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Two New Fixed Point Results for Generalized Wardowski Type Contractions

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Abstract: In this paper, we introduce the concept of Branciari $F-$Contractions, multiplicative F-contractions and establish new fixed point theorems for such contractions. Our results improve and generalize fixed point results in the literature. Some examples are given here to illustrate the usability of the obtained results.

Keywords: Branciari metric space, multiplicative metric space, fixed point, Branciari F-contraction, multiplicative F-contraction.

1 Introduction

In metric fixed point theory the contractive conditions on underlying functions play an important role for finding solution of fixed point problems. Banach contraction principle [8] is a fundamental result in metric fixed point theory. Due to its importance and simplicity, several authors have obtained many interesting extensions and generalizations of the Banach contraction principle. In 1973, Geraghty [20] studied a generalization of Banach contraction principle. Ciric [7], introduced quasi contraction, which is a generalization of Banach contraction principle. Then a lot of generalization of Banach principle has been given in the literature. Over the years, it has been generalized in different directions by several mathematicians(see [1-35]).

In 2000, Branciari [6] introduced the concept of generalized metric spaces, where the triangle inequality is replaced by the inequality $d(x,y) \leq d(x,u)+d(u,v)+d(v,y)$ for all pairwise distinct points $x,y,u,v \in X$. Various fixed point results were established on such spaces, see ([9],[10],[12]-[17]) and the references therein.

Definition 1.*[6] Let X be a non-empty set and $d : X \times X \longrightarrow [0,\infty)$ be a mapping such that for all $x,y \in X$ and all distinct points $u,v \in X$, each of them different from x and y, one has*
(i) $d(x,y) = 0 \Longleftrightarrow x = y$,
(ii) $d(x,y) = d(y,x)$,
(iii) $d(x,y) \leq d(x,u)+d(u,v)+d(v,y)$.

Then (X,d) is called a Branciari metric space(or for short BMS).

Definition 2.*Let (X,d) be a BMS, $\{x_n\}$ be a sequence in X and $x \in X$, we say that $\{x_n\}$ is convergent to x if and only if $d(x_n,x) \longrightarrow 0$ as $n \longrightarrow \infty$. We denote this by $x_n \longrightarrow x$.*

Definition 3.*Let (X,d) be a BMS and $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is Cauchy sequence if and only if $d(x_n,x_m) \longrightarrow 0$ as $n,m \longrightarrow \infty$.*

Definition 4.*Let (X,d) be a BMS. We say that (X,d) is complete if and only if every Cauchy sequence in X converges to some element in X.*

On other hand, in 2012, Wardowski [25] introduce a new type of contractions called F-contractions and prove a new fixed point theorem concerning F-contractions. He generalized the Banach contraction principle in a different aspect from the well-known results from the literature. Wardowski defined the F-contraction as follows.

Definition 5.*[25] Let (X,d) be a complete metric space. A mapping $T : X \to X$ is said to be an F contraction if there exists $\tau > 0$ such that*

$$\forall\, x,y \in X,\ d(Tx,Ty) > 0 \Rightarrow \tau + F(d(Tx,Ty)) \leq F(d(x,y)) \quad (1)$$

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where $F:\mathbb{R}_+\to\mathbb{R}$ is a mapping satisfying the following conditions:

(F1) F is strictly increasing, i.e. for all $x,y\in\mathbb{R}_+$ such that $x<y$, $F(x)<F(y)$;
(F2) For each sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive numbers, $\lim_{n\to\infty}\alpha_n=0$ if and only if
$\lim_{n\to\infty}F(\alpha_n)=-\infty$;
(F3) There exists $k\in(0,1)$ such that $\lim\alpha\to 0^+\alpha^k F(\alpha)=0$.

Let $\mathscr{F}$ denote the family of all functions $F:\mathbb{R}_+\longrightarrow\mathbb{R}$ which satisfy conditions $(F1),(F2)$ and $(F3)$.

Wardowski [25] stated a modified version of the Banach contraction principle as follows.

Theorem 1.[25] *Let (X,d) be a complete metric space and let $T:X\to X$ be an F contraction. Then T has a unique fixed point $z\in X$ and for every $x\in X$ the sequence $\{T^n x\}_{n\in\mathbb{N}}$ converges to z.*

Example 1.[25] Let $F:\mathbb{R}_+\to\mathbb{R}$ be given by the formula $F(\alpha)=\ln\alpha$. It is clear that F satisfied (F1)-(F2)-(F3) for any $k\in(0,1)$. Each mapping $T:X\to X$ satisfying (1) is an F-contraction such that

$$d(Tx,Ty)\le e^{-\tau}d(x,y),\ \text{for all } x,y\in X,\ Tx\neq Ty.$$

It is clear that for $x,y\in X$ such that $Tx=Ty$ the inequality $d(Tx,Ty)\le e^{-\tau}d(x,y)$, also holds, i.e. T is a Banach contraction.

Example 2.[25] If $F(\alpha)=\ln\alpha+\alpha$, $\alpha>0$ then F satisfies (F1)-(F3) and the condition (1.1) is of the form

$$\frac{d(Tx,Ty)}{d(x,y)}e^{d(Tx,Ty)-d(x,y)}\le e^{-\tau},\ \text{for all } x,y\in X,\ Tx\neq Ty.$$

Remark.[25] From $(F1)$ and (1.1) it is easy to conclude that every $F-$contraction is necessarily continuous.

On the other hand, Let X be a nonempty set. Multiplicative metric [35] is a mapping $d:X\times X\longrightarrow\mathbb{R}$ satisfying the following conditions:
(m1) $d(x,y)>1$ for all $x,y\in X$ and $d(x,y)=1$ if and only if $x=y$,
(m2) $d(x,y)=d(y,x)>1$ for all $x,y\in X$,
(m3) $d(x,y)\le d(x,z).d(z,y)$for all $x,y,z\in X$ (multiplicative triangle inequality).

Also (X,d) is called a multiplicative metric space.

Ozavsar and Cervikel [31] generalized the celebrated Banach contraction mapping principle in the setup of multiplicative metric spaces.

Definition 6.[31] *Let (X,d) be a multiplicative metric space, $x\in X$ and $\varepsilon>1$. We now define a set*

$$B_\varepsilon(x)=\{y\in X\setminus d(x,y)<\varepsilon\},$$

which is called a multiplicative open ball of radius ε with center x. Similarly, one can describe a multiplicative closed ball as

$$\overline{B_\varepsilon(x)}=\{y\in X\setminus d(x,y)\le\varepsilon\}.$$

Definition 7.[31] *Let (X,d) be a multiplicative metric space, $\{x_n\}$ be a sequence in X, and $x\in X$. If, for every multiplicative open ball $B_\varepsilon(x)$, there exists a natural number Nsuch that $n\ge N\Longrightarrow x_n\in B_\varepsilon(x)$, then the sequence $\{x_n\}$ is said to be multiplicative convergent to x, denoted by $x_n\longrightarrow x$ as $n\longrightarrow\infty$.*

Definition 8.[31] *Let (X,d) be a multiplicative metric space and $\{x_n\}$ be a sequence in X. The sequence $\{x_n\}$ is called a multiplicative Cauchy sequence if, for all $\varepsilon>1$, there exists $N\in\mathbb{N}$ such that $d(x_m,x_n)<\varepsilon$ for $m,n\ge N$.*

Definition 9.[31] *Let (X,d) be a multiplicative metric space. The multiplicative metric paces X is said to be complete if and only if every Cauchy sequence $\{x_n\}$ in X for all $n\in\mathbb{N}$ converges in X.*

Definition 10.[31] *Let (X,d) be a multiplicative metric space. A self mapping $T:X\to X$ is said to be multiplicative contraction if there exists $\lambda\in[0,1)$ such that*

$$d(Tx,Ty)\le(d(x,y))^{\lambda}$$

$\forall x,y\in X$.

Theorem 2.[31] *Let (X,d) be a complete multiplicative metric space and $T:X\to X$ be multiplicative contraction, then T has a unique fixed point.*

They also extended Kannan and Chatterjea results from complete metric space to complete multiplicative metric spaces. Later on He et al. [32] extended the results in [31] to two pair of self-mappings satisfying certain commutative conditions on a multiplicative metric space. Abbas et al.[33] proved the results of He et al.[32] for local conractions. Yamaod et al.[34] gave the concept of cyclic (α,β)-admissible mapping in multiplicative metric spaces and proved some fixed point results for these mappings. For more details in multiplicative metric spaces we refer the reader to [35,36].

The aim of this article is to introduce the notion of Branciari F-contraction, multiplicative F-contractions and establish new fixed point theorems for such contractions. Throughout this article, $\mathbb{N},\mathbb{R}^+,\mathbb{R}$ denote the set of natural numbers, the set of positive real numbers and the set of real numbers, respectively.

The following lemmas will be needed in the sequel.

Lemma 1.[10] *Let (X,d) be a BMS and $\{x_n\}$ be a Cauchy sequence in (X,d) such that and $x\in X$, we say that $d(x_n,x)\longrightarrow 0$ as $n\longrightarrow\infty$ for some $x\in X$. Then $d(x_n,y)\longrightarrow d(x,y)$ as $n\longrightarrow\infty$ for all $y\in X$. In particular, $\{x_n\}$ does not converge to y if $y\neq x$.*

Lemma 2.[31] *Let (X,d) be a multiplicative metric space, $\{x_n\}$ be a sequence in X and $x\in X$. Then $x_n\longrightarrow x$ as $n\longrightarrow\infty$ if and only if $d(x_n,x)\longrightarrow 1$ as $n\longrightarrow\infty$.*

Lemma 3.[31] *Let (X,d) be a multiplicative metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a multiplicative Cauchy sequence if and only if $d(x_m,x_n)\longrightarrow 1$ as $m,n\longrightarrow\infty$.*

2 Branciari F-contractions.

In this paper, we introduce the notion of Branciari F-rational contraction and establish new fixed point theorems for such contractions in the setting of complete Branciari metric spaces.

Definition 11. *Let (X,d) be a BMS. Then, $T:X\longrightarrow X$ is said to be Branciari F-rational contraction, there exist $F\in\mathscr{F}$ and $\tau>0$ such that*

$$\forall x,y\in X,\ d(Tx,Ty)>0\Rightarrow \tau+F\left(d(Tx,Ty)\right)\le F\left(M(x,y)\right), \tag{2}$$

where

$$M(x,y)=\max\left\{\begin{array}{c} d(x,y),d(x,Tx),d(y,Ty), \\ \dfrac{d(x,Tx)d(y,Ty)}{1+d(x,y)},\dfrac{d(x,Tx)d(y,Ty)}{1+d(Tx,Ty)}\end{array}\right\}.$$

Theorem 3. *Let (X,d) be a complete BMS and $T:X\longrightarrow X$ be a Branciari F-rational contraction. If T or F is continuous, then T has a unique fixed point in X.*

Proof. Let x be an arbitrary point in X. If for some $n\in\mathbb{N}$ we have $T^nx=T^{n+1}x$, Then T^nx will be a fixed point of T. So with out loss of generality, we can assume that

$$d(T^nx,T^{n+1}x)>0,\forall\, n\in\mathbb{N}.$$

Now, from (2), for all $n\in\mathbb{N}$, we have

$$\tau+F\left(d(T^nx,T^{n+1}x)\right)\le F\left(M(T^{n-1}x,T^nx)\right), \tag{3}$$

where

$$\begin{aligned} M(T^{n-1}x,T^nx) &= \max\left\{\begin{array}{l} d(T^{n-1}x,T^nx), \\ d(T^{n-1}x,TT^{n-1}x), \\ d(T^nx,TT^nx), \\ \frac{d(T^{n-1}x,TT^{n-1}x)d(T^nx,TT^nx)}{1+d(T^{n-1}x,T^nx)}, \\ \frac{d(T^{n-1}x,TT^{n-1}x)d(T^nx,TT^nx)}{1+d(TT^{n-1}x,TT^nx)} \end{array}\right\} \\ &= \max\left\{\begin{array}{l} d(T^{n-1}x,T^nx), \\ d(T^{n-1}x,T^nx), \\ d(T^nx,T^{n+1}x), \\ \frac{d(T^{n-1}x,T^nx)d(T^nx,T^{n+1}x)}{1+d(T^{n-1}x,T^nx)}, \\ \frac{d(T^{n-1}x,T^nx)d(T^nx,T^{n+1}x)}{1+d(T^nx,T^{n+1}x)} \end{array}\right\} \\ &= \max\{d(T^{n-1}x,T^nx),d(T^nx,T^{n+1}x)\}. \end{aligned}$$

Now if, $M(T^{n-1}x,T^nx)=d(T^nx,T^{n+1}x)$, then inequality (3) turns into

$$\tau+F\left(d(T^nx,T^{n+1}x)\right)\le F\left(d(T^nx,T^{n+1}x)\right),$$

which is contradiction with $\tau>0$. Thus we conclude that

$$\max\{d(T^{n-1}x,T^nx),d(T^nx,T^{n+1}x)\}=d(T^{n-1}x,T^nx)$$

for all $n\in\mathbb{N}$.Hence, the inequality (3) turns into

$$F\left(d(T^nx,T^{n+1}x)\right)\le F(d(T^{n-1}x,T^nx))-\tau\ \ \forall n\in\mathbb{N}. \tag{4}$$

Iteratively, we find that

$$\begin{aligned} F\left(d(T^nx,T^{n+1}x)\right) &\le F(d(T^{n-1}x,T^nx))-\tau \\ &\le F(d(T^{n-2}x,T^{n-1}x))-2\tau \\ &\le F(d(T^{n-3}x,T^{n-2}x))-3\tau \\ &\ \ \vdots \\ &\le F(d(x,Tx))-n\tau\ \ \forall n\in\mathbb{N}. \end{aligned} \tag{5}$$

Since $F\in\mathscr{F}$, so by taking limit as $n\longrightarrow\infty$ in (5), we deduce

$$\lim_{n\longrightarrow\infty}F(d\left(T^nx,T^{n+1}x\right))=-\infty \iff \lim_{n\longrightarrow\infty}d\left(T^nx,T^{n+1}x\right)=0. \tag{6}$$

Now from (F_3), there exists $0<k<1$ such that

$$\lim_{n\longrightarrow\infty}\left[d\left(T^nx,T^{n+1}x\right)\right]^kF(d\left(T^nx,T^{n+1}x\right))=0. \tag{7}$$

By (5), we have

$$\begin{aligned} & d\left(T^nx,T^{n+1}x\right)^kF(d\left(T^nx,T^{n+1}x\right)) \\ & -d\left(T^nx,T^{n+1}x\right)^kF\left(d\left(x,Tx\right)\right) \\ & \le d\left(T^nx,T^{n+1}x\right)^k\left[F\left(d\left(x,Tx\right)-n\tau\right)\right] \\ & -d\left(T^nx,T^{n+1}x\right)^kF\left(d\left(x,Tx\right)\right) \\ & = -n\tau\left[d\left(T^nx,T^{n+1}x\right)\right]^k\le 0. \end{aligned} \tag{8}$$

Letting $n\longrightarrow\infty$ in (8) and applying (6) and (7), we have,

$$\lim_{n\longrightarrow\infty}n\left[d\left(T^nx,T^{n+1}x\right)\right]^k=0, \tag{9}$$

and hence

$$\lim_{n\longrightarrow\infty}n^{\frac{1}{k}}d\left(T^nx,T^{n+1}x\right)=0. \tag{10}$$

Then there exists $n_1\in\mathbb{N}$ such that $n\left(d\left(T^nx,T^{n+1}x\right)\right)^k\le 1$ for all $n\ge n_1$, this implies

$$d\left(T^nx,T^{n+1}x\right)\le\frac{1}{n^{\frac{1}{k}}}. \tag{11}$$

Now, we will prove that T has a periodic point. Suppose that it is not the case, then $T^nx\ne T^mx$ for all $n,m\in\mathbb{N}$ such that $n \neq m$. Using (2), we get

$$\begin{aligned} &\tau + F\left(d\left(T^n x, T^{n+2}x\right)\right) \\ &\leq F\left(\max\left\{\begin{array}{l} d\left(T^{n-1}x, T^{n+1}x\right), \\ d\left(T^{n-1}x, TT^{n-1}x\right), \\ d\left(T^{n+1}x, TT^{n+1}x\right), \\ \dfrac{d\left(T^{n-1}x, TT^{n-1}x\right) d\left(T^{n+1}x, TT^{n+1}x\right)}{1+d\left(T^{n-1}x, T^{n+1}x\right)}, \\ \dfrac{d\left(T^{n-1}x, TT^{n-1}x\right) d\left(T^{n+1}x, TT^{n+1}x\right)}{1+d\left(TT^n x, TT^{n+1}x\right)} \end{array}\right\}\right) \\ &= F\left(\max\left\{\begin{array}{l} d\left(T^{n-1}x, T^{n+1}x\right), \\ d\left(T^{n-1}x, T^n x\right), \\ d\left(T^{n+1}x, T^{n+2}x\right), \\ \dfrac{d\left(T^{n-1}x, T^n x\right) d\left(T^{n+1}x, T^{n+2}x\right)}{1+d\left(T^{n-1}x, T^{n+1}x\right)}, \\ \dfrac{d\left(T^{n-1}x, T^n x\right) d\left(T^{n+1}x, T^{n+2}x\right)}{1+d\left(T^n x, T^{n+2}x\right)} \end{array}\right\}\right) \\ &= F\left(\max\left\{\begin{array}{c} d\left(T^{n-1}x, T^{n+1}x\right), d\left(T^{n-1}x, T^n x\right), \\ d\left(T^{n+1}x, T^{n+2}x\right) \end{array}\right\}\right). \end{aligned} \tag{12}$$

Since F is increasing, we obtain from (12)

$$\tau + F\left(d\left(T^n x, T^{n+2}x\right)\right) \leq \max\left\{\begin{array}{l} F\left(d\left(T^{n-1}x, T^{n+1}x\right)\right), \\ F\left(d\left(T^{n-1}x, T^n x\right)\right), \\ F\left(d\left(T^{n+1}x, T^{n+2}x\right)\right) \end{array}\right\}. \tag{13}$$

Let I be the set of $n \in \mathbb{N}$ such that

$$\begin{aligned} u_n &= \max\left\{\begin{array}{l} F\left(d\left(T^{n-1}x, T^{n+1}x\right)\right), \\ F\left(d\left(T^{n-1}x, T^n x\right)\right), \\ F\left(d\left(T^{n+1}x, T^{n+2}x\right)\right) \end{array}\right\} \\ &= F\left(d\left(T^{n-1}x, T^{n+1}x\right)\right). \end{aligned}$$

If $|I| < \infty$ then there $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\begin{aligned} &\max\left\{\begin{array}{l} F\left(d\left(T^{n-1}x, T^{n+1}x\right)\right), \\ F\left(d\left(T^{n-1}x, T^n x\right)\right), \\ F\left(d\left(T^{n+1}x, T^{n+2}x\right)\right) \end{array}\right\} \\ &= \max\left\{\begin{array}{l} F\left(d\left(T^{n-1}x, T^n x\right)\right), \\ F\left(d\left(T^{n+1}x, T^{n+2}x\right)\right) \end{array}\right\}. \end{aligned}$$

In this case, we get from (13)

$$\begin{aligned} &\tau + F\left(d\left(T^n x, T^{n+2}x\right)\right) \\ &\leq \max\left\{\begin{array}{l} F\left(d\left(T^{n-1}x, T^n x\right)\right), \\ F\left(d\left(T^{n+1}x, T^{n+2}x\right)\right) \end{array}\right\} \end{aligned}$$

for all $n \geq N$. Letting $n \longrightarrow \infty$ in the above inequality and using (6), we obtain

$$\lim_{n \longrightarrow \infty} F\left(d\left(T^n x, T^{n+2}x\right)\right) = -\infty.$$

If $|I| = \infty$, we can find a subsequence of $\{u_n\}$, then we denote also by $\{u_n\}$, such that

$$u_n = F\left(d\left(T^{n-1}x, T^{n+1}x\right)\right) \quad \text{for } n \text{ large enough.}$$

In this case, we obtain from (13)

$$\tau + F\left(d\left(T^n x, T^{n+2}x\right)\right) \leq F\left(d\left(T^{n-1}x, T^{n+1}x\right)\right)$$

Iteratively, we find that

$$\begin{aligned} F\left(d(T^n x, T^{n+2}x)\right) &\leq F(d(T^{n-1}x, T^{n+1}x)) - \tau \\ &\leq F(d(T^{n-2}x, T^n x)) - 2\tau \\ &\leq F(d(T^{n-3}x, T^{n-1}x)) - 3\tau \\ &\;\;\vdots \\ &\leq F(d(x, T^2 x)) - n\tau \quad \forall n \in \mathbb{N}. \end{aligned} \tag{14}$$

Letting $n \longrightarrow \infty$ in the above inequality, we obtain

$$\lim_{n \longrightarrow \infty} F\left(d\left(T^n x, T^{n+2}x\right)\right) = -\infty. \tag{15}$$

Then in all cases, (15) holds. Using (15) and the property $(F2)$, we have

$$\lim_{n \longrightarrow \infty} d\left(T^n x, T^{n+2}x\right) = 0. \tag{16}$$

Now from (F_3), there exists $0 < k < 1$ such that

$$\lim_{n \longrightarrow \infty} \left[d\left(T^n x, T^{n+2}x\right)\right]^k F\left(d\left(T^n x, T^{n+2}x\right)\right) = 0. \tag{17}$$

By (14), we have

$$\begin{aligned} &d\left(T^n x, T^{n+2}x\right)^k F\left(d\left(T^n x, T^{n+2}x\right)\right) - d\left(T^n x, T^{n+2}x\right)^k F\left(d\left(x, T^2 x\right)\right) \\ &\leq d\left(T^n x, T^{n+2}x\right)^k \left[F\left(d\left(x, T^2 x\right) - n\tau\right)\right] - d\left(T^n x, T^{n+2}x\right)^k F\left(d\left(x, T^2 x\right)\right) \\ &= -n\tau\left[d\left(T^n x, T^{n+2}x\right)\right]^k \leq 0. \end{aligned} \tag{18}$$

Letting $n \longrightarrow \infty$ in (18) and applying (16) and (17), we have,

$$\lim_{n \longrightarrow \infty} n\left[d\left(T^n x, T^{n+2}x\right)\right]^k = 0, \tag{19}$$

and hence

$$\lim_{n \longrightarrow \infty} n^{\frac{1}{k}} d\left(T^n x, T^{n+2}x\right) = 0. \tag{20}$$

Then there exists $n_2 \in \mathbb{N}$ such that

$$d\left(T^n x, T^{n+2}x\right) \leq \frac{1}{n^{\frac{1}{k}}} \quad \text{for all } n \geq n_2. \tag{21}$$

Let $h = \max\{n_0, n_1\}$. we consider two cases.
Case 1: If $m > 2$ is odd, then writing $m = 2L+1, L \geq 1$, using (11), for all $n \geq h$, we obtain

$$\begin{aligned} d\left(T^n x, T^{n+m}x\right) &\leq d\left(T^n x, T^{n+1}x\right) + d\left(T^{n+1}x, T^{n+2}x\right) \\ &\quad + \ldots + d\left(T^{n+2L}x, T^{n+2L+1}x\right) \\ &\leq \frac{1}{n^{\frac{1}{k}}} + \frac{1}{(n+1)^{\frac{1}{k}}} + \ldots + \frac{1}{(n+2L)^{\frac{1}{k}}} \\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}. \end{aligned}$$

Case 2: If $m > 2$ is even, then writing $m = 2L, L \geq 2$, using (11) and (21), for all $n \geq h$, we have

$$\begin{aligned} d\left(T^n x, T^{n+m} x\right) &\leq d\left(T^n x, T^{n+2} x\right) + d\left(T^{n+2} x, T^{n+3} x\right) \\ &\quad + \ldots + d\left(T^{n+2L-1} x, T^{n+2L} x\right) \\ &\leq \frac{1}{n^{\frac{1}{k}}} + \frac{1}{(n+2)^{\frac{1}{k}}} + \ldots + \frac{1}{(n+2L-1)^{\frac{1}{k}}} \\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}. \end{aligned}$$

Thus, combining all cases, we have

$$d\left(T^n x, T^{n+m} x\right) \leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}} \text{ for all } n \geq h,\, m \in \mathbb{N}.$$

Since the series $\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}$ is convergent (since $\frac{1}{k} > 1$), we deduce that $\{T^n x\}$ is a Cauchy sequence. From the completeness of X, there $z \in X$ such that $T^n x \longrightarrow z$ as $n \longrightarrow \infty$. Now we assume that T is continuous. Hence, we have

$$z = \lim_{n \longrightarrow \infty} T^{n+1} x = \lim_{n \longrightarrow \infty} T\left(T^n x\right) = T\left(\lim_{n \longrightarrow \infty} T^n x\right) = Tz.$$

Next, we assume that F is continuous. Without restriction of the generality, we can suppose that $T^n x \neq z$ for all n. Suppose that $d(z, Tz) > 0$, we have

$$\begin{aligned} &\tau + F\left(d\left(T^{n+1} x, Tz\right)\right) \\ &\leq F\left(\max\left\{\begin{array}{l} d\left(T^n x, z\right), \\ d\left(T^n x, T^{n+1} x\right), \\ d\left(z, Tz\right), \\ \dfrac{d\left(T^n x, T^{n+1} x\right) d\left(z, Tz\right)}{1 + d\left(T^n x, z\right)}, \\ \dfrac{d\left(T^n x, T^{n+1} x\right) d\left(z, Tz\right)}{1 + d\left(T^{n+1} x, Tz\right)} \end{array}\right\}\right) \\ &= F\left(\max\left\{\begin{array}{l} d\left(T^n x, z\right), d\left(T^n x, T^{n+1} x\right), \\ d\left(z, Tz\right) \end{array}\right\}\right). \end{aligned}$$

Which implies

$$\begin{aligned} &\tau + F\left(d\left(T^{n+1} x, Tz\right)\right) \\ &\leq F\left(\max\left\{\begin{array}{l} d\left(T^n x, z\right), \\ d\left(T^n x, T^{n+1} x\right), \\ d\left(z, Tz\right) \end{array}\right\}\right). \end{aligned}$$

Letting $n \longrightarrow \infty$ in the above inequality, using Lemma 16, we obtain

$$\tau + F\left(d\left(z, Tz\right)\right) \leq F\left(d\left(z, Tz\right)\right).$$

This implies,

$$d\left(z, Tz\right) < d\left(z, Tz\right),$$

which is a contradiction. Thus we have $z = Tz$, which is also a contradiction with the assumption: T does not have a periodic point. Thus T has a periodic point, say z of period q. Suppose that the set of fixed points of T is empty. Then we have

$$q > 1 \text{ and } d\left(z, Tz\right) > 0.$$

By using (2), we get

$$\tau + F\left(d\left(z, Tz\right)\right) = \tau + F\left(d\left(T^q z, T^{q+1} z\right)\right) \leq F\left(d\left(T^{q-1} z, T^q z\right)\right).$$

This implies

$$\begin{aligned} F\left(d\left(z, Tz\right)\right) &\leq F\left(d\left(T^{q-1} z, T^q z\right)\right) - \tau \\ &\leq \ldots \leq F\left(d\left(z, Tz\right)\right) - q\tau < F\left(d\left(z, Tz\right)\right), \end{aligned}$$

which is a contradiction. Thus the set of fixed points of T is non-empty (that is, T has at least one fixed point). Now we suppose that $z, u \in X$ are two fixed points of T such that $d(z, u) = d(Tz, Tu) > 0$. From the hypothesis, we obtain

$$\tau + F\left(d\left(z, u\right)\right) = F\left(d\left(Tz, Tu\right)\right) \leq F\left(d\left(z, u\right)\right),$$

it is a contradiction. Therefore T has a unique fixed point.

Since a metric space is a Branciari metric space, we can obtain the following result.

Definition 12. *Let (X, d) be a metric space. Then, $T : X \longrightarrow X$ is said to be F-rational contraction, there exist $F \in \mathscr{F}$ and $\tau > 0$ such that for all $x, y \in X$*

$$d(Tx, Ty) > 0 \Rightarrow \tau + F\left(d(Tx, Ty)\right) \leq F\left(M(x, y)\right), \quad (22)$$

where

$$M(x, y) = \max\left\{\begin{array}{l} d(x, y), d(x, Tx), d(y, Ty), \\ \dfrac{d(x, Tx) d(y, Ty)}{1 + d(x, y)}, \\ \dfrac{d(x, Tx) d(y, Ty)}{1 + d(Tx, Ty)} \end{array}\right\}.$$

Theorem 4. *Let (X, d) be a complete metric space and $T : X \longrightarrow X$ be a F-rational rational contraction map. If T or F is continuous, then T has a unique fixed point in X.*

Now we give the following definition.

Definition 13. *Let (X, d) be a BMS. Then $T : X \longrightarrow X$ is said to be a Branciari F-contraction map, there exist $F \in \mathscr{F}$ and $\tau > 0$ such that*

$$\forall x, y \in X,\ d(Tx, Ty) > 0 \Rightarrow \tau + F\left(d(Tx, Ty)\right) \leq F\left(d(x, y)\right).$$

Theorem 5. *Let (X, d) be a complete BMS and $T : X \longrightarrow X$ be a Branciari F-contraction map. Then T has a unique fixed point in X*

Corollary 1. *[25] Let (X, d) be a complete metric space and $T : X \longrightarrow X$ be a F-contraction map. Then T has a unique fixed point in X.*

*Example 3.*Let $X = \{1,2,3,4\}$. Define $d : X\times X \longrightarrow \mathbb{R}_+$ as follows

$$d(1,2) = d(2,1) = 3,$$
$$d(2,3) = d(3,2) = d(1,3) = d(3,1) = 1,$$
$$d(1,4) = d(4,1) = d(2,4) = d(4,2) = d(3,4) = d(4,3) = 4.$$

It is clear that (X,d) is a complete BMS, but it is not metric space because d does not satisfy triangle inequality on X. Indeed,

$$3 = d(1,2) > d(1,3) + d(3,2) = 1+1 = 2.$$

Let $T : X \longrightarrow X$ be the mapping defined by

$$T(x) = \begin{cases} 2 \text{ if } x \in \{1,2,3\}, \\ 1 \text{ if } x = 4. \end{cases}$$

Define $F : \mathbb{R}_+ \to \mathbb{R}$ by $F(\alpha) = \ln\alpha$.Now, for $x \in \{1,2,3\}$,$y = 4$, where $\tau = 1$, we have

$$\begin{aligned}\tau + F(d(T(x),T(4))) &= 1 + F(d(2,1)) \\ &\leq F(d(x,4)).\end{aligned}$$

So *Branciari* F-contraction, T has a unique fixed point (that is, 2).

*Example 4.*Let $X = \{0,\frac{8}{3},7\}$ endowed with the usual metric $d(x,y) = |x-y|$ for all $x,y \in X$. Define a mapping, $T : X \to X$ by,

$$Tx = \begin{cases} \frac{8}{3} & x \in \{0,\frac{8}{3}\}, \\ 0 & x = 7. \end{cases}$$

It is clear that (X,d) is a complete metric space. Now, since, T is not continuous, so T is not *Branciari* F-contraction (or F-contraction) by Remark 9.

Next, For $x \in \{0,\frac{8}{3}\}$ and $y = 7$, we have

$$d(Tx,T7) = d\left(\frac{8}{3},0\right) = \left|\frac{8}{3}-0\right| = \frac{8}{3} > 0,$$

and

$$\max\left\{\begin{array}{l} d(x,7), d(x,Tx), d(7,T7), \\ \dfrac{d(x,Tx)d(7,T7)}{1+d(x,7)}, \\ \dfrac{d(x,Tx)d(7,T7)}{1+d(Tx,T7)} \end{array}\right\} = 7.$$

So, by choosing, $F(\alpha) = \ln\alpha + \alpha \in \mathscr{F}$ and $\tau \in (0,4.965]$, we see that

$$\tau + F(d(Tx,Ty)) \leq F(M(x,y)), \ \forall x,y \in X,\ Tx \neq Ty.$$

Therefore, *Branciari* F*-rational* contraction (or F*-rational* contraction) and hence, T has a unique fixed point (that is, $\frac{8}{3}$).

3 Multiplicative F-contractions

In this section, we give the concept multiplicative F-contractions and introduce new fixed point theorem for such contractions. We support these contractions by providing some example in the context of a multiplicative metric spaces.

Definition 14.*Let (X,d) be a multiplicative metric space and $T : X \to X$ be a self mapping. Then T is said to be multiplicative F-contraction if there exists $\lambda \in (0,1)$ such that for all $x,y \in X$,*

$$d(Tx,Ty) > 0 \Rightarrow F(d(Tx,Ty)) \leq [F(d(x,y))]^{\lambda}, \quad (23)$$

where $F : (1,\infty) \to (1,\infty)$ is a mapping satisfying the following conditions:

(F1*) F is strictly increasing, *i.e.* for all $x,y \in (1,\infty)$ such that $x < y$, we have $F(x) < F(y)$;

(F2*) for each sequence $\{\alpha_n\}_{n=1}^{\infty} \subset (1,\infty)$, $\lim_{n\to\infty} \alpha_n = 1^+$ if and only if $\lim_{n\to\infty} F(\alpha_n) = 1$;

(F3*) F is continuous.

We denote with $\mathscr{F}$ the family of all functions F that satisfy the conditions (F1*)-(F3*).

We support this idea by the following examples.

*Example 5.*Let $F : (1,\infty) \to (1,\infty)$ be given by the formula $F(\alpha) = \sqrt{x}$for $\alpha > 1$. It is clear that F satisfied (F1*)-(F3*) for any $k \in (0,1)$. Each mapping $T : X \to X$ satisfying (3.1) is an multiplicative F-ontraction such that

$$d(Tx,Ty) \leq [d(x,y)]^k, \text{ for all } x,y \in X,\ Tx \neq Ty.$$

It is clear that for $x,y \in X$ such that $Tx = Ty$ the inequality $d(Tx,Ty) - 1 \leq e^{-\tau}\left(d(x,y)^{\lambda} - 1\right)$, also holds, i.e. T is a multiplicative Banach contraction.

*Remark.*From (F1*) and (23) it is easy to conclude that every multiplicative F-contraction is necessarily multiplicative contractive mapping i.e

$$d(Tx,Ty) < d(x,y)^{\lambda} \quad \text{for all } x,y \in X,\ Tx \neq Ty.$$

Thus every multiplicative F-contraction is a continuous mapping.

Now we prove the main result of the paper.

Theorem 6.*Let (X,d) be a complete multiplicative metric space and $T : X \to X$ be multiplicative F-contraction. Then T has a unique fixed point that is there exists $z \in X$ such that $z = Tz$. And for every $x_0 \in X$, the sequence $\{Tx_0\}_{n\in\mathbb{N}}$ converges to z.*

*Proof.*Let $x_0 \in X$ be an arbitrary but fixed. We define a sequence $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}\cup\{0\}$with x_0 as initial point. If there exists some $n_0 \in \mathbb{N}\cup\{0\}$ such that $x_{n_0+1} = x_{n_0}$, then $Tx_{n_0} = x_{n_0}$ and we are nothing to prove that is x_{n_0} is a fixed point of T. So we suppose that $x_{n+1} \neq x_n$ for all

$n \in \mathbb{N} \cup \{0\}$. Then $d(x_{n+1}, x_n) > 0$ for all $n \in \mathbb{N} \cup \{0\}$. It follows from (23) that for each $n \in \mathbb{N}$

$$\begin{aligned} F(d(x_n, x_{n+1})) &= F(d(Tx_{n-1}, Tx_n)) \\ &\leq [F(d(x_{n-1}, x_n))]^{\lambda} \\ &\leq [F(d(x_{n-2}, x_{n-1}))]^{\lambda^2} \\ &\leq ... \leq [F(d(x_0, x_1))]^{\lambda^n}. \end{aligned}$$

which implies that

$$F(d(x_n, x_{n+1})) \leq [F(d(x_0, x_1))]^{\lambda^n}. \tag{24}$$

Taking limit as $n \to +\infty$, we get $\lim_{n \longrightarrow \infty} F(d(Tx_{n-1}, Tx_n)) = 1$, which together with $(F2^*)$ gives us

$$\lim_{n \longrightarrow \infty} d(x_n, x_{n+1}) = 1. \tag{25}$$

Now , we claim that $\{x_n\}_{n=1}^{\infty}$ is a cauchy sequence. Arguing by contradiction, we have that there exists $\varepsilon > 0$ and sequence $\{p(n)\}_{n=1}^{\infty}$ and $\{q(n)\}_{n=1}^{\infty}$ of natural numbers such that for all $n \in \mathbb{N}$

$$p(n) > q(n) > n, \ d(x_{p(n)}, x_{q(n)}) \geq \varepsilon, d(x_{p(n)-1}, x_{q(n)}) < \varepsilon. \tag{26}$$

So, we have

$$\begin{aligned} \varepsilon &\leq d(x_{p(n)}, x_{q(n)}) \leq d(x_{p(n)}, x_{p(n)-1}).d(x_{p(n)-1}, x_{q(n)}) \quad (27) \\ &\leq d(x_{p(n)}, x_{p(n)-1}).\varepsilon \\ &= d(x_{p(n)-1}, Tx_{p(n)-1}).\varepsilon. \end{aligned}$$

Letting $n \longrightarrow \infty$ in (27) and using (25), we obtain

$$\lim_{n \longrightarrow \infty} d(x_{p(n)}, x_{q(n)}) = \varepsilon. \tag{28}$$

Also, from (25) there exists a natural number $n_1 \in \mathbb{N}$ such that

$$d(x_{p(n)}, Tx_{p(n)}) < \frac{\varepsilon}{4} \text{ and } d(x_{q(n)}, Tx_{q(n)}) < \frac{\varepsilon}{4}, \ \forall n \geq n_1. \tag{29}$$

Next, we claim that

$$d(Tx_{p(n)}, Tx_{q(n)}) = d(x_{p(n)+1}, x_{q(n)+1}) > 1 \ \forall n \geq n_1. \tag{30}$$

Arguing by contradiction, there exists $m \geq n_1$ such that

$$d(x_{p(m)+1}, x_{q(m)+1}) = 1. \tag{31}$$

It follows from (26), (29) and (31) that

$$\begin{aligned} \varepsilon &\leq d(x_{p(m)}, x_{q(m)}) \leq d(x_{p(m)}, x_{p(m)+1}).d(x_{p(m)+1}, x_{q(m)}) \\ &\leq d(x_{p(m)}, x_{p(m)+1}).d(x_{p(m)+1}, x_{q(m)+1}).d(x_{q(m)+1}, x_{q(m)}) \\ &= d(x_{p(m)}, Tx_{p(m)}).d(x_{p(m)+1}, x_{q(m)+1}).d(x_{q(m)}, Tx_{q(m)}) \\ &< \frac{\varepsilon}{4}.1.\frac{\varepsilon}{4} = . \end{aligned}$$

This contradiction establishes the relation (30) it follows from (30) and (23) that,

$$F\left(d\left(x_{P(n)+1}, x_{q(n)+1}\right)\right) \leq \left[F\left(d\left(x_{p(n)}, x_{q(n)}\right)\right)\right]^k \ \forall n \geq n_1. \tag{32}$$

So from $(F3^*)$, (28) and (32), we have

$$F(\varepsilon) \leq [F(\varepsilon)]^k < F(\varepsilon).$$

This contradiction show that $\{x_n\}$ is a multiplicative Cauchy sequence. From the completeness of X, there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$. Finally, the continuity of T yields

$$d(z, Tz) = \lim_{n \longrightarrow \infty} d(x_n, Tx_n) = \lim_{n \longrightarrow \infty} d(x_n, x_{n+1}) = d(z, z) = 1.$$

Thus z is a fixed point of T. Now, we show that z is the unique fixed point of T. Assume that y is another fixed point of T such that $Tz = z \neq y = Ty$, then we get

$$\begin{aligned} F(d(z, y)) &= F(d(Tz, Ty)) \\ &\leq [F(d(z, y))]^k < F(d(z, y)) \end{aligned}$$

a contradiction, which implies that $z = y$.

Observe that the multiplicative Banach contraction principle follows immediately from Theorem 31. Indeed, if T is a multiplicative Banach contraction, i.e., there exists $\lambda \in [0, 1)$ such that

$$d(Tx, Ty) \leq [d(x, y)]^{\lambda}, \text{ for all } x, y \in X \text{ and } d(Tx, Ty) > 0,$$

then, we have

$$(d(Tx, Ty))^2 \leq \left[(d(x, y))^2\right]^{\lambda}, \text{ for all } x, y \in X \text{ and } d(Tx, Ty) > 0.$$

Clearly the function $F : (1, \infty) \longrightarrow (1, \infty)$ defined by $F(t) = t^2$, $t > 1$ belongs to $\mathscr{F}$. So, the existence and uniqueness of the fixed point follows from Theorem 31. In the following example, we show that Theorem 31 is a real generalization of the multiplicative Banach contraction.

*Example 6.*Let $X = \{0, 1, 3\}$ and $d : \mathbb{R}^2 \longrightarrow \mathbb{R}^+$ a multiplicative metric space defined by $d(x, y) = e^{|x-y|}$. Note that $(\mathbb{R}, d)$ is a complete multiplicative metric space. Define $T : X \longrightarrow X$ by

$$T(x) = \begin{cases} 1 & \text{if } x \in \{0, 1\}, \\ 0 & \text{if } \quad x = 3. \end{cases}$$

T is not a multiplicative Banach contraction as it is not a continuous mapping. But, for $x \in \{0, 1\}$, $y = 3$, $\lambda \in \left[\frac{1}{5}, 1\right)$, by putting $F(t) = \sqrt{t}$, with $t > 1$, we get

$$\begin{aligned} F(d(T(x), T(1))) &= \sqrt{d(T(x), T(1))} \\ &= \sqrt{e} \end{aligned}$$

and

$$[F(d(x, 1))]^{\lambda} = \left[\sqrt{d(x, 1)}\right]^{\lambda} = \left[\sqrt{e^{|x-1|}}\right]^{\lambda}.$$

Clearly we have

$$F(d(T(x), T(y))) \leq [F(d(x, y))]^{\lambda} \text{ for all } x, y \in X \text{ and } Tx \neq Ty.$$

Thus T is a multiplicative F-contraction and T has a unique fixed point.

4 Conclusion

In this connection, the main aim of our paper is to present the concept of Branciari $F-$Contractions, multiplicative F-contractions. Existence of fixed point results of such type of F-contractions are established. The study of results is very useful in the sense that it generalizes the F-contraction given in [25]. The new concepts lead to further investigations and applications.

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