Mathematical Sciences Letters
An International Journal

http://dx.doi.org/10.12785/msl/030209

Lipschitz Compactness

Zargham Bahmani*
Department of Mathematics, Islamic Azad University of Genaveh Branch,Genaveh, Iran

Received: 16 Dec. 2013, Revised: 6 Mar. 2014; Accepted: 9 Mar. 2014
Published online: 1 May 2014

E. mail: *bahmani.math@gmail.com*

Abstract: After defining the concept of Lipschits compactness by Bahmani and Khorshidvandpour [Z.Bahmani and S.Khorshidvandpour, Advances and Applications in Mathematical Sciences 12, 7 (2013)],this paper presents some complementary results on Lipschitz compact spaces. Also, We introduce the concept of C-lipschitz compact space. As an important result, we prove that $Se(X)$, the normed space of all sequence in a normed space X, is Lipschitz compact when X is C-lipschitz compact.

Keywords: Lipschitz compactness, C-lipschitz compactness, Lipschitz map.

I. INTRODUCTIONAND PRELIMINARIES

Recently, many authors have studied Lipschitz maps and Lipschitz spaces. Miyata and Watanabe [1] have presented many properties of Lipschitz functions. Jouini [2] has completely discussed on generalized Lipschitz functions. In [3], Albiac studied the Lipschitz space and dual of Lipschitz space. Furthermore, he obtained many useful results on Lipschitz embedding and uniformly homeomorphisms. Also, Aronszajn [4] and Cheeger [5], have studied Lipschitz differentiability. Authors in [6] have proven some interesting results on the Lipschitz structure of quasi-Banach spaces. Some another results on Lipschitz functions can be found in [7-10].
In[11], We have introduced the concept of Lipschitz compactness. In this work , First, we prove some results on Lipschitz compactness and then the concept of C-lipschitz compactness is introduced .More specially, we manage to prove that the normed space $Se(X)$ is Lipschitz compact when X is C-lipschitz compact. First of all we need the following definitions. Throughout of this paper X is a normed space over $\mathrm{F}(=\mathbb{R}$or $\mathbb{C}$); unless the contrary is specified.

Definition1.1. Let X and Y be normed spaces.We say that a map $f: X \to Y$ is a Lipschitz map, if there is a positive constant C so that

$$\|f(x)-f(y)\| \leq C\|x-y\|$$

For all $x, y \in X$. The constant C is called Lipschitz constant of f. In this paper, $Lip(X)$denotes the normed space of all Lipschitz mappings on X with following norm:

$$\|f\|_{Lip} = sup\left\{\frac{\|f(x)-f(y)\|}{\|x-y\|} : x, y \in X, x \neq y\right\}$$

Definition1.2. We say that X is Lipschitz compact, when for each sequence (x_n)in X, there exist a nonzero $f \in Lip(X)$ such that $(f(x_n))$ have a convergent subsequence.

In [11], we defined the following set:

$$MI(X) = \{f: X \to X \mid f(x) = \lambda x \quad \text{for some } \lambda \in \mathrm{F}\}$$

Indeed, we showed that if X is Lipschitz compact and$Lip(X) = MI(X)$, then X is sequentially compact.
In the next section we improve this result.

II. LIPSCHITZ COMPACTNESS

In the section, First we improve Theorem 2.10 of [11] and then prove some theorems on a Lipschitz compact space X.

Theorem 2.1. Let $Lip(X) = MI(X)$.Then X is Lipschitz compact if and only if it is sequentially compact.

Theorem 2.2.let$X$ be Lipschitz compact and Y a subspace of X. Suppose further, $Lip(Y) = MI(Y)$. Then Y is closed.

Proof. Let $(y_n) \subseteq Y$ be a sequence which $y_n \to x \in X$.We show that $x \in Y$. There is a nonzero real number λ such that (λy_n) has a convergent subsequence, say(λy_{n_k}).Clearly, (λy_{n_k}) converges to $\frac{x}{\lambda}$. Hence $\lambda = 1$ and $x \in X$,as claimed.

Theorem2.3. let X be Lipschitz compact Banach space , $Lip(X) = MI(X)$ and $Lip(E) = \{f|E: f \in Lip(X)\}$ for every infinite subset E of X .Then X is compact.

Proof.Let E be an infinite subset of X and (x_n) a sequence in E. There is a nonzero Lipschitz function f on E such that $(f(x_n))$ has a convergent subsequence, say $(f(x_{n_k}))$.Let $f(x_{n_k}) \to x \in X$ as $k \to \infty$.By hypotheses $x_{n_k} \to f^{-1}(x)$as $k \to \infty$..Therefore, $f^{-1}(x) \in E'$,where E' is the set of all limit points of E .

In[11], we proved that if $f: X \to Y$, where X *and* Y are normed spaces, is a continuous function such that $fog = gof$ for all $g \in Lip(X)$ and X is Lipschitz compact, Then $f(X)$ is also.In the following theorem we give a similar result but with different assumptions.

Theorem2.4. Let $f: X \to Y$be a continuous bijective operator and X Lipschitz compact such that $Lip(X) = MI(X)$. Then Y is Lipschitz compact.

Proof. Assume that (y_n) be a sequence in Y.there is a sequence (x_n) in X such that $f(x_n) = y_n$. There is a nonzero real number λ so that $(\lambda x_n)(= (\lambda f^{-1}(y_n)))$ has a convergent subsequence, namely $(\lambda x_{n_k})(= (\lambda f^{-1}(y_{n_k})))$. It follows that (y_{n_k}) is convergent in Y.

III. Sequential Banach Space

In the section we define a new Banach space so-called sequential Banach space. First of all, we have a definition.

Definition 3.1. X be a linear space over F .A function$\|.\|: X \to [0, +\infty]$ is called a G-norm if it satisfies in the following conditions:

(i)$\|x\| = 0$if and only if $x = 0$;

(ii)$\|\alpha x\| = |\alpha|\|x\|$for all $x \in X$ and $\alpha \in F$;

(iii)$\|x + y\| \le \|x\| + \|y\|$for all $x, y \in X$.

Then $(X, \|.\|)$ is called a G-normed space.

Definition 3.2. Let X be a normed space over F. By $Se(X)$, we denote the set of sequences in X. $Se(X)$together with the following operations is a linear space:

$$(x_n) + (y_n) = (x_n + y_n)$$
$$\alpha(x_n) = (\alpha x_n)$$

For all (x_n),$(y_n) \in Se(X)$ and $\alpha \epsilon$F. $Se(X)$endowed with the following G-norm is a G-normed space:

$$\|(x_n)\| = \sup\{\|x_j\|: j \in \mathbb{N}\}$$

Definition 3.3. We consider a function f on the G-normed space $Se(X)$as $f = (f_1, f_2, \dots)$,wheref_i is a function on X, for every $i \in \mathbb{N}$.

Evidently, f is continuous if and only if any of $f_i s$ is continuous.

Definition3.4. We say that X is C-Lipschitz compact, when for each sequence (x_n)in X, there exist a nonzero contraction f on X such that $(f(x_n))$ has a convergent subsequence.

The next theorem is in the focus of our attention.

Theorem 3.5. If X is C-lipschitz compact, then $Se(X)$ is also.

Proof.Let (y_n) be a sequence in $Se(X)$. Assume that $(y_n) = (x_n{}^{(1)}, x_n{}^{(2)}, \dots, x_n{}^{(k)}, \dots)$.Since X is C-lipschitz compact, so there is a contraction f_1 on X such that $(f_1(x_n{}^{(1)}))$ has a covergent subsequence, say $(f_1(x_{n_{i_1}}{}^{(1)}))$.Simillary, there is a contraction f_2 on X such that $(f_2(x_n{}^{(2)}))$ has a covergent subsequence, say $(f_2(x_{n_{i_2}}{}^{(2)}))$..By continouuing the the process, one can find a sequence (f_k) of contraction on X .by setting $f = (f_1, f_2, \dots, f_k, \dots)$ we deduce that f is a contraction on $Se(X)$. Also

$$f(y_{n_i}) = (f_1(x_{n_{i_1}}{}^{(1)}, f_2(x_{n_{i_2}}{}^{(2)}, \dots, f_k\left(x_{n_{i_k}}{}^{(k)}\right), \dots)$$

Is a convergent subsequence of $f(y_n)$.

References

[1] T. Miyata and T. Watanabe, Lipschitz functions and approximate resolutions, Topol. Appl. 122 (2002), 353-375.

[2] M. Jouini, Generalized lipschitz functions. Nonlinear Anal. 41 (2000), 371-382.

[3] F. Albiac, Nonlinear structure of some classical quasi-Banach spaces and *F*-spaces, J. Math. Anal. Appl. 340 (2008), 1312-1325.

[4] N. Aronszajn, Differentiability of Lipschitzian mappings between Banach Spaces, Studia Math. 57(2) (1976), 147-190.

[5] J. Cheeger, Differentiability of Lipschitz functions on metric measure Spaces, Geom. Funct. Anal. 9(3) (1999), 428-517.

[6] F. Albiac and N. J. Kalton, Lipschitz structure of quasi-Banach spaces, Israel Journal of Mathematics 170 (2009), 317-335.

[7] P.W.Jones, Lipschitz and bi-Lipschitz Functions, Revista Mathematica Iberoamericana ,4(1),1988,115-121.

[8] E.Durand-Cartagena and J.A.Jaramillo,Pointwise Lipschitz functions on metric spaces,Journal of Mathematical Analysis and Applications ,363(2), 2010,525–548.

[9] M.I. Garrido, J.A. Jaramillo, Lipschitz-type functions on metric spaces, J. Math. Anal. Appl.**340** (2008), 282 – 290.

[10] I. Garrido, J.A. Jaramillo and Y.C. Rangel,Lip-density and Algebras of Lipschitz Functions on Metric Spaces,extracta mathematicae Vol. **25**, No. 3, 2010, 249-261.

[11] Z.Bahmani and S.Khorshidvandpour, Compactness in Lipschitz Spaces,Advances and Applications in Mathematical Sciences,12(7),2013,411-414.