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Some Fixed Point and Common Fixed Point Theorems in Generalized D*-metric Spaces

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Abstract: In this paper, we prove some fixed point and common fixed point theorems in generalized D^*-metric spaces by using the idea of A. Branciari [8] of integral type contraction.

Keywords: Generalized D^*-metric space; fixed point; common fixed point; integral type contractive mapping.

1 Introduction

The notion of cone metric space was introduced by Haung and Zhang [18] in 2007. They replace an ordered Banach space for the real numbers and proved some fixed point theorems of contractive mappings in cone metric space. In 2003, Mustafa and Sims [23] introduced a new concept of generalized metric spaces, which are known as G-metric spaces. In 2000, Dhage [15] defined the concept of D-metric spaces as a generalization of metric spaces and proved some important results in such spaces. Shabnam et al. [29] modify D-metric space and thus gave the idea of D^*-metric spaces. In 2010, Aage and Salunke introduced generalized D^*-metric space by replacing R by a real Banach space in D^*-metric spaces and proved some fixed point theorems in generalized D^*-metric space. Moreover, In 2002, Branciari [8] gave the idea of integral type contractive mappings in complete metric spaces and study the existence of fixed points for mappings which is defined on complete metric space satisfies integral type contraction. Recently R. Shah et al. [25], proved some fixed point theorems in cone b-metric space by using the idea of A. Branciari [8] of integral type contraction. In this paper, by using the same idea given by A. Branciari [8] of integral type contraction we prove some fixed point and common fixed point theorems of integral type contractive mappings in setting of generalized D^*-metric space. We recommend some other references to the reader see, [3,2,4,5,6,7,8,9,10,12,13,14,17,20,21,24,26,27,28,30,31].

2 Preliminaries

We will need the following definitions and results in this paper.

Definition 2.1[18] *Let $\mathbb{Q}$ be a real Banach space and P be a subset of $\mathbb{Q}$. Then P is called cone if and only if:*
(i) P is closed, nonempty and $P \neq \{0\}$;
(ii) $cp + dq \in P$ for all $p,q \in P$ where c,d are non-negative real numbers;
(iii) $P \cap -P = \{0\}$.

Definition 2.2[18] *Suppose P be a cone in real Banach space $\mathbb{Q}$, we define a partial ordering $\leq$ with respect to P by $p \leq q$ iff $q - p \in P$. We shall write $p < q$ to indicate that $p \leq q$ but $p \neq q$, while $p \ll q$ will stand for $q - p \in intP$.*

Definition 2.3[18] *The cone P is called normal if there is number $K > 0$ such that for all $p,q \in \mathbb{Q}$, $0 \leq p \leq q$ implies $\|p\| \leq K\|q\|$.*
The least positive number K satisfying the above inequality is called the normal constant of cone.

In the following we always suppose that $\mathbb{Q}$ is a Banach space, P is a cone in $\mathbb{Q}$ with $intP \neq \emptyset$ and $\leq$ is a partial ordering with respect to P.

Definition 2.4[1] *Let Y be a non-empty set. A generalized D^*-metric space on Y is a function, $D^* : Y \times Y \times Y \to \mathbb{Q}$, that satisfies the following conditions for all $u,v,w,a \in Y$:*
(1) $D^*(u,v,w) \geq 0$,
(2) $D^*(u,v,w) = 0$ if and only if $u = v = w$,

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(3) $D^*(u,v,w) = D^*(p\{u,v,w\})$, *where* p *is a permutation function,*
(4) $D^*(u,v,w) \leq D^*(u,v,a) + D^*(a,w,w)$,
Then the function D^* *is called a generalized* D^**-metric and the pair* (Y,D^*) *is called a generalized* D^**-metric space.*

Example 2.5[1] *Let* $\mathbb{Q} = R^2$, $P = \{(u,v) \in \mathbb{Q} : u,v \geq 0\}$, $Y = R$ *and* $D^* : Y \times Y \times Y \to \mathbb{Q}$ *defined by* $D^*(u,v,w) = (|u-v| + |v-w| + |u-w|, \alpha(|u-v| + |v-w| + |u-w|))$, *where* $\alpha \geq 0$ *is constant. Then* (Y,D^*) *is generalized* D^**-metric space.*

In 2002, Branciari in [8] gave the idea of integral type contraction and introduced a general contractive condition of integral type as follows.

Theorem 2.6[8] *Let* (Y,d) *be a complete metric space,* $\alpha \in (0,1)$ *and* $f : Y \to Y$ *is a mapping such that for all* $x,y \in Y$,

$$\int_0^{d(f(x),f(y))} \phi(t)dt \leq \alpha \int_0^{d(x,y)} \phi(t)dt$$

where $\phi : [0,+\infty) \to [0,+\infty)$ *is nonnegative and Lebesgue-integrable mapping which is summable (i.e., with finite integral) on each compact subset of* $[0,+\infty)$ *such that for each* $\varepsilon > 0$, $\int_0^{\varepsilon} \phi(t)dt > 0$, *then* f *has a unique fixed point* $a \in Y$, *such that for each* $x \in Y$, $\lim_{n\to\infty} f^n(x) = a$.

In [22], Khojasteh et al. presented the new concept of integral with respect to a cone and introduced the Branciaris result in cone metric spaces.

Definition 2.7*Suppose that* P *is a normal cone in* $\mathbb{G}$. *Let* $a,b \in P$ *and* $a < b$. *We define*

$$[a,b] := \{x \in \mathbb{G} : x = tb + (1-t)a, \text{ for some } t \in [0,1]\},$$

$$[a,b) := \{x \in \mathbb{G} : x = tb + (1-t)a, \text{ for some } t \in [0,1)\}.$$

Definition 2.8*The set* $\{a = x_0, x_1, \ldots, x_n = b\}$ *is called a partition for* $[a,b]$ *if and only if the sets* $\{[x_{i-1},x_i)]\}_{i=1}^n$ *are pairwise disjoint and* $[a,b] = \{\bigcup_{i=1}^n [x_{i-1},x_{)})\} \bigcup \{b\}$.

Definition 2.9*For each partition* P *of* $[a,b]$ *and each increasing function* $\phi : [a,b] \to P$, *we define cone lower summation and cone upper summation as*

$$L_n^{Con}(\phi,P) := \sum_{i=0}^{n-1} \phi(x_i)\|x_i - x_{i+1}\|$$

$$U_n^{Con}(\phi,P) := \sum_{i=0}^{n-1} \phi(x_{i+1})\|x_i - x_{i+1}\|$$

respectively.

Definition 2.10*Suppose that* P *is a normal cone in* $\mathbb{G}$. $\phi : [a,b] \to P$ *is called an integrable function on* $[a,b]$ *with respect to cone* P *or to simplicity, Cone integrable function, if and only if for all partition* P *of* $[a,b]$

$$\lim_{n\to\infty} L_n^{Con}(\phi,P) = S^{Con} = \lim_{n\to\infty} U_n^{Con}(\phi,P)$$

where S^{Con} *must be unique.*

We show the common value S^{Con} by

$$\int_a^b \phi(x)d_P(x) \text{ or simply by } \int_a^b \phi d_P.$$

Let $\mathscr{L}^1([a,b],P)$ denotes the set of all cone integrable functions.

Lemma 1.[22] *Let* $f,g \in \mathscr{L}^1([a,b],P)$. *The following two statements hold.*

- *(1) If* $[a,b] \subset [a,c]$, *then* $\int_a^b f d_P \leq \int_a^c f d_P$, *for* $f \in \mathscr{L}^1([a,b],P)$.
- *(2)* $\int_a^b (\alpha f + \beta g) d_P = \alpha \int_a^b f d_P + \beta \int_a^b g d_P$, *for* $\alpha,\beta \in \mathbb{R}$.

Definition 2.11[22] *The function* $\varphi : P \to P$ *is called subadditive cone integrable function if and only if for all* $c,d \in P$

$$\int_0^{c+d} \varphi d_P \leq \int_0^c \varphi d_P + \int_0^d \varphi d_P$$

Example 2.12[22] *Let* $\mathbb{Q} = Y = R$, $d(u,v) = |u-v|$, $P = [0,+\infty)$, *and* $\varphi(t) = \frac{1}{t+1}$ *for all* $t > 0$ *then for all* $c,d \in P$,

$$\int_0^{c+d} \frac{dt}{t+1} = ln(c+d+1),$$

$$\int_0^c \frac{dt}{t+1} = ln(c+1),$$

$$\int_0^d \frac{dt}{t+1} = ln(d+1)$$

since $cd \geq 0$, *then* $c+d+1 \leq c+d+1+cd = (c+1)(d+1)$ *Therefore,*

$$ln(c+d+1) \leq ln((c+1)(d+1)) = ln(c+1) + ln(d+1).$$

Which showing that φ *is subadditive cone integrable function.*

Lemma 2.[1] *Let* (Y,D^*) *be a generalized* D^**-metric space,* P *be a normal cone. Let* $\{u_n\}$ *be a sequence in* Y. *Then* $\{u_n\}$ *converges to* u *if and only if* $D^*(u_m,u_n,u) \to 0$ *as* $m,n \to \infty$.

Lemma 3.[1] *Let* (Y,D^*) *be a generalized* D^**-metric space then the following are equivalent.*
(i) $\{u_n\}$ *is* D^**-convergent to* u.
(ii) $D^*(u_n,u_n,u) \to 0$, *as* $u \to \infty$.
(iii) $D^*(u_n,u,u) \to 0$, *as* $u \to \infty$.

Lemma 4.[1] *Let* (Y,D^*) *be a generalized* D^**-metric space,* P *be a normal cone. Let* $\{u_n\}$ *be a sequence in* Y. *If* $\{u_n\}$ *converges to* u *and* $\{u_n\}$ *converges to* v, *then* $u = v$. *That is limit is unique.*

Lemma 5.[1] *Let* (Y,D^*) *be a generalized* D^**-metric space,* $\{u_n\}$ *be sequence in* Y. *If* $\{u_n\}$ *converges to* u, *then* $\{u_n\}$ *is a Cauchy sequence.*

Lemma 6.[1] *Let* (Y,D^*) *be a generalized* D^**-metric space,* P *be a normal cone. Let* $\{u_n\}$ *be a sequence in* Y*. Then* $\{u_n\}$ *is a Cauchy sequence if and only if* $D^*(u_m,u_n,u_l) \to 0$ *as* $m,n,l \to \infty$*.*

Proposition 1.[3] *Let* f *and* g *be weakly compatible self maps of a set* Y*. If* f *and* g *have a unique point of coincidence* $w = fu = gu$*, then* w *is the unique common fixed point of* f *and* g*.*

Theorem 2.13[22] *Let* (Y,d) *be a complete regular cone metric space and* H *be a mapping on* Y*. Suppose that there exist a function* θ *from* P *into itself which satisfies:*
(i) $\theta(0) = 0$ *and* $\theta(t) \gg 0$ *for all* $t \gg 0$*.*
(ii) *The function* θ *is nondecreasing and continuous. Moreover, its inverse is also continuous.*
(iii) *For all* $0 \neq \varepsilon \in P$*, there exist* $\delta \gg 0$ *such that for all* $a,b \in Y$

$$\theta(d(a,b)) < \varepsilon + \delta \text{ implies } \theta(d(Ha,Hb)) < \varepsilon. \tag{2.1}$$

(iv) *For all* $a,b \in Y$

$$\theta(a+b) \leq \theta(a) + \theta(b). \tag{2.2}$$

Then the function H *has a unique fixed point.*

Remark 2.14[22] *If* $\varphi : P \to P$ *is a non-vanishing map and a sub-additive cone integrable on each* $[a,b] \subset P$ *such that for each* $\varepsilon \gg 0$, $\int_0^\varepsilon \varphi d_P \gg 0$ *and* $\theta(x) = \int_0^x \varphi d_P$ *must have the continuous inverse, then* θ *is satisfies in all conditions in Theorem 2.13.*

3 Main Results

In this section, we prove some fixed point and common fixed point theorems in generalize D^*-metric space by using integral type contractive mappings. Our first main result is stated as:

Theorem 3.1*Let* (Y,D^*) *be a generalized* D^**-metric space and* P *be a normal cone. Let* $M,H : Y \to Y$ *be two mappings which satisfy the following conditions,*
$(i)H(Y) \subset M(Y)$
$(ii)H(Y)$ *or* $M(Y)$ *is complete and*
(iii)

$$\begin{aligned}\int_0^{D^*(Hu,Hv,Hw)} \varphi d_p \;&\leq\; a\int_0^{D^*(Mu,Mv,Mw)} \varphi d_p + b\int_0^{D^*(Mu,Hv,Hv)} \varphi d_p \\ &+\; c\int_0^{D^*(Mv,Hv,Hv)} \varphi d_p + d\int_0^{D^*(Mw,Hw,Hw)} \varphi d_p\end{aligned}$$

for all $u,v,w \in Y$*, where* $a,b,c,d \geq 0$, $a+b+c+d < 1$*. Suppose* $\varphi : P \to P$ *is a Lebesgue integrable mapping which is summable, non-negative and such that for each* $\varepsilon \gg 0$, $\int_0^\varepsilon \varphi d_p \gg 0$ *having continuous inverse. Then* M *and* H *have a unique point of coincidence in* Y*. Moreover if* M *and* H *are weakly compatible, then* M *and* H *have a unique common fixed point.*

Proof. Let $u_0 \in Y$. Choose $u_1 \in Y$ such that $Hu_0 = Mu_1$ with $Hu_{n-1} = Mu_n$.
We have

$$\begin{aligned}\int_0^{D^*(Mu_n,Mu_{n+1},Mu_{n+1})} \varphi d_p \;&=\; \int_0^{D^*(Hu_{n-1},Hu_n,Hu_n)} \varphi d_p \\ &\leq\; a\int_0^{D^*(Mu_{n-1},Mu_n,Mu_n)} \varphi d_p \\ &+\; b\int_0^{D^*(Mu_{n-1},Hu_{n-1},Hu_{n-1})} \varphi d_p \\ &+\; c\int_0^{D^*(Mu_n,Hu_n,Hu_n)} \varphi d_p \\ &+\; d\int_0^{D^*(Mu_n,Hu_n,Hu_n)} \varphi d_p \\ &=\; a\int_0^{D^*(Mu_{n-1},Mu_n,Mu_n)} \varphi d_p \\ &+\; b\int_0^{D^*(Mu_{n-1},Mu_n,Mu_n)} \varphi d_p \\ &+\; c\int_0^{D^*(Mu_n,Mu_{n+1},Mu_{n+1})} \varphi d_p \\ &+\; d\int_0^{D^*(Mu_n,Mu_{n+1},Mu_{n+1})} \varphi d_p \\ &=\; (a+b)\int_0^{D^*(Mu_{n-1},Mu_n,Mu_n)} \varphi d_p \\ &+\; (c+d)\int_0^{D^*(Mu_n,Mu_{n+1},Mu_{n+1})} \varphi d_p\end{aligned}$$

This implies

$$\int_0^{D^*(Mu_n,Mu_{n+1},Mu_{n+1})} \varphi d_p \leq q\int_0^{D^*(Mu_{n-1},Mu_n,Mu_n)} \varphi d_p$$

where $q = \frac{(a+b)}{1-(c+d)}$, repeating this process, we get

$$\int_0^{D^*(Mu_n,Mu_{n+1},Mu_{n+1})} \varphi d_p \leq q^n\int_0^{D^*(Mu_0,Mu_1,Mu_1)} \varphi d_p$$

Then for all $m,n \in N$, $n < m$. we have,

$$\begin{aligned}\int_0^{D^*(Mu_n,Mu_m,Mu_m)} \varphi d_p \quad&\leq\quad \int_0^{D^*(Mu_n,Mu_n,Mu_{n+1})} \varphi d_p \\ &+\quad \int_0^{D^*(Mu_{n+1},Mu_{n+1},Mu_{n+2})} \varphi d_p \\ &+\quad \int_0^{D^*(Mu_{n+2},Mu_{n+2},Mu_{n+3})} \varphi d_p \\ &+\cdots+ \int_0^{D^*(Mu_{m-1},Mu_{m-1},Mu_m)} \varphi d_p \\ &\leq\quad (q^n + q^{n+1} + \cdots + q^{m-1})\int_0^{D^*(Mu_0,Mu_1,Mu_1)} \varphi d_p \\ &\leq\quad \frac{q^n}{1-q}\int_0^{D^*(Mu_0,Mu_1,Mu_1)} \varphi d_p \to 0.\end{aligned}$$

Thus

$$\lim_{n,m\to\infty} D^*(Mu_n,Mu_m,Mu_m) = 0.$$

So, $\{Mu_n\}$ is D^*- Cauchy sequence, since $M(Y)$ is D^* complete, there exist $j \in M(Y)$ such that $\{Mu_n\} \to j$ as $n \to \infty$, there exist $l \in Y$ such that $M_l = j$. If $H(Y)$ is complete, then there exist $j \in H(Y)$ such that $Mu_n \to j$, as $H(Y) \subset M(Y)$, we have $j \in M(Y)$. Then there exist

$l \in Y$ such that $M_l = j$. We claim that $H_l = j$,

$$\begin{aligned}\int_0^{D^*(H_l,j,j)} \varphi d_p &= \int_0^{D^*(H_l,H_l,j)} \varphi d_p \\ &\leq \int_0^{D^*(H_l,H_l,Hu_n)} \varphi d_p + \int_0^{D^*(Hu_n,j,j)} \varphi d_p \\ &\leq a\int_0^{D^*(M_l,M_l,Mu_n)} \varphi d_p + b\int_0^{D^*(M_l,H_l,H_l)} \varphi d_p \\ &+ c\int_0^{D^*(M_l,H_l,H_l)} \varphi d_p + d\int_0^{D^*(Mu_n,Hu_n,Hu_n)} \varphi d_p \\ &+ \int_0^{D^*(Mu_{n+1},j,j)} \varphi d_p \\ &\leq a\int_0^{D^*(j,j,Mu_n)} \varphi d_p + b\int_0^{D^*(j,H_l,H_l)} \varphi d_p \\ &+ c\int_0^{D^*(j,H_l,H_l)} \varphi d_p + d\int_0^{D^*(Mu_n,Mu_{n+1},Mu_{n+1})} \varphi d_p \\ &+ \int_0^{D^*(Mu_{n+1},j,j)} \varphi d_p\end{aligned}$$

This implies that

$$\begin{aligned}\int_0^{D^*(H_l,H_l,j)} \varphi d_p \leq \frac{a}{1-(b+c)} &\int_0^{D^*(j,j,Mu_n)} \varphi d_p \\ + \frac{d}{1-(b+c)} &\int_0^{D^*(Mu_n,Mu_{n+1},Mu_{n+1})} \varphi d_p \\ + \frac{1}{1-(b+c)} &\int_0^{D^*(Mu_{n+1},j,j)} \varphi d_p\end{aligned}$$

This implies,
$D^*(H_l,H_l,j) = 0$ as $n \to \infty$. So, $H_l = j$. i.e. $H_l = M_l$ and l is a point of coincidence point of M and H. Next we show the uniqueness.
For this, suppose that there exists a point q in Y such that $M_q = H_q$. Now

$$\begin{aligned}\int_0^{D^*(H_l,H_l,H_q)} \varphi d_p &\leq a\int_0^{D^*(M_l,M_l,M_q)} \varphi d_p + b\int_0^{D^*(M_l,H_l,H_l)} \varphi d_p \\ &+ c\int_0^{D^*(M_l,H_l,H_l)} \varphi d_p + d\int_0^{D^*(M_q,H_q,H_q)} \varphi d_p \\ &= a\int_0^{D^*(H_l,H_l,H_q)} \varphi d_p\end{aligned}$$

we have

$$\int_0^{D^*(H_l,H_l,H_q)} \varphi d_p \leq a\int_0^{D^*(H_l,H_l,H_q)} \varphi d_p$$

Since $a+b+c+d < 1$. Hence $D^*(H_l,H_l,H_q) = 0$ i.e. $H_l = H_q$. Thus l is a unique point of coincidence of M and H. So, M and H have a unique common fixed point.

Corollary 3.2*Let (Y,D^*) be a generalized D^*-metric space and P be a normal cone. Let $H : Y \to Y$ be a mapping which satisfy condition,*

$$\begin{aligned}\int_0^{D^*(Hu,Hv,Hw)} \varphi d_p &\leq a\int_0^{D^*(u,v,w)} \varphi d_p + b\int_0^{D^*(u,Hu,Hu)} \varphi d_p \\ &+ c\int_0^{D^*(v,Hv,Hv)} \varphi d_p + d\int_0^{D^*(w,Hw,Hw)} \varphi d_p\end{aligned}$$

for all $u,v,w \in Y$, where $a,b,c,d \geq 0$, $a+b+c+d < 1$. Suppose $\varphi : P \to P$ is a Lebesgue integrable mapping which is summable, non-negative and such that for each $\varepsilon \gg 0$, $\int_0^{\varepsilon} \varphi d_p \gg 0$ having continuous inverse. Then H have a unique fixed point in Y.

Proof. The proof uses Result 3.1 by replacing M by identity mapping.

Theorem 3.3*Let (Y,D^*) be a generalized D^*-metric space and P be a normal cone. Let $M,H : Y \to Y$ be two mappings which satisfy the following conditions,*
$(i) H(Y) \subset M(Y)$
$(ii) H(Y)$ or $M(Y)$ is complete and
(iii)

$$\begin{aligned}\int_0^{D^*(Hu,Hv,Hw)} \varphi d_p &\leq a\left(\int_0^{[D^*(Mu,Hv,Hv)+D^*(Mv,Hu,Hu)]} \varphi d_p\right) \\ &+ b\left(\int_0^{[D^*(Mv,Hw,Hw)+D^*(Mw,Hv,Hv)]} \varphi d_p\right) \\ &+ c\left(\int_0^{[D^*(Mu,Hw,Hw)+D^*(Mw,Hu,Hu)]} \varphi d_p\right)\end{aligned}$$

for all $u,v,w \in Y$, where $a,b,c \geq 0$, $2a+2b+2c < 1$. Suppose $\varphi : P \to P$ is a Lebesgue integrable mapping which is summable, non-negative and such that for each $\varepsilon \gg 0$, $\int_0^{\varepsilon} \varphi d_p \gg 0$ having continuous inverse. Then M and H have a unique point of coincidence in Y. Moreover if M and H are weakly compatible, then M and H have a unique common fixed point.

Proof. Let $u_0 \in Y$. Choose $u_1 \in Y$ such that $Hu_0 = Mu_1$ with $Hu_n = Mu_{n+1}$.
We have

$$\begin{aligned}\int_0^{D^*(Mu_n,Mu_{n+1},Mu_{n+1})} \varphi d_p &= \int_0^{D^*(Hu_{n-1},Hu_n,Hu_n)} \varphi d_p \\ &\leq a\left(\int_0^{[D^*(Mu_{n-1},Hu_n,Hu_n)+D^*(Mu_n,Hu_{n-1},Hu_{n-1})]} \varphi d_p\right) \\ &+ b\left(\int_0^{[D^*(Mu_n,Hu_n,Hu_n)+D^*(Mu_n,Hu_n,Hu_n)]} \varphi d_p\right) \\ &+ c\left(\int_0^{[D^*(Mu_{n-1},Hu_n,Hu_n)+D^*(Mu_n,Hu_{n-1},Hu_{n-1})]} \varphi d_p\right) \\ &= (a+c)\int_0^{[D^*(Mu_{n-1},Mu_n,Mu_n)+D^*(Mu_n,Mu_{n+1},Mu_{n+1})]} \varphi d_p \\ &+ 2b\int_0^{D^*(Mu_n,Mu_{n+1},Mu_{n+1})} \varphi d_p\end{aligned}$$

This implies that

$$\int_0^{D^*(Mu_n,Mu_{n+1},Mu_{n+1})} \varphi d_p \leq q\int_0^{D^*(Mu_{n-1},Mu_n,Mu_n)} \varphi d_p$$

where $\frac{(a+b)}{1-(a+2b+c)}$, $q \in [0,1)$, repeating this process, we get,

$$\int_0^{D^*(Mu_n,Mu_{n+1},Mu_{n+1})} \varphi d_p \leq q^n\int_0^{D^*(Mu_0,Mu_1,Mu_1)} \varphi d_p$$

Then for all $m,n \in N$, $n < m$.
we have,

$$\begin{aligned}\int_0^{D^*(Mu_m,Mu_n,Mu_n)} \varphi d_p &\le \int_0^{D^*(Mu_m,Mu_{m+1},Mu_{m+1})} \varphi d_p \\ &+ \int_0^{D^*(Mu_{m+1},Mu_n,Mu_n)} \varphi d_p \\ &\le \int_0^{D^*(Mu_m,Mu_{m+1},Mu_{m+1})} \varphi d_p \\ &+ \int_0^{D^*(Mu_{m+1},Mu_{m+2},Mu_{m+2})} \varphi d_p \\ &+\cdots+ \int_0^{D^*(Mu_{n-1},Mu_n,Mu_n)} \varphi d_p \\ &\le (q^m + q^{m+1} + \cdots + q^{n-1}) \int_0^{D^*(Mu_0,Mu_1,Mu_1)} \varphi d_p \\ &\le \frac{q^m}{1-q} \int_0^{D^*(Mu_0,Mu_1,Mu_1)} \varphi d_p \to 0.\end{aligned}$$

Thus

$$\lim_{n,m\to\infty} D^*(Mu_m, Mu_n, Mu_n) = 0.$$

So, $\{Mu_n\}$ is D^*- Cauchy sequence, since $M(Y)$ is D^* complete, there exist $j \in M(Y)$ such that $\{Mu_n\} \to j$ as $n \to \infty$, there exist $l \in Y$ such that $M_l = j$. If $H(Y)$ is complete, then there exist $j \in H(Y)$ such that $Mu_n \to j$, as $H(Y) \subset M(Y)$, we have $j \in M(Y)$. Then there exist $l \in Y$ such that $M_l = j$. We claim that $H_l = j$,

$$\begin{aligned}\int_0^{D^*(H_l,H_l,j)} \varphi d_p &\le \int_0^{D^*(H_l,H_l,Hu_n)} \varphi d_p + \int_0^{D^*(Hu_n,j,j)} \varphi d_p \\ &\le a\left(\int_0^{[D^*(M_l,H_l,H_l)+D^*(M_l,H_l,H_l)]} \varphi d_p\right) \\ &+ b\left(\int_0^{[D^*(M_l,Hu_n,Hu_n)+D^*(Mu_n,H_l,H_l)]} \varphi d_p\right) \\ &+ c\left(\int_0^{[D^*(M_l,Hu_n,Hu_n)+D^*(Mu_n,H_l,H_l)]} \varphi d_p\right) \\ &+ \int_0^{D^*(Hu_n,j,j)} \varphi d_p\end{aligned}$$

This implies that

$$\begin{aligned}\int_0^{D^*(H_l,H_l,j)} \varphi d_p \le& \frac{b+c}{1-(2a+b+c)} \int_0^{[D^*(j,Mu_{n+1},Mu_{n+1})+D^*(j,Mu_n,Mu_n)]} \varphi d_p \\ &+ \frac{1}{1-(2a+b+c)} \int_0^{D^*(Mu_{n+1},j,j)} \varphi d_p\end{aligned}$$

This implies,
$D^*(H_l,H_l,j) = 0$ as $n \to \infty$. So, $H_l = j$. i.e. $H_l = M_l$ and l is a point of coincidence point of M and H. Next we show the uniqueness.
For this, suppose that there exists a point q in Y such that $M_q = H_q$. Now

$$\begin{aligned}\int_0^{D^*(H_l,H_l,H_q)} \varphi d_p &\le a\left(\int_0^{[D^*(M_l,H_l,H_l)+D^*(M_l,H_l,H_l)]} \varphi d_p\right) \\ &+ b\left(\int_0^{[D^*(M_l,H_q,H_q)+D^*(M_q,H_l,H_l)]} \varphi d_p\right) \\ &+ c\left(\int_0^{[D^*(M_l,H_q,H_q)+D^*(M_q,H_l,H_l)]} \varphi d_p\right) \\ &= b\left(\int_0^{[D^*(H_l,H_l,H_q)+D^*(H_l,H_l,H_q)]} \varphi d_p\right) \\ &+ c\left(\int_0^{[D^*(H_l,H_l,H_q)+D^*(H_l,H_l,H_q)]} \varphi d_p\right) \\ &= (2b+2c) \int_0^{D^*(H_l,H_l,H_q)} \varphi d_p\end{aligned}$$

we have

$$\int_0^{D^*(H_l,H_l,H_q)} \varphi d_p \le (2b+2c) \int_0^{D^*(H_l,H_l,H_q)} \varphi d_p.$$

since $2a+2b+2c < 1$. Hence $D^*(H_l,H_l,H_q) = 0$. Thus $H_l = H_q$. Also $M_l = M_q$, since $H_l = M_l$. Hence l is a unique point of coincidence of M and H and l is a unique common fixed point of M and H in Y.

Corollary 3.4*Let (Y,D^*) be a generalized D^*-metric space and P be a normal cone. Let $H : Y \to Y$ be a mappings which satisfy the condition,*

$$\begin{aligned}\int_0^{D^*(Hu,Hv,Hw)} \varphi d_p &\le a\left(\int_0^{[D^*(u,Hv,Hv)+D^*(v,Hu,Hu)]} \varphi d_p\right) \\ &+ b\left(\int_0^{[D^*(v,Hw,Hw)+D^*(w,Hv,Hv)]} \varphi d_p\right) \\ &+ c\left(\int_0^{[D^*(u,Hw,Hw)+D^*(w,Hu,Hu)]} \varphi d_p\right)\end{aligned}$$

for all $u,v,w \in Y$, where $a,b,c \ge 0$, $2a+2b+2c < 1$. Suppose $\varphi : P \to P$ is a Lebesgue integrable mapping which is summable, non-negative and such that for each $\varepsilon \gg 0$, $\int_0^{\varepsilon} \varphi d_p \gg 0$ having continuous inverse. Then H has a unique fixed point in Y.

Proof. The proof follows from Theorem 3.3 by replacing M by identity mapping.

Theorem 3.5*Let (Y,D^*) be a generalized D^*-metric space and P be a normal cone. Let $M,H : Y \to Y$ be two mappings which satisfy the following conditions,*
$(i) H(Y) \subset M(Y)$
$(ii) H(Y)$ or $M(Y)$ is complete and
(iii)

$$\begin{aligned}\int_0^{D^*(Hu,Hv,Hv)} \varphi d_p &\le a\left(\int_0^{[D^*(Mv,Hv,Hv)+D^*(Mu,Hv,Hv)]} \varphi d_p\right) \\ &+ b\int_0^{D^*(Mv,Hu,Hu)} \varphi d_p\end{aligned}$$

for all $u,v,w \in Y$*, where* $a,b \geq 0$, $3a+b<1$*. Suppose* $\varphi : P \to P$ *is a Lebesgue integrable mapping which is summable, non-negative and such that for each* $\varepsilon \gg 0$, $\int_0^{\varepsilon} \varphi d_p \gg 0$ *having continuous inverse. Then* M *and* H *have a unique point of coincidence in Y. Moreover if* M *and* H *are weakly compatible, then* M *and* H *have a unique common fixed point.*

Proof. Let $u_0 \in Y$. Choose $u_1 \in Y$ such that $Hu_0 = Mu_1$ with $Hu_n = Mu_{n+1}$.
We have

$$\begin{aligned}
\int_0^{D^*(Mu_n,Mu_{n+1},Mu_{n+1})} \varphi d_p &= \int_0^{D^*(Hu_{n-1},Hu_n,Hu_n)} \varphi d_p \\
&\leq a\Big(\int_0^{[D^*(Mu_n,Hu_n,Hu_n)+D^*(Mu_{n-1},Hu_n,Hu_n)]} \varphi d_p\Big) \\
&+ b\int_0^{D^*(Mu_n,Hu_{n-1},Hu_{n-1})} \varphi d_p \\
&\leq a\Big(\int_0^{[D^*(Mu_n,Mu_{n+1},Mu_{n+1})+D^*(Mu_{n-1},Mu_{n+1},Mu_{n+1})]} \varphi d_p\Big) \\
&+ b\int_0^{D^*(Mu_n,Mu_n,Mu_n)} \varphi d_p \\
&\leq a\int_0^{D^*(Mu_n,Mu_{n+1},Mu_{n+1})} \varphi d_p \\
&+ a\int_0^{D^*(Mu_{n-1},Mu_n,Mu_n)} \varphi d_p \\
&+ a\int_0^{D^*(Mu_n,Mu_{n+1},Mu_{n+1})} \varphi d_p
\end{aligned}$$

This implies that

$$\int_0^{D^*(Mu_n,Mu_{n+1},Mu_{n+1})} \varphi d_p \leq r\int_0^{D^*(Mu_{n-1},Mu_n,Mu_n)} \varphi d_p$$

where $r = \frac{a}{1-2a}$ and $r \in [0,1)$, repeating this process we get,

$$\int_0^{D^*(Mu_n,Mu_{n+1},Mu_{n+1})} \varphi d_p \leq r^n\int_0^{D^*(Mu_0,Mu_1,Mu_1)} \varphi d_p$$

Then for all $m,n \in N$, $n<m$ we have,

$$\begin{aligned}
\int_0^{D^*(Mu_m,Mu_n,Mu_n)} \varphi d_p &\leq \int_0^{D^*(Mu_m,Mu_{m+1},Mu_{m+1})} \varphi d_p \\
&+ \int_0^{D^*(Mu_{m+1},Mu_n,Mu_n)} \varphi d_p \\
&\leq \int_0^{D^*(Mu_m,Mu_{m+1},Mu_{m+1})} \varphi d_p \\
&+ \int_0^{D^*(Mu_{m+1},Mu_{m+2},Mu_{m+2})} \varphi d_p \\
&+\cdots+ \int_0^{D^*(Mu_{n-1},Mu_n,Mu_n)} \varphi d_p \\
&\leq (r^m+r^{m+1}+\cdots+r^{n-1})\int_0^{D^*(Mu_0,Mu_1,Mu_1)} \varphi d_p \\
&\leq \frac{r^m}{1-r}\int_0^{D^*(Mu_0,Mu_1,Mu_1)} \varphi d_p \to 0.
\end{aligned}$$

Thus

$$\lim_{n,m\to\infty} D^*(Mu_m,Mu_n,Mu_n) = 0.$$

So, $\{Mu_n\}$ is D^*- Cauchy sequence, since $M(Y)$ is D^* complete, there exist $j \in M(Y)$ such that $\{Mu_n\} \to j$ as $n \to \infty$, there exist $l \in Y$ such that $M_l = j$. If $H(Y)$ is complete, then there exist $j \in H(Y)$ such that $Mu_n \to j$, as $H(Y) \subset M(Y)$, we have $j \in M(Y)$. Then there exist $l \in Y$ such that $M_l = j$. We claim that $H_l = j$,

$$\begin{aligned}
\int_0^{D^*(H_l,H_j,j)} \varphi d_p &\leq \int_0^{D^*(H_l,H_l,Mu_{n-1})} \varphi d_p + \int_0^{D^*(Mu_{n-1},j,j)} \varphi d_p \\
&\leq a\Big(\int_0^{[D^*(M_l,H_l,H_l)+D^*(M_l,H_l,H_l)]} \varphi d_p\Big) \\
&+ b\int_0^{D^*(H_l,H_l,Hl)} \varphi d_p + \int_0^{D^*(Mu_{n-1},j,j)} \varphi d_p \\
&= a\int_0^{[D^*(H_l,H_l,j)+D^*(H_l,H_l,j)]} \varphi d_p \\
&+ b\int_0^{D^*(H_l,H_l,j)} \varphi d_p + \int_0^{D^*(Mu_{n-1},j,j)} \varphi d_p
\end{aligned}$$

This implies that

$$\int_0^{D^*(H_l,H_j,j)} \varphi d_p \leq \frac{1}{1-(2a+b)}\int_0^{D^*(Mu_{n-1},j,j)} \varphi d_p$$

This implies,
$D^*(H_l,H_l,j) = 0$ as $n \to \infty$. So, $H_l = j$. i.e. $H_l = M_l$ and l is a point of coincidence point of M and H. Next we show the uniqueness.
For this, suppose that there exists a point q in Y such that $M_q = H_q$. Now

$$\begin{aligned}
\int_0^{D^*(H_l,H_q,Hq)} \varphi d_p &\leq a\Big(\int_0^{[D^*(M_l,H_l,H_l)+D^*(M_l,H_l,H_l)]} \varphi d_p\Big) \\
&+ b\int_0^{D^*(M_q,H_l,Hl)} \varphi d_p \\
&= a\int_0^{D^*(H_l,H_q,Hq)} \varphi d_p + b\int_0^{D^*(H_l,H_q,Hq)} \varphi d_p \\
&= (a+b)\int_0^{D^*(H_l,H_q,Hq)} \varphi d_p
\end{aligned}$$

Hence $D^*(H_l,H_q,H_q) = 0$. Hence $H_l = H_q$. Also $M_l = M_q$, since $H_l = M_l$. Hence l is a unique point of coincidence of M and H and l is a unique common fixed point of M and H in Y.

Corollary 3.6*Let* (Y,D^*) *be a generalized* D^**-metric space and* P *be a normal cone. Let* $H : Y \to Y$ *be a mappings which satisfy the condition,*
(iii)

$$\int_0^{D^*(Hu,Hv,Hv)} \varphi d_p \leq a\Big(\int_0^{[D^*(v,Hv,Hv)+D^*(u,Hv,Hv)]} \varphi d_p\Big) + b\int_0^{D^*(v,Hu,Hu)} \varphi d_p$$

for all $u,v,w \in Y$*, where* $a,b \geq 0$, $a+b<1$*. Suppose* $\varphi : P \to P$ *is a Lebesgue integrable mapping which is summable, non-negative and such that for each* $\varepsilon \gg 0$, $\int_0^{\varepsilon} \varphi d_p \gg 0$ *having continuous inverse. Then* H *has a unique fixed point in* Y*.*

Proof. The proof follows from Theorem 3.5 by replacing M by identity mapping.

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