is said to be monotone mapping on H if $\langle x-y, u-v\rangle \geq 0$ for all $(x,u),(y,v) \in grA$. A monotone mapping A on H is said to be maximal if grA is not properly contained in the graph of any other monotone mapping on H. For a maximal monotone A on H and $r > 0$, we define a single-valued mapping $J_r = (Id + rA)^{-1} : H \to domA$, which is called the resolvent of A for $r > 0$. The Yoside approximation of A of index $r > 0$ is $A_r = \frac{1}{2}(Id - J_r)$. From [24], we have that: $A_r x \in AJ_r x$, for all $x \in H$ and $r > 0$. For details see [[28], [18], [29], [30]] .

Remarks 2.1[31]. Let A be a maximal monotone mapping on H and let $A^{-1}0 = \{x \in H : 0 \in AX\}$:
(i) $A^{-1}0 = Fix(J_r)$ for all $r > 0$,
(ii) J_r is firmly nonexpansive,
(iii) if $s, r \in R$ with $s \geq r > 0$ and $x \in H$, we have $\|x - J_s x\| \geq \|x - J_r x\|$.

Lemma 2.8.([32]) Let K be a closed convex subset of a real Hilbert space H. Let R be a nonexpansive nonself-mapping of K into H such that $F(R) \neq \emptyset$. Then $F(R) = F(P_K R)$.

Lemma 2.9.([33]) Let $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be three sequences satisfying follows:

$$x_{n+1} \leq (1+y_n) + z_n, \forall n \geq n_0$$

where n_0 is some nonnegative integer, $\sum_{n=0}^{\infty} z_n < \infty$ and $\sum_{n=0}^{\infty} y_n < \infty$. Then $\lim_{n\to\infty} x_n$ exists.

3 Weak convergence Theorems of explicit iterative process with errors

In this section, we prove our new weak convergence theorems for families of firmly nonexpansive mappings in Hilbert spaces.

Theorem 3.1. Let H be a Hilbert space, K be a nonempty, closed and convex subset of H. Consider $\{R_n\} : K \to H$ be a sequence of firmly nonexpansive mappings with $S := \bigcap_{n=1}^{\infty} F(R_n) \neq \phi$ and $\{R_n\}$ satisfies resolvent property. Let $\{\alpha_n\}$ be a sequence of real numbers in (0,1) which satisfies that: $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\{e_n\}$ be a bounded sequence in K which satisfies that: $\sum_{n=1}^{\infty} \|e_n\| < \infty$. For a sequence $\{x_n\}$ of K which generated defined as in (2).
Then $x_n \rightharpoonup \overline{x}$, where $\overline{x} \in \bigcap_{n=1}^{\infty} F(R_n)$.

Proof. Let $w \in S$, then $R_n(w) = w$ for all $n \in N$. And since R_n is firmly nonexpansive , then it is nonexpansive for all $n \in N$. From lemma 2.8, we get that, if $w \in S$ then:

$$F(R_n) = F(P_K R_n),\ \forall\, n \in N.$$

Thus, for all $n \in N$ we have

$$P_K(R_n w) = P_K(w) = w. \tag{7}$$

Therefore, by (7) and (ii) in lemma 2.2, we obtain that

$$\begin{aligned}
&\|x_{n+1} - w\|^2 \\
&= \|P_K((1-\alpha_n)x_n + \alpha_n R_n x_n + e_n) - P_K(w)\|^2 \leq \|(1-\alpha_n)x_n + \alpha_n R_n x_n + e_n - w\|^2 \\
&= \|(1-\alpha_n)(x_n - w + e_n) + \alpha_n(R_n x_n - w + e_n)\|^2 \\
&= (1-\alpha_n)\|x_n - w + e_n\|^2 + \alpha_n \parallel R_n x_n - w + e_n \parallel^2 - (1-\alpha_n)\alpha_n\|R_n x_n - x_n\|^2 \\
&\leq (1-\alpha_n)(\|x_n - w\| + \|e_n\|)^2 + \alpha_n(\|R_n x_n - w\| + \|e_n\|)^2 - (1-\alpha_n)\alpha_n\|R_n x_n - x_n\|^2.
\end{aligned}$$

Since,

$$\|R_n x_n - w\| = \|R_n x_n - R_n w\| \leq \|x_n - w\|, \tag{8}$$

then we get,

$$\|x_{n+1} - w\|^2 \leq (1-\alpha_n)(\|x_n - w\| + \|e_n\|)^2 + \alpha_n(\|x_n - w\| + \|e_n\|)^2 - (1-\alpha_n)\alpha_n\|R_n x_n - x_n\|^2.$$

Also, we get that

$$\|x_{n+1} - w\|^2 \leq (\|x_n - w\| + \|e_n\|)^2 - (1-\alpha_n)\alpha_n\|R_n x_n - x_n\|^2 \tag{9}$$

which implies that

$$\|x_{n+1} - w\| \leq \|x_n - w\| + \|e_n\|$$

Hence, $\lim_{n\to\infty} \|x_n - w\|$ exist and $\{x_n\}$ is bounded sequence. Let $\lim_{n\to\infty} \|x_{n+1} - w\| = L$. Since, $\lim_{n\to\infty} \|e_n\| = 0$, then from (9), we obtain that

$$L^2 \leq L^2 - (1-\alpha_n)\alpha_n\|R_n x_n - x_n\|^2$$

Hence,

$$\lim_{n\to\infty} \alpha_n(1-\alpha_n)\|R_n x_n - x_n\|^2 = 0. \tag{10}$$

Since $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. Therefore, we get that

$$\lim_{n\to\infty} \|R_n x_n - x_n\| = 0. \tag{11}$$

Since $\{R_n\}$ satisfies the resolvent property, there exist a nonexpansive mapping $R : K \to K$ and $n_0, k \in N$ such that:

$$\|x - Rx\| \leq k\|x - R_n x\|, \tag{12}$$

for all $x \in K$ and $n \geq n_0$ and $F(R) = \bigcap_{n\in N} F(R_n)$. Put $x = x_n$ in (10) , we get,

$$\|Rx_n - x_n\| \leq k\|x_n - R_n x_n\|, \forall\, n \geq n_0. \tag{13}$$

From (13), we have that

$$\lim_{n\to\infty} \|x_n - Rx_n\| = 0. \tag{14}$$

Since $\{x_n\}$ is bounded, there exist subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $u \in K$ such that $x_{n_k} \rightharpoonup u$. By (12) and lemma 2.6 we have $u \in S$. Now, we show that $\{x_n\}$ converges to a point $\bar{x} \in S$. Let $\{x_{n_l}\}$ and $\{x_{n_m}\}$ be subsequences of $\{x_n\}$ which converge weakly to $u, v \in K$ respectively. If $u \neq v$, then by the opial property, we have

$$\begin{aligned} &\lim_{l\to\infty} \|x_{n_l} - u\| < \lim_{l\to\infty} \|x_{n_l} - v\| = \lim_{m\to\infty} \|x_{n_m} - v\| \\ &< \lim_{m\to\infty} \|x_{n_m} - u\| = \lim_{l\to\infty} \|x_{n_l} - u\||. \end{aligned} \tag{15}$$

Then must be $u = v$. Therefore, $x_n \rightharpoonup \bar{x}$.

Corollary 3.1. Let H be a Hilbert space, K be a nonempty, closed and convex subset of H. Consider $R : K \to H$ be a firmly nonexpansive mappings with $F(R) \neq \phi$. Let $\{\alpha_n\}$ be a sequence of real numbers in (0,1) which satisfies that: $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\{e_n\}$ be a bounded sequence in K which satisfies that: $\sum_{n=1}^{\infty} \|e_n\| < \infty$. For a sequence $\{x_n\}$ of K which generated defined as follows::

$$\begin{cases} x_1 \in K \text{ is chosen arbitrarily} \\ x_{n+1} := P_K((1-\alpha_n)x_n + \alpha_n R x_n + e_n), \forall\, n \in N. \end{cases}$$

Then $x_n \rightharpoonup \bar{x}$, where $\bar{x} \in F(R)$.

Theorem 3.2. Let H be a Hilbert space, K be a nonempty, closed and convex subset of H. Consider $\{R_n\} : K \to H$ be a sequence of firmly nonexpansive mappings from K into itself which satisfies AKTT-condition. Let $R : K \to K$ be a mapping defined by $Rz = \lim_{n\to\infty} R_n z$ for all $z \in K$. Suppose $S := \bigcap_{n=1}^{\infty} F(R_n) \neq \phi$. Let $\{\alpha_n\}$ be a sequence of real numbers in (0,1) which satisfies that: $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\{e_n\}$ be a bounded sequence in K which satisfies that: $\sum_{n=1}^{\infty} \|e_n\| < \infty$. For a sequence $\{x_n\}$ of K which generated defined as in (2). Then $x_n \rightharpoonup \bar{x}$, where $\bar{x} \in \bigcap_{n=1}^{\infty} F(R_n)$.

Proof. Since $\{R_n\}$ are firmly nonexpansive mappings. Then we have

$$\begin{aligned} &\|Rx - Ry\| = \|\lim_{n\to\infty} R_n x - \lim_{n\to\infty} R_n y\| \\ &= \lim_{n\to\infty} \|R_n x - R_n y\| \leq \lim_{n\to\infty} \|x - y\|. \end{aligned} \tag{16}$$

For all $x, y \in K$. Hence R is a nonexpansive. Since $\{R_n\}$ satisfies AKTT-condition and $Rz = \lim_{n\to\infty} R_n z$ for all $z \in K$, then by lemma 2.4 we get that

$$\lim_{n\to\infty} \sup\{\|Rz - R_n z\| : z \in B\} = 0, \tag{17}$$

for each bounded subset B of K. Then the same argument of Theorem 3.1, we get that

$$\|x_{n+1} - w\|^2 \leq (\|x_n - w\| + \|e\|)^2 - \alpha_n(1-\alpha_n)\|R_n x_n - x_n\|^2. \tag{18}$$

Thus by (17) we have that

$$\lim_{n\to\infty} \|Rx_n - R_n x_n\| = 0. \tag{19}$$

Following the argument of Theorem 3.1. One see that

$$\lim_{n\to\infty} \|x_n - R_n x_n\| = 0. \tag{20}$$

Using (19) and (20), we get that

$$\|x_n - Rx_n\| \leq \|x_n - R_n x_n\| + \|R_n x_n - Rx_n\|.$$

Thus,

$$\lim_{n\to\infty} \|x_n - Rx_n\| = 0$$

Then the same argument as the proof of Theorem 3.1 leads to the proof of Theorem 3.2.

4 Weak convergence of explicit iteration with errors terms for nonexpansive mappings

Let K be a nonempty closed convex subset of a Hilbert space H and let $\{R_n\}$ and Γ be two families of nonexpansive mappings of K into itself such that: $\emptyset \neq F(\Gamma) = \bigcap_{n=1}^{\infty} F(R_n)$, where $F(R_n)$ is the set of all fixed points of $\{R_n\}$ and $F(\Gamma)$ is the set of all common fixed points of Γ. Nakajo et.al [34] gave the following two definitions:

Definition 4.1 [34]. R_n is said to satisfy the NST-condition (I) with Γ if for each bounded sequence $\{z_n\} \subset K$, $\lim_{n\to\infty} \|z_n - R_n z_n\| = 0$ implies that $\lim_{n\to\infty} \|z_n - Rz_n\| = 0$ for all $R \in \Gamma$. In particular, if $\Gamma = \{R\}$, i.e. Γ consists of one mapping R, then $\{R_n\}$ is said to satisfy the NST-condition (I) with R.

Definition 4.2 [34]. $\{R_n\}$ is said to satisfy the NST-Condition (II) if for each bounded sequence $\{z_n\} \subset K$, $\lim_{n\to\infty} \|z_{n+1} - R_n z_n\| = 0$ implies that $\lim_{n\to\infty} \|z_n - R_m z_n\| = 0$ for all $m \in N$.

Theorem 4.1. Let H be a Hilbert space, K be a nonempty, closed and convex subset of H. Consider $\{R_n\} : K \to H$ be a sequence of firmly nonexpansive mappings from K into itself. Let Γ be a family of nonexpansive mappings of K into itself, which satisfies $\emptyset \neq F(\Gamma) = \bigcap_{n=1}^{\infty} F(R_n)$ and NST-condition (I). Let $\{\alpha_n\}$ be a sequence of real numbers in (0,1) which satisfies that: $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\{e_n\}$ be a bounded sequence in K which satisfies that: $\sum_{n=1}^{\infty} \|e_n\| < \infty$. For a sequence $\{x_n\}$ of K which generated defined as in (2).

Then $x_n \rightharpoonup \bar{x}$, where $\bar{x} \in \bigcap_{n=1}^{\infty} F(R_n)$.

Proof. By doing the same steps as in the proof of Theorem 3.1, we get $\{x_n\}$ is bounded and

$$\lim_{n\to\infty} \|x_n - R_n x_n\| = 0.$$

By NST-condition (I),

$$\lim_{n\to\infty} \|x_n - Rx_n\| = 0,$$

for all $R \in \Gamma$. Since $\{x_n\}$ is bounded , there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $u \in K$ such that: $x_{n_k} \to u$. By lemma 2.6, we have that $u \in F(R)$ for all $T \in \Gamma$. Thus we have that: $u \in F(\Gamma) \subseteq \bigcap_{n=1}^{\infty} F(R_n)$. Then the same steps as in the proof of Theorem 3.1 lead to $x_n \rightharpoonup \bar{x}$, where $\bar{x} \in \bigcap_{n=1}^{\infty} F(R_n)$.

Lemma 4.1 [34]. Let K be a nonempty, closed and convex subset of a Hilbert space H. Let S and R be two nonexpansive mappings of K into itself such that: $F(R) \bigcap F(S) \neq \emptyset$. Let $\{\gamma_n\} \subseteq [a,b]$ for some $a,b \in (0,1)$ with $a \leq b$. For each $n \in N$ let $R_n := \gamma_n S + (1-\gamma_n)R$ and $\Gamma := \{\frac{S+R}{2}\}$. Then $\{R_n\}$ and Γ satisfies NST-condition(I) and $\bigcap_{n=1}^{\infty} F(R_n) = F(\Gamma) = F(S) \bigcap F(R)$.

By using Theorem 4.1 and Lemma 4.1, we prove the following Theorem.

Theorem 4.2. Let H be a Hilbert space, K be a nonempty, closed and convex subset of H. Let S and R be two nonexpansive mappings of K into itself $F(R) \bigcap F(S) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence of real numbers in (0,1) which satisfies that: $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\{e_n\}$ be a bounded sequence in K which satisfies that: $\sum_{n=1}^{\infty} \|e_n\| < \infty$. For a sequence $\{x_n\}$ of K which generated defined as follows:

$$\begin{cases} x_1 \in K \text{ is chosen arbitrarily} \\ y_n := \frac{1}{2}x_n + \frac{1}{2}(\gamma_n S x_n + (1-\gamma_n) R x_n) \\ x_{n+1} := P_K((1-\alpha_n)x_n + \alpha_n y_n + e_n), \forall n \in N. \end{cases}$$

Then $x_n \rightharpoonup \bar{x}$, where $\bar{x} \in F(S) \bigcap F(T)$.

Proof . Define, $S_1 = \frac{1}{2}Id + \frac{1}{2}S$ and $R_1 = \frac{1}{2}Id + \frac{1}{2}R$. Then, S_1, R_1 are firmly nonexpansive. Let $R_n = \gamma_n S_1 + (1-\gamma_n)R_1$, then for all $x,y \in K$, by Lemma 2.2 (ii) we have that:

$$\begin{aligned} &\|R_n x - R_n y\|^2 = \|\gamma_n(S_1 x - S_1 y) + (1-\gamma_n)(R_1 x - R_1 y))\|^2 \\ &= \gamma_n \|S_1 x - S_1 y\|^2 + (1-\gamma_n)\|R_1 x - R_1 y\|^2 - \gamma_n(1-\gamma_n) \\ &\ |(S_1 x - S_1 y) - (R_1 x - R_1 y))\|^2 \\ &\leq \gamma_n \langle S_1 x - S_1 y, x-y\rangle + (1-\gamma_n)\langle R_1 x - R_1 y, x-y\rangle \\ &= \langle R_n x - R_n y, x-y\rangle. \end{aligned} \tag{21}$$

Thus R_n is firmly nonexpansive. Therefore we have that $R_n := \frac{1}{2}Id + \frac{1}{2}(\gamma_n S + (1-\gamma_n R)$. Let $\Gamma := \{\frac{S_1+R1}{2}\}$. We have $\{R_n\}$ and Γ satisfies NST-condition (I) and $\bigcap_{n=1}^{\infty} F(R_n) = F(\Gamma) = F(S_1) \bigcap F(R_1) = F(S) \bigcap F(R)$. By doing the same steps as in the Theorem 4.1 we get $x_n \rightharpoonup \bar{x}$, where $\bar{x} \in F(S) \bigcap F(R)$.

Lemma 4.2 [34]. Let K be a nonempty closed convex subset of a Hilbert space H and let $S = \{T(s) : 0 \leq s \leq \infty\}$ be a one-parameter nonexpansive semigroup on K with $F(S) \neq \emptyset$. Let $\{t_n\}$ be a sequence of real numbers with $0 < t_n < \infty$ such that $\lim_{n\to\infty} t_n = \infty$, for all $n \in N$, define a mapping R_n of K into itself by

$$T_n x = \frac{1}{t_n}\int_0^{t_n} T(s)x\, ds$$

for all $x \in K$. Then, $\{T_n\}$ satisfies the NST-condition (I) with $S = \{T(s) : 0 \leq s \leq \infty\}$.

Using Lemma 2.4 and Theorem 4.1 , we prove the following theorem.

Theorem 4.3. Let K be a nonempty closed convex subset of a Hilbert space H. Let $S = \{T(s) : 0 \leq s \leq \infty\}$ be a one-parameter nonexpansive semigroup on K with $F(S) \neq \emptyset$. Let $\{t_n\}$ be a sequence of real numbers with $0 < t_n < \infty$ such that $\lim_{n\to\infty} t_n = \infty$, for all $n \in N$. Let $\{\alpha_n\}$ be a sequence of real numbers in (0,1) which satisfies that: $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\{e_n\}$ be a bounded sequence in K which satisfies that: $\sum_{n=1}^{\infty} \|e_n\| < \infty$. For a sequence $\{x_n\}$ of K which generated defined as follows:

$$\begin{cases} x_1 \in K \text{ is chosen arbitrarily} \\ y_n := \frac{1}{2}x_n + \frac{1}{t_n}\int_0^{t_n} T(s)x_n\, ds \\ x_{n+1} := P_K((1-\alpha_n)x_n + \alpha_n y_n + e_n), \forall\, n \in N \end{cases}$$

Then $x_n \rightharpoonup \bar{x}$, where $\bar{x} \in F(S)$.

Proof. For each $n \in N$, let T_n as follows,

$$T_n x = \frac{1}{t_n}\int_0^{t_n} T(s)x\, ds$$

for all $x \in K$. Since $S \neq \emptyset, F(S) \neq \emptyset$ and $\{t_n\}$ be a sequence of real number such that $\lim_{n\to\infty} t_n = \infty$, $\{T_n\}$ and by Lemma 4.2 , S satisfy NST-condition (I) and $\bigcap_{n=1}^{\infty} F(T_n) = F(S)$, for each $n \in N$. Define U_n as follows:$U_n := \frac{1}{2}(Id + T_n)$, then for all $x,y \in K$ and $n \in N$

$$\begin{aligned} &\|T_n x - T_n y\| = \|\frac{1}{t_n}\int_0^{t_n} T(s)(x-y)\, ds\| \\ &=\leq \frac{1}{t_n}\int_0^{t_n} \|T(s)(x-y)\|\, ds \\ &\leq \frac{1}{t_n}\int_0^{t_n} \|x-y\|\, ds = \|x-y\|. \end{aligned} \tag{22}$$

From equation (20), we have that T_n is nonexpansive . Since $T_n = 2U_n - Id$ and again by lemma 2.1 (ii) we have that U_n is firmly nonexpansive for each $n \in N$. Let $x \in \bigcap_{n=1}^{\infty} F(T_n)$, then $x \in F(T_n) \forall n \in N$, therefore $U_n(x) = \frac{1}{2}(Id + T_n)(x) = x$, thus $x \in \bigcap_{n=1}^{\infty} F(U_n)$. Conversely, let $x \in \bigcap_{n=1}^{\infty} F(U_n)$, then we have $x \in F(U_n) \forall n \in N$, therefore $T_n(x) = 2U_n(x) - Id(x) = x$, thus $x \in \bigcap_{n=1}^{\infty} F(T_n)$. Then, we get that: $\bigcap_{n=1}^{\infty} F(T_n) = \bigcap_{n=1}^{\infty} F(U_n) = F(S)$. Now, we prove that: $\{U_n\}$ and S satisfy NST-condition (I). Let $\{z_n\}$ is bounded sequence such that:

$$\lim_{n\to\infty} \|z_n - U_n z_n\| = 0$$

Thus, we have that:

$$\lim_{n\to\infty} \|z_n - T_n z_n\| = \lim_{n\to\infty} \|z_n - (2U_n - Id)z_n\| = \lim_{n\to\infty} 2\|z_n - U_n z_n\| = 0.$$

Since $\{T_n\}$ and S satisfy NST-condition (I), we have that $\lim_{n\to\infty} \|z_n - T(t)z_n\| = 0$ for every $T(t) \in S$. Thus $\{U_n\}$ and S satisfy NST-condition (I). By Theorem 4.1, we get the proof of the Theorem.

5 Applications to error analysis of some Algorithms

In this section we will give the weak convergence of explicit iterative algorithms with the error analysis for solving variational inclusion problems, equilibrium problems and split feasibility problems in Hilbert spaces. These applications play an important role in a lot of applications specifically in signal and image processing, see, e.g. ([35]-[41]).

5.1 Variational inclusion problem with errors

Let H be a Hilbert space and A be a set-valued mapping with domain *domA*. Chuang and Takahashi [18] stated the variational inclusion problem as follows: Find $x \in H$ such that $0 \in A(x)$. They also gave the the weak convergence theorem for finding a solution of variational inclusion problem using explicit iterative process. Now, we consider the following weak convergence theorem for implicit iterative process for solving variational inclusion problem.

Theorem 5.1.1. Let H be a Hilbert space. Let A be a maximal monotone mapping on H with $A^{-1}0 \neq \emptyset$. Let $\{\beta_n\}$ be a sequence in $(0,\infty)$ and let $\{\alpha_n\}$ be a sequence of real numbers in (0,1) which satisfies that: $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\{e_n\}$ be a bounded sequence in K which satisfies that: $\sum_{n=1}^{\infty} \|e_n\| < \infty$. For a sequence $\{x_n\}$ of K which generated defined as follows:

$$\begin{cases} x_1 \in K \text{ is chosen arbitrarily} \\ x_{n+1} := (1-\alpha_n)x_n + \alpha_n J_{\beta_n} x_n + e_n, \forall\, n \in N. \end{cases}$$

Then $x_n \rightharpoonup \bar{x}$, where $\bar{x} \in A^{-1}0$.

Proof. Let $\beta_n > \beta$ for some $\beta > 0$. From Remarks 2.1(iii) , we have that:

$$\|x - J_{\beta_n}\| \geq \|x - J_{\beta}\|.$$

Therefore, $\{J_{\beta_n}\}$ is firmly nonexpansive for all $n \in N$. Thus $\{J_{\beta_n}\}$ satisfies the resolvent property. Following the argument of Theorem 3.1, we get that: $x_n \rightharpoonup \bar{x} \in \bigcap_{n=1}^{\infty} F(J_{\beta_n})$. Since we have that: $A^{-1}0 = F(J_r)$ for all $r > 0$. Thus we get that: $\bar{x} \in A^{-1}0$.

5.2 Equilibrium problems with errors in Hilbert spaces

Let K be a nonempty, closed and convex subset of H. The equilibrium problem can be stated as follows: Find $x \in K$ such that $f(x,y) \geq 0$ for all $y \in K$ where $f : K \times K \to R$ is bifunction. In this section we use $EP(f)$ to denote the set of such $x \in K$, i.e. $EP(f) = \{x \in K : f(x,y) \geq 0, \forall\, y \in K\}$.

Combettes and Hirstoga [28], gave algorithms for solving Equilibrium problem used the following assumptions.

Condition 5.2.1 [28]. The bifunction $f : K \times K \to R$ is such that:
(i) $f(x,x) = 0, \forall\, x \in K$,
(ii) $f(x,y) + f(y,x) \leq 0, \forall (x,y) \in K^2$,
(iii) For every $x \in K$, $f(x,.) : K \to R$ is lower semicontinuous and convex,
(iv) $\limsup_{\varepsilon\to 0^+} f((1-\varepsilon)+\varepsilon z, y) \leq f(x,y), \forall\, (x,y,z) \in K^3$.

Then we introduce the following two Lemmas, which shows the uniqueness of solution of the equilibrium problems.

Lemma 5.2.1 [28]. Let $f : K \times K \to R$ be a bifunction satisfying Condition 5.2.1 Then for $r > 0$ and $x \in H$, there exists $z \in K$ such that:

$$f(z,x) + \frac{1}{r}\langle y-z, z-x\rangle \geq 0, \ \forall\, y \in K.$$

Define a mapping $T_r : H \to K$ as follows:

$$R_r x = \{z \in K : f(z,x) + \frac{1}{r}\langle y-z, z-x\rangle \geq 0, \forall y \in K\}, \forall\, x \in H,$$

Then the following statements are hold:
(i) R_r is single valued,
(ii) R_r is firmly nonexpansive,
(iii) $F(R_r) = EP(f)$,
(iv) $EP(f)$ is closed and convex.

Lemma 5.2.2 ([42]). Let H be a Hilbert space and let K be a nonempty, closed and convex subset of H. Let $f : K \times K \to R$ satisfy Condition 5.2.1 Let A_f be a multivalued mapping of H into itself defined by :

$$A_f x = \begin{cases} \{z \in H : f(x,y) \geq \langle y-x, z\rangle, \forall\, y \in K\}, & x \in K, \\ \emptyset, & \forall\, x \notin K. \end{cases}$$

Then, $EP(f) = A_f^{-1}0$ and A_f is a maximal operator with $domA_f \subset K$. Further, for any $x \in H$ and $r > 0$, the resolvent R_r of f coincides with J_r the resolvent of A_f, i.e. $R_r x = J_r x$ for all $x \in H$ and $r > 0$.

Thus from Lemma 5.2.2 we get that: the solution of the equilibrium problem can be funded by the the following scheme :

$$\begin{cases} x_1 \in K & \text{is chosen arbitrarily} \\ x_{n+1} := (1-\alpha_n)x_n + \alpha_n R_{\beta_n} x_n + e_n, \forall n \in N \end{cases}$$

where the term e_n was represented the error of computations. Therefore from Theorem 5.1.1 and Lemma 5.2.2 we get the following weak convergence theorem for finding the equilibrium problem by the explicit iterative process with error.

Theorem 5.2.1. Let K be a nonempty, closed and convex subset of a Hilbert space H . Let $f : K \times K \to R$ be a bifunction which satisfied Condition 5.2.1 and $EP(f) \neq \emptyset$. Let $\{\beta_n\}$ be a sequence in $(0,\infty)$ and $\{\alpha_n\}$ be a sequence a of real numbers in (0,1) which satisfies that: $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\{e_n\}$ be a bounded sequence in K which satisfies that: $\sum_{n=1}^{\infty} \|e_n\| < \infty$. For a sequence $\{x_n\}$ of K which generated defined as follows:

$$\begin{cases} x_1 \in K & \text{is chosen arbitrarily} \\ x_{n+1} := (1-\alpha_n)x_n + \alpha_n R_{\beta_n} x_n + e_n, \forall n \in N \end{cases}$$

Then $x_n \rightharpoonup \overline{x}$, where $\overline{x} \in EP(f)$.

5.3 Split feasibility problems with errors

Censor and Elfving [35] presented the split feasibility problem in R^n. Chuang and Takahashi [18] presented generalized split feasibility problem in any Hilbert space as: let K and M be nonempty, closed and convex subsets of Hilbert space H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator, and let A^* be the adjoint of A. Find $x \in H_1$ such that $x \in K$ and $Ax \in M$.

Let $\Omega := \{x \in K : Ax \in M\}$ is the set of solutions of the split feasibilty problem. Suppose $\Omega \neq \emptyset$ and let $\rho > 0$. Byrne [36] considered the solution of split feasibility problem as:

$$\overline{x} \in \Omega \Leftrightarrow P_K(\overline{x} + \rho A^*(P_M - Id)A\overline{x}) = \overline{x},$$

and proposed the following implicit algorithm with errors of computations to solve the split feasibility problem:
Algorithm 5.3.1([36]). Let $x_1 \in H_1$ be arbitrary. Choose $\rho \in (0, \frac{2}{\|A\|^2})$ and $\{\alpha_n\}$ in $(0,1)$. Suppose $Rx = \frac{1}{2}x + \frac{1}{2}P_K(x + \rho A^*(P_M - Id)Ax)$, for all $x \in H_1$. For $n = 1, 2, ...$, let

$$x_{n+1} := (1 - \alpha_n)x_n + \alpha_n Rx_n.$$

Chuang and Takahashi [18], proved the weak convergence for Algorithm 5.3.1. In fact,when we study the convergence of iterations required for a solution of the some problems by the explicit iterative process , we also must study the error of computer computations . We now propose an algorithm for solving the split feasibility problem in the explicit iterative process with error. The proposed algorithm 5.3.1 can be written in implicit form as:
Algorithm 5.3.2. Let K and M be nonempty, closed and convex subsets of Hilbert space H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator, and let A^* be the adjoint of A. suppose $\rho \in (0, \frac{2}{\|A\|^2})$ and $\Omega \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence of real numbers in (0,1) which satisfies that: $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$ and $\{e_n\}$ be a bounded sequence in K which satisfies that: $\sum_{n=1}^{\infty} \|e_n\| < \infty$. For a sequence $\{x_n\}$ of K which generated defined as follows:

$$\begin{cases} x_1 \in K \text{ is chosen arbitrarily} \\ x_{n+1} := (1 - \alpha_n)x_n + \alpha_n Rx_n + e_n, \forall\, n \in N. \end{cases}$$

Where $Rx = \frac{1}{2}x + \frac{1}{2}P_K(x + \rho A^*(P_M - Id)Ax)$, for all $x \in H_1$.
Theorem 5.3. Suppose the sequence $\{x_n\}$ generated by the implicit method as in Algorithm 5.3.2. Then $x_n \rightharpoonup \overline{x}$, where $\overline{x} \in \Omega$.
Proof. Since P_K is firmly nonexpansive. Then P_K is nonexpansive. Therefore we can write R as $R = \frac{1}{2}(Id + W)$, where $Wx := P_K(x + \rho A^*(P_M - Id)Ax)$. Then we get that $W = 2r - Id$. Thus, from Lemma 2.1 (iii) , R is firmly nonexpansive and $F(R) = F(W)$. Following Corollary 3.1, we have that $x_n \rightharpoonup \overline{x}$, where $\overline{x} \in \Omega$.

6 Conclusion

We introduced a new explicit metric projection iteration scheme of finding a common fixed point of infinite families of nonlinear mappings in a Hilbert space and we proved weak convergence theorems for finding common fixed points of these families of firmly nonexpansive mappings. The error of computations sequence of this iterative process was considered in our work. Also. we given weak convergence theorems for finding a common fixed point of families of nonexpansive mappings in a Hilbert space. Finally, a common solution of equilibrium problems and split feasibility problems are established in the framework of real Hilbert spaces and their weak convergence theorems are obtained under certain assumptions.
Competing interests
The authors declare that they have no competing interests.
Authors contributions
The authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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