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Characterization and Estimation of Weibull-Rayleigh Distribution with Applications to Life Time Data

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Abstract: In this paper, a new family of distributions called *T-X* distribution is defined. Some of its properties and special cases are discussed. A member of the family, namely, the three-parameter Weibull-Rayleigh distribution is defined and studied. Some of its properties including distribution shapes, limit behavior, hazard function, moments, and characteristic function are discussed. The method of maximum likelihood estimation, method of moments and L-moment estimator is used for estimating the model parameters and the observed Fisher's Information matrix is derived. The flexibility of the Weibull-Rayleigh distribution is assessed by applying it to the real data set and comparing it with other distributions.

Keywords: Weibull-Rayleigh distribution, Hazard function, Moments, Order statistics, Maximum likelihood estimation, R Software.

1 Introduction

Statistical distributions are important for parametric inferences and applications to fit real world phenomena. Although many distributions have been developed, there are many methods for generating statistical distributions in the literature. Some well-known methods in the early days for generating univariate continuous distributions include methods based on differential equations developed by Pearson [1], methods of translation developed by Johnson [2], and the methods based on quantile functions developed by Tukey [3]. The interest in developing new methods for generating new or more flexible distributions continues to be active in the modern decades. Lee et al. [4] indicated that the majority of methods developed after 1980s are the methods of 'combination' for the reason that these new methods are based on the idea of combining two existing distributions or by adding additional parameters to an existing distribution to generate a new family of distributions. As a result, many new families of distributions have been developed and studied by researchers.

Mudholkar and Srivastava [5] proposed the exponentiated Weibull distribution to analyze bathtub failure data. Gupta et al. [6] defined the exponentiated exponential distribution by taking $F(x)$ to be the cumulative distribution function (CDF) of an exponential distribution. The exponentiated Weibull distribution in Mudholkar and Srivastava [5] is a member of the class of exponentiated distributions by taking $F(x)$ to be the CDF of a Weibull distribution. Eugene et al. [7] introduced a new class of distributions generated from the beta distribution. The cumulative distribution function $F(x)$, the class of beta-generated distributions is defined as

$$G_B(x) = \frac{1}{B(\alpha,\beta)} \int_0^{F(x)} t^{\alpha-1}(1-t)^{\beta-1}\,dt$$

where X is any continuous random variable with CDF $F(x)$. Eugene et al. [2] developed and studied the beta-normal distribution by taking $F(x)$ to be the CDF of a normal distribution.

Alzaatreh, Lee and Famoye [8] proposed a method for generating new distributions, namely, the T-Xfamily. Let $r(t)$ be the PDF of a non-negative continuous random variable T defined on $[0,\infty)$, and let $F(x)$ denote the CDF of a random variable X. Then the CDF for the T-X class of distributions for a random variable X is

$$G(x) = \int_0^{-\log(1-F(x))} r(t)\,dt = R\{-\log(1-F(x))\} \quad (1)$$

where $R(t)$ is the CDF of the random variable T. The corresponding PDF of the exponentiated T-X distribution is given by

$$g(x) = \frac{f(x)}{1-F(x)}\, r\{-\log(1-F(x))\} \quad (2)$$

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In this new class, the distribution of the random variable *T* is the generator. The new family of distributions generated from (1) is called "*T-X* distribution". Alzaatreh, Famoye and Lee [9] defined the Weibull-Pareto distribution from (1) by taking $r(t)$ to be the Weibull distribution and $F(x)$ to be the Pareto distribution. Note that the upper limit for generating the *T-X* distribution is $-\log(1-F(x))$. It is clear that one can define a different upper limit for generating different types of *T-X* distributions. Some continuous distributions of the T-X families that have been studied are Weibull-exponential distribution (Alzaghal et al. [10]) and Kareema and Boshi [11], developed the Exponential Pareto distribution.

In this article we present a new generalization of the Rayleigh distribution called the Weibull-Rayleigh distribution. This Rayleigh model was first introduced by Rayleigh [12]. The Rayleigh distribution has a wide range of applications including life testing experiments, operations research reliability analysis, applied statistics, agriculture and clinical studies. This distribution is a special case of the two parameter Weibull distribution with the shape parameter equal to 2. Siddiqui [13] discussed the origin and properties of the Rayleigh distribution. Merovci et al [14] developed the transmuted Rayleigh distribution while Ahmad et al [15] studied the transmuted inverse Rayleigh distribution using Quadratic transmutation map and discussed some properties of this family. Several authors have contributed to this model, namely, Howlader and Hossian [16] and Abd Elfattah et al. [17]. The paper is outlined as follows. In Section 2, we define the cumulative, density and hazard functions of the Weibull-Rayleigh (WR) distribution. In Section 3, we introduced the statistical properties include, skewness and kurtosis, *rth* moment and moment generating function. The distribution of order statistics is expressed in Section 4. Finally, the Section 5 gives the estimation of the model parameters using Least squares and weighted least squares estimators, method of moments and the Maximum likelihood estimation.

2 T-X Weibull-Rayleigh Distribution

If the random variable *T* follows the Weibull distribution with parameter α and λ then its probability density function (PDF) is given as

$$r(t)=\frac{\alpha}{\lambda}\left(\frac{t}{\lambda}\right)^{\alpha-1}e^{\left(-\frac{t}{\lambda}\right)^{\alpha}}\ ;t>0,\alpha,\lambda>0 \qquad (3)$$

where α and λ are shape and scale parameter respectively. Thus by using the cumulative distribution function (CDF) of Weibull distribution the cdf of Weibull-X family using (1) is defined as

$$G(x)\ =1-\exp\left\{-\left(\frac{-\log(1-F(x))}{\lambda}\right)^{\alpha}\right\} \qquad (4)$$

and the corresponding probability density function is given by

$$g(x)=\frac{\alpha}{\lambda}\frac{f(x)}{1-F(x)}\left\{\frac{-\log(1-F(x))}{\lambda}\right\}^{\alpha-1}\exp\left\{-\left(\frac{-\log(1-F(x))}{\lambda}\right)^{\alpha}\right\} \qquad (5)$$

Thus the CDF of the Weibull-Rayleigh distribution (WRD) when *X* follows the Rayleigh distribution in equation (5) is given by

$$G(x)\ =1-\exp\left(-\left(\frac{x^2}{2\lambda\theta^2}\right)^{\alpha}\right) \qquad (6)$$

and the corresponding PDF of the Weibull-Rayleigh distribution is given by

$$g(x)=\frac{\alpha}{\lambda}\frac{x}{\theta^2}\left(\frac{x^2}{2\lambda\theta^2}\right)^{\alpha-1}\exp\left(-\left(\frac{x^2}{2\lambda\theta^2}\right)^{\alpha}\right) \qquad (7)$$

when $\alpha=1$ and $\lambda=1$, the WRD reduces to the Rayleigh distribution with parameter θ.

Figuer (1.1) and Figure (1.2) represents pdf's and cdf's of TX-Weibull-Rayleigh distribution

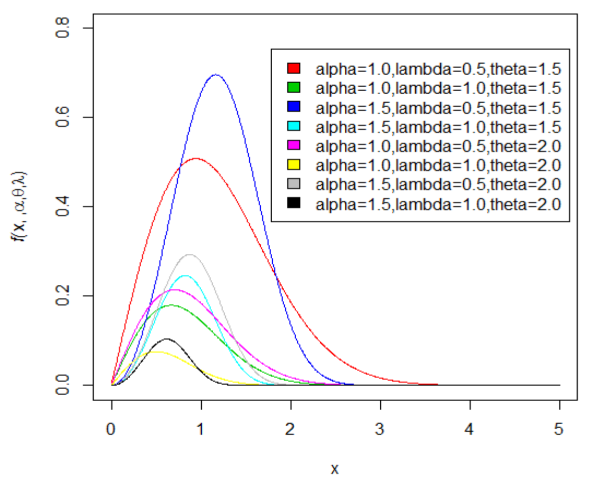

Figure 1.1: Pdf of TX-WR Distribution under different values to parameters

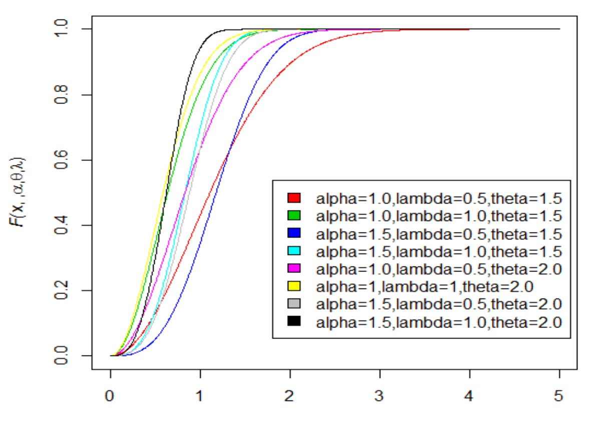

Figure 1.2: Cdf of TX-WR Distribution under different values to parameters

Figure 1.1 and 1.2 illustrates some of the possible shapes of Weibull-Rayleigh distribution for different values of the parameters α , θ and λ . Figure 1.1 shows that the density function of Weibull-Rayleigh is unimodal and, for fixed α and θ , it becomes more and more peaked as the value of λ is decreased.

2.1 Survival and hazard functions

The WRD can be a useful characterization of the survival time of a given system because of it analytical structure. The survival function is given by $S(x) = 1 - G(x)$. Thus using (6),

$$S(x) = \exp\left(-\frac{x^2}{2\lambda\theta^2}\right)^{\alpha} \qquad (8)$$

Another characteristics of interest of a random variable is the hazard function defined by

$$h(x) = \frac{g(x)}{S(x)}$$

Thus using (6) and (8), the hazard function is given by

$$h(x) = \frac{\alpha}{\lambda}\frac{x}{\theta^2}\left(\frac{x^2}{2\lambda\theta^2}\right)^{\alpha-1} \qquad (9)$$

From the hazard function the following can be observed:

By setting $\alpha = \lambda = 1$, the hazard function in (9) reduces to the hazard function of the Rayleigh distribution.

3 Statistical Properties

In this section, we present the statistical properties of WRD especially mean, variance, coefficient of variation, moment, Skewness, Kurtosis, Moment generating function and Characteristic function.

3.1 Moments

Theorem 3.1: If X is a random variable distributed as a $WRD(X;\alpha,\lambda,\theta)$, then the r^{th} non-central moment is given by

$$\mu_r' = (2\lambda)^{\frac{r}{2}}\theta^r\Gamma\left(\frac{r}{2\alpha}+1\right)$$

Proof: $\mu_r' = \int_0^\infty x^r g(x;\alpha,\theta,\lambda)\,dx$

$$= \int_0^\infty x^r \frac{\alpha}{\lambda}\frac{x}{\theta^2}\left(\frac{x^2}{2\lambda\theta^2}\right)^{\alpha-1} \exp\left(-\left(\frac{x^2}{2\lambda\theta^2}\right)^{\alpha}\right)$$

$$= \frac{\alpha}{\lambda\theta^2}\int_0^\infty x^{r+1}\left(\frac{x^2}{2\lambda\theta^2}\right)^{\alpha-1} \exp\left(-\left(\frac{x^2}{2\lambda\theta^2}\right)^{\alpha}\right)dx$$

Substituting $t = \left(\frac{x^2}{2\lambda\theta^2}\right)$, we have

$$\mu_r' = \alpha(2\lambda)^{\frac{r}{2}}\theta^r\int_0^\infty t^{\frac{r}{2}+\alpha-1}\exp(-t^\alpha)\,dx$$

$$\mu_r' = (2\lambda)^{\frac{r}{2}}\theta^r\Gamma\left(\frac{r}{2\alpha}+1\right) \qquad (10)$$

which completes the proof.

Substitute r=1,2 in equation (10) we get mean and variance for Weibull Rayleigh distribution.

Mean = $\mu_1' = \sqrt{2\lambda}\,\theta\,\Gamma\left(\frac{1}{2\alpha}+1\right)$ (11)

$$\mu_2' = 2\lambda\theta^2\,\Gamma\left(\frac{1}{\alpha}+1\right)$$

Variance = $\mu_2 = 2\lambda\theta^2\left(\Gamma\left(\frac{1}{\alpha}+1\right) - \left(\Gamma\left(\frac{1}{2\alpha}+1\right)\right)^2\right)$

By putting $\alpha = \lambda = 1$, in equation (11) we get mean of the Rayleigh distribution.

3.2 Moment generating function

In this sub section we derived the moment generating function of Weibull-Rayleigh distribution.

Theorem 3.2: If X has the $WRD(X;\alpha,\lambda,\theta)$ then the moment generating function $M_X(t)$ has the following form

$$M_X(t) = \sum_{j=0}^{\infty}\frac{t^j}{j!}(2\lambda)^{\frac{j}{2}}\theta^j\Gamma\left(\frac{j}{2\alpha}+1\right)$$

Proof: We begin with the well known definition of the moment generating function given by

$$M_X(t) = E\left(e^{tx}\right) = \int_0^\infty e^{tx}\, g(x;\alpha,\theta,\lambda)\,dx$$

$$= \int_0^\infty\left[1 + t\,x + \frac{(t\,x)^2}{2!} + ...\right] g(x;\alpha,\theta,\lambda)\,dx$$

$$= \int_0^\infty \sum_{j=0}^\infty \frac{t^j}{j!} x^j g(x;\alpha,\theta,\lambda)\,dx$$

$$= \sum_{j=0}^\infty \frac{t^j}{j!} \mu_j'$$

$$\Rightarrow M_X(t) = \sum_{j=0}^\infty \frac{t^j}{j!} (2\lambda)^{\frac{j}{2}} \theta^j \Gamma\left(\frac{j}{2\alpha}+1\right)$$

3.3 Characteristic function

In this sub section we derived the characteristic function of Weibull-Rayleigh distribution.

Theorem 3.3: If X has the $WRD(X;\alpha,\lambda,\theta)$ then the Characteristic function $\phi_X(t)$ has the following form

$$\varphi_X(t) = \sum_{j=0}^\infty \frac{(it)^j}{j!} (2\lambda)^{\frac{j}{2}} \theta^j \Gamma\left(\frac{j}{2\alpha}+1\right)$$

Proof: We begin with the well known definition of the characteristic function given by

$$\varphi_X(t) = E\left(e^{itx}\right) = \int_0^\infty e^{itx} g(x;\alpha,\theta,\lambda)\,dx$$

$$= \int_0^\infty \left[1 + it\,x + \frac{(it\,x)^2}{2!} + ...\right] g(x;\alpha,\theta,\lambda)\,dx$$

$$= \int_0^\infty \sum_{j=0}^\infty \frac{(it)^j}{j!} x^j g(x;\alpha,\theta,\lambda)\,dx$$

$$= \sum_{j=0}^\infty \frac{(it)^j}{j!} \mu_j'$$

$$\Rightarrow \phi_X(t) = \sum_{j=0}^\infty \frac{(it)^j}{j!} (2\lambda)^{\frac{j}{2}} \theta^j \Gamma\left(\frac{j}{2\alpha}+1\right)$$

4 Order Statistics

In this section, we derive closed form expressions for the pdfs of the k[th] order statistic of the Weibull-Rayleigh distribution. In statistics, the k[th] order statistic of a statistical sample is equal to its k[th] smallest value. Together with rank statistics, order statistics are among the most fundamental tools in non-parametric statistics and inference. We know that if $X_{(1)}, X_{(2)},..., X_{(n)}$ denotes the order statistics of a random sample $X_1, X_2,\dots, X_n$ from a continuous population with cdf $G_X(x)$ and pdf $g_X(x)$, then the pdf of rth order statistics $X_{(r)}$ is given by

$$G_{X(k)}(x) = \frac{n!}{(k-1)!(n-k)!} g_X(x) \left(G_X(x)\right)^{k-1} \left(1-G_X(x)\right)^{n-k} \quad (12)$$

Substituting equation (6) and equation (7) in equation (12), we get pdf of k[th] order statistic given as

$$g_{X(k)}(x) = \frac{n!}{(k-1)!(n-k)!} \frac{\alpha}{\lambda} \frac{x}{\theta^2} \left(\frac{x^2}{2\lambda\theta^2}\right)^{\alpha-1} \Im(x,\lambda,\theta) \left(1-\Im(x,\lambda,\theta)\right)^{k-1} \left(\Im(x,\lambda,\theta)\right)^{n-k} \quad (13)$$

where $\Im(x,\lambda,\theta) = \exp\left(-\left(\frac{x^2}{2\lambda\theta^2}\right)^\alpha\right)$

Note that at $\alpha = \lambda = 1$, (13) yields the pdf of the k[th] order statistic of Rayleigh distribution.

Therefore, the pdf of the first (smallest) order statistic $X_{(1)}$ is given by

$$g_{X(1)}(x) = \frac{n\alpha}{\lambda} \frac{x}{\theta^2} \left(\frac{x^2}{2\lambda\theta^2}\right)^{\alpha-1} \Im(x,\lambda,\theta) \left(\Im(x,\lambda,\theta)\right)^{n-1} \quad (14)$$

and the pdf of the largest order statistic $X_{(n)}$ is given by

$$g_{X(n)}(x) = \frac{n\alpha}{\lambda} \frac{x}{\theta^2} \left(\frac{x^2}{2\lambda\theta^2}\right)^{\alpha-1} \Im(x,\lambda,\theta) \left(1-\Im(x,\lambda,\theta)\right)^{n-1} \quad (15)$$

5 Estimation of Parameters

In this section, we discuss the various methods of estimation like moment method, L moment estimator, and Maximum likelihood estimation for Weibull-Rayleigh distribution and verifying their efficiencies.

5.1 Method of Moment Estimators

In this section, we study the method of moment estimators (MMEs) of the parameters of Weibull-Rayleigh distribution. If X follows WRD (α,λ,θ), then

$$\textbf{Mean} = \mu = \mu_1' = \sqrt{2\lambda}\,\theta\,\Gamma\left(\frac{1}{2\alpha}+1\right) \quad (16)$$

$$\mu_2' = 2\lambda\theta^2\,\Gamma\left(\frac{1}{\alpha}+1\right)$$

Variance =

$$\sigma^2 = \mu_2 = 2\lambda\theta^2\left(\Gamma\left(\frac{1}{\alpha}+1\right) - \left(\Gamma\left(\frac{1}{\alpha}+1\right)\right)^2\right) \quad (17)$$

It is well known that the principle of the moment's method is to equate the sample moments with the subsequent population moments.

From (16) and (17), we obtain the coefficient of variation (C.V) as

$$C.V=\frac{\sigma}{\mu}=\frac{\sqrt{\left(\Gamma\left(\frac{1}{\alpha}+1\right)-\left(\Gamma\left(\frac{1}{2\alpha}+1\right)\right)^{2}\right)}}{\Gamma\left(\frac{1}{2\alpha}+1\right)} \qquad (18)$$

The C.V. is independent of the parameters λ and θ. Therefore, equating the sample C.V. with the population C.V., we obtain

$$\frac{S}{\overline{X}}=\frac{\sqrt{\left(\Gamma\left(\frac{1}{\alpha}+1\right)-\left(\Gamma\left(\frac{1}{2\alpha}+1\right)\right)^{2}\right)}}{\Gamma\left(\frac{1}{2\alpha}+1\right)} \qquad (19)$$

Where $S^2=\frac{1}{n}\sum_{i=1}^{n}(X_i-\overline{X})^2$ and $\overline{X}=\frac{1}{n}\sum_{i=1}^{n}X_i$. We need to solve (19) to obtain the estimate of α, say $\hat{\alpha}_{MME}$. Once we estimate α, we can use (16) and (18) to estimate λ and θ say $\hat{\lambda}_{MME}$ and $\hat{\theta}_{MME}$

5.2 L-Moment Estimator

In this section, we propose a method of estimating the unknown parameter of Weibull- Rayleigh distribution by L-moments estimators, which can be obtained as the linear combination of order statistics. The L-moments estimators were originally proposed by Hosking [18], and it is observed that the L-moments estimators are more robust than the usual moment estimators. It is observed (see, Gupta and Kundu [19] that the LMEs have certain advantages over the conventional moment estimators.

The standard method to compute the L-moment estimators is same as the common moment estimators, i.e. to equate the sample L-moments with the population L-moments.

The first two sample L-moments are

$$l_1=\frac{1}{n}\sum_{i=1}^{n}x_{(i)} \quad \text{and}$$

$$l_2=\frac{2}{n(n-1)}\sum_{i=1}^{n}(i-1)x_{(i)}-l_1$$

and the first two population L-moments are

$$\lambda_1=E(X_{1:1})=E(X)$$

$$\lambda_2=\frac{1}{2}\left(E(X_{1:1})-E(X_{2:2})\right)=\int_{-\infty}^{\infty}x[2G(x)-1]\,g(x)\,dx$$

Thus for Weibull-Rayleigh distribution, we obtain

$$\lambda_1=\sqrt{2\lambda}\,\theta\,\Gamma\left(\frac{1}{2\alpha}+1\right)$$

$$\lambda_2=\sqrt{2\lambda}\,\theta\,\Gamma\left(\frac{1}{2\alpha}+1\right)\left(1-\frac{1}{2^{\frac{1}{2\alpha}}}\right)$$

Analogously to the usual method of moments, the L-moment method also consists of equating the first few population L-moments (λ_r) to the corresponding sample L-moments (l_r), thus obtaining as many equations as are needed to solve for the unknown population parameters,

i.e.

$$\lambda_r=l_r \quad r=1,2,..,p$$

for the p parameters.

Therefore, LMEs can be obtained by solving the following three equations. Substituting the values of the population L-moments λ_1, λ_2, and λ_3 for the sample L-moments, we get

$$l_1=\sqrt{2\lambda}\,\theta\,\Gamma\left(\frac{1}{2\alpha}+1\right) \qquad (20)$$

$$l_2=\sqrt{2\lambda}\,\theta\,\Gamma\left(\frac{1}{2\alpha}+1\right)\left(1-\frac{1}{2^{\frac{1}{2\alpha}}}\right) \qquad (21)$$

$$l_3=\sqrt{2\lambda}\,\theta\,\Gamma\left(\frac{1}{2\alpha}+1\right)\left(\frac{2}{3^{\frac{1}{2\alpha}}}-\frac{3}{2^{\frac{1}{2\alpha}}}+1\right) \qquad (22)$$

Solving these equations do not yield explicit solution for the estimates of parameters, we used the L-skewness measure to estimate α

First, we obtain the LME of α, say $\hat{\alpha}_{LME}$, as the solution of the following nonlinear equation

$$\frac{l_2}{l_1}=\left(\frac{1}{2^{\frac{1}{2\alpha}}}-1\right) \qquad (23)$$

Once $\hat{\alpha}_{LME}$ is obtained, the LME of λ and θ say $\hat{\lambda}_{LME}$

and $\hat{\theta}_{LME}$, can be obtained from (21) and (22)

5.3 Maximum Likelihood Estimation

Let $X_1, X_2, ..., X_n$ be a random sample of size n from Weibull-Rayleigh distribution. Then the likelihood function is given by

$$L=\prod_{i=1}^{n}\left\{\frac{\alpha}{\lambda}\frac{x_i}{\theta^2}\left(\frac{x_i^{\,2}}{2\lambda\theta^2}\right)^{\alpha-1}\exp\left(-\left(\frac{x_i^{\,2}}{2\lambda\theta^2}\right)^{\alpha}\right)\right\} \tag{24}$$

By taking logarithm of (24), we find the log likelihood function

$$\log L = n\log\alpha - n\log\lambda - 2n\log\theta + \sum_{i=1}^{n}\log x_i + (\alpha-1)\sum_{i=1}^{n}\log\left(\frac{x_i^{\,2}}{2\lambda\theta^2}\right) - \sum_{i=1}^{n}\left(\frac{x_i^{\,2}}{2\lambda\theta^2}\right)^{\alpha} \tag{25}$$

Therefore, the MLE of α, θ *and* λ which maximizes (25) must satisfy the following normal equations

$$\frac{\partial\log L}{\partial\alpha}=\frac{n}{\alpha}+\sum_{i=1}^{n}\log\left(\frac{x_i^{\,2}}{2\lambda\theta^2}\right)-\sum_{i=1}^{n}\left(\frac{x_i^{\,2}}{2\lambda\theta^2}\right)^{\alpha}\log\left(\frac{x_i^{\,2}}{2\lambda\theta^2}\right)$$

$$\frac{\partial\log L}{\partial\lambda}=-\frac{n}{\lambda}-\frac{n(\alpha-1)}{\lambda}+\frac{\alpha}{2\lambda^2\theta^2}\sum_{i=1}^{n}\left(\frac{x_i^{\,2}}{2\lambda\theta^2}\right)^{\alpha-1}x_i^{\,2}$$

$$\frac{\partial\log L}{\partial\theta}=-\frac{2n}{\theta}-\frac{2n(\alpha-1)}{\theta}+\frac{\alpha}{\lambda\theta^3}\sum_{i=1}^{n}\left(\frac{x_i^{\,2}}{2\lambda\theta^2}\right)^{\alpha-1}x_i^{\,2}$$

The solution of the non-linear system of equations obtained by differentiating equation (25) with respect to α, θ *and* λ gives the maximum likelihood estimates of the model parameters. The solution can also be obtained directly by using R software when data sets are available. In order to compute the standard error and asymptotic confidence interval we use the usual large sample approximation in which the maximum likelihood estimator of a parametercan be treated as being approximately multivariate normal. Hence as $n \to \propto$ the asymptotic distribution of the maximum likelihood estimators $(\hat{\alpha}, \hat{\theta}, \hat{\lambda})$ is given by

$$\begin{pmatrix}\hat{\alpha}\\ \hat{\theta}\\ \hat{\lambda}\end{pmatrix} \sim N\left[\begin{pmatrix}\alpha\\ \theta\\ \lambda\end{pmatrix},\begin{pmatrix}\hat{v}_{\alpha\alpha} & \hat{v}_{\alpha\theta} & \hat{v}_{\alpha\lambda}\\ \hat{v}_{\theta\alpha} & \hat{v}_{\theta\theta} & \hat{v}_{\theta\lambda}\\ \hat{v}_{\lambda\alpha} & \hat{v}_{\lambda\theta} & \hat{v}_{\lambda\lambda}\end{pmatrix}\right]$$

where $\begin{pmatrix}v_{\alpha\alpha} & v_{\alpha\theta} & v_{\alpha\lambda}\\ v_{\theta\alpha} & v_{\theta\theta} & v_{\theta\lambda}\\ v_{\lambda\alpha} & v_{\lambda\theta} & v_{\lambda\lambda}\end{pmatrix}=\begin{pmatrix}I_{\alpha\alpha} & I_{\alpha\theta} & I_{\alpha\lambda}\\ I_{\theta\alpha} & I_{\theta\theta} & I_{\theta\lambda}\\ I_{\lambda\alpha} & I_{\lambda\theta} & I_{\lambda\lambda}\end{pmatrix}^{-1}$ is the approximate variance covariance matrix with its elements obtained from

$$I_{\alpha\alpha}=-E\left(\frac{\partial^2}{\partial\alpha^2}\log L\right)\quad I_{\alpha\theta}=-E\left(\frac{\partial^2}{\partial\alpha\,\partial\theta}\log L\right)$$

$$I_{\alpha\lambda}=-E\left(\frac{\partial^2}{\partial\alpha\,\partial\lambda}\log L\right)\quad I_{\theta\theta}=-E\left(\frac{\partial^2}{\partial\theta^2}\log L\right)$$

$$I_{\theta\lambda}=-E\left(\frac{\partial^2}{\partial\theta\,\partial\lambda}\log L\right)\quad I_{\lambda\lambda}=-E\left(\frac{\partial^2}{\partial\lambda^2}\log L\right)$$

6 Applications

To compare the flexibility of the Weibull-Rayleigh distribution over the well-known Rayleigh and sub models, two real data sets are used and analysis performed with the help of R software.

Data set I: The first data set (n=63) is on the strengths of 1.5 cm glass fibres. The data was originally obtained by workers at the UK National Physical Laboratory and it has been used by Smith and Naylor [20] and Bourguignon *et al*. [21] applied the Weibull G family to fit the data. The data is as follows:

0.55, 0.74, 0.77, 0.81, 0.84, 0.93, 1.04, 1.11, 1.13, 1.24, 1.25, 1.27, 1.28, 1.29, 1.30, 1.36, 1.39, 1.42, 1.48, 1.48, 1.49, 1.49, 1.50, 1.50, 1.51, 1.52, 1.53, 1.54, 1.55, 1.55, 1.58, 1.59, 1.60, 1.61, 1.61, 1.61, 1.61, 1.62, 1.62, 1.63, 1.64, 1.66, 1.66, 1.66, 1.67, 1.68, 1.68, 1.69, 1.70, 1.70, 1.73, 1.76, 1.76, 1.77, 1.78, 1.81, 1.82, 1.84, 1.84, 1.89, 2.00, 2.01, 2.24.

The summary of the data is given in Table 1. The MLEs of WRD parameters and the goodness of fit statistics are reported in Table 2.

Table 2, displays the Maximum Likelihood estimates of the model parameters. It was obvious that T-X Weibull-Rayleigh provides a better fit as compared to other Rayleigh models since it has lowest value of -2logL, Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC). Hence, the Weibull Rayleigh distribution performed better than other generalizations of Rayleigh distribution. The distribution of the data is skewed to the left with skewness -0.878. This shows that T-X Weibull-Rayleigh has the ability to fit the left skewed data.

Table 1: Data summary for strength of 1.5 cm glass fibres

Statistics	Values	Statistics	Values
n	63	Median	1.590
Minimum	0.550	Maximum	2.240
First Quartile	1.375	Third Quartile	1.685
Mean	1.507	Variance	0.105
Skewness	-0.878	Kurtosis	3.923

Table 2: Estimates and Performance of the distributions

Distribution	Parameters (S.E)	-2log$L$	AIC	BIC
Weibull Rayleigh distribution	α =2.8903 (0.2880) λ =0.3926 (2.7012) θ =1.8372 (6.3200)	30.4136	36.4136	47.2096
Rayleigh distribution	θ =1.0894 (0.0686)	99.5817	101.5817	103.1803
Exponentiated Rayleigh distribution	α =2.4564 (155.247) θ =2.3925 (151.208) λ =5.4859 (1.1848)	47.8575	53.8575	58.6535
Transmuted Rayleigh distribution	α =2.1673 (76.1721) θ =1.3342 (57.3636) λ =-1.000 (0.3813)	54.3135	57.3135	65.2307

Data set II: The data set is on the breaking stress of carbon fibres of 50 mm length (GPa). The data has been previously used by Cordeiro and Lemonte [22] and Al-Aqtash *et al.* [23]. The data is as follows:

0.39, 0.85, 1.08, 1.25, 1.47, 1.57, 1.61, 1.61, 1.69, 1.80, 1.84, 1.87, 1.89, 2.03, 2.03, 2.05, 2.12, 2.35, 2.41, 2.43, 2.48, 2.50, 2.53, 2.55, 2.55, 2.56, 2.59, 2.67, 2.73, 2.74, 2.79, 2.81, 2.82, 2.85, 2.87, 2.88, 2.93, 2.95, 2.96, 2.97, 3.09, 3.11, 3.11, 3.15, 3.15, 3.19, 3.22, 3.22, 3.27, 3.28, 3.31, 3.31, 3.33, 3.39, 3.39, 3.56, 3.60, 3.65, 3.68, 3.70, 3.75, 4.20, 4.38, 4.42, 4.70, 4.90.

The summary of the data is given in Table 3. The MLEs of WRD parameters and the goodness of fit statistics are reported in Table 4.

Table 4, displays the Maximum Likelihood estimates of the model parameters. It was obvious that T-X Weibull-Rayleigh provides a better fit as compared to other Rayleigh models since it has lowest value of -2logL, Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC). Hence, the Weibull Rayleigh distribution performed better than other generalizations of Rayleigh distribution. The distribution of the data is skewed to the left with skewness -0.128. This shows that T-X Weibull-Rayleigh has the ability to fit the left skewed data.

Table 3: Data summary

Statistics	Values	Statistics	Values
N	66	Median	2.835
Minimum	0.390	Maximum	4.900
First Quartile	2.178	Third Quartile	3.278
Mean	2.760	Variance	0.794
Skewness	-0.128	Kurtosis	3.222

Table 4: Estimates and Performance of the distributions

Distribution	Parameters (S.E)	-2log$L$	AIC	BIC
Weibull Rayleigh distribution	α =1.7205 (0.1654) λ =1.2534 (23.5972) θ =1.9340 (18.2054)	172.1352	178.1352	189.0524
Rayleigh distribution	θ =2.0491 (0.1261)	196.4168	198.4168	202.0558
Exponentiated Rayleigh distribution	α =2.1908 (51.5533) θ =0.4205 (9.8959) λ =2.3483 (0.4311)	177.2735	183.2735	188.1907
Transmuted Rayleigh distribution	α =1.6653 (27.3984) λ =0.2844 (5.1687) θ =-0.9587 (0.0929)	177.7488	183.7488	188.666

7 Conclusion

In this article we propose a new model of T-X family, called the Weibull-Rayleigh distribution which extends the Rayleigh distribution in the analysis of data with real support. An obvious reason for generalizing a standard distribution is because the generalized form provides larger flexibility in modeling real data. We derive expansions for the moments, moment generating function, characteristic function and Order statistics. The estimation of parameters is approached by the method of L-moment estimator, method of moments and maximum likelihood estimation. An application of the Weibull-Rayleigh distribution to real data is provided which show that the new distribution can be used quite effectively to provide better fits than the Rayleigh distribution, Exponentiated Rayleigh distribution and Transmuted Rayleigh distribution.

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