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Bayes Estimation of the Scale Parameter of the Logistic Distribution Based on Progressive First-Failure Censored Scheme

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Abstract: In this paper we develop approximate Bayes estimators of the scale parameter of the logistic distribution, based on a new life test plan called a progressive first-failure censored plan introduced by [22]. We consider the maximum likelihood and Bayesian inference of the unknown parameter of the model, as well as the reliability and hazard rate functions. Lindley's approximation [12] and Markov Chain Monte Carlo (MCMC) methods such as importance sampling procedure are applied. The Bayes estimators have been obtained relative to both symmetric (squared error) and asymmetric (linex and general entropy) loss functions. Finally, to assess the performance of the proposed estimators, some numerical results using Monte Carlo simulation study were reported.

Keywords: Logistic distribution, progressive first-failure censored, general uniform distribution, loss functions, Lindley's approximation, importance sampling technique.

1 Introduction

The logistic function is one of the most popular and widely used for growth models in demographic studies. The logistic distribution arises frequently in statistical modelling. It has been used in the analysis of survival data, graduation of mortality statistics and is used in some applications as a substitute for the normal distribution [3]. The logistic distribution has been applied in studies of population growth, physicochemical phenomena, bio-assay and a life test data, see [1]. [18] compared the logistic distribution and weibull distribution for modeling wind speed data. Many researchers have used asymmetric loss function applied to several statistical models ([6] and [20]). Life testing experiments are usually time consuming and costly. We therefore, use various types of censoring schemes to cut short the experiment. The censoring scheme in an experiment may also arise naturally without the control of the experimenter. For example, in medical studies a patient may drop out of a study before its completion. Initially, the popular censoring schemes were conventional type I and type II. They had a drawback that an item could not be withdrawn before the completion of the experiment. See [15] and [19]. [7] thought over this point and introduced progressive type II censoring scheme which allows removal of items from the experiment before the final termination point. [2] compiled the work done on progressive censoring up-to year 1999. Progressive censoring has also been studied by many authors like [16], [9], [10] and references cited therein. There are situations in real life where lifetimes of items are very high and test facilities are limited. If the test material is comparatively cheaper, one can put $k \times n$ items on test instead of only n units. In this case n sets or groups each consisting of k items are put on test separately. In each set only first-failure is observed and the progressive censoring is applied to n groups. [8] studied this type of grouping of units and observing only first-failure. Some other studies on first-failure are by [5], [21] and [23]. The combination of first-failure and progressive censoring is known as progressive first-failure censoring scheme. This concept was given by [22]. They described estimation methods in case of a Weibull distribution using this new censoring plan. More recent references can be found in [13], [11] and [14]. We shall now describe the progressive first-failure censoring scheme. Assume that $k \times n$ items are put on test in n independent groups with k items in each group. We prefix the progressive censoring scheme $\underline{R} = (R_1, R_2, ..., R_m)$. Upon the first-failure of

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a unit, we remove that group in which first-failure occurred and R_1 additional groups randomly from the remaining $(n-1)$ groups in the experiment. As soon as second failure takes place we remove that group and additional R_2 groups randomly from remaining $(n-R_1-2)$ groups and so on. This procedure continues till the $m-^{th}$ failure occurs when the remaining R_m groups and the group in which last failure took place are removed. Obviously, $\sum_{i=1}^{m} R_i + m = n$. Also, if $R_1 = R_2 = ... = R_m = 0$, the progressive first-failure censoring scheme reduces to first-failure censoring scheme and if $R_1 = R_2 = ... = R_{m-1} = 0$ and $R_m = n-m$, it reduces to first-failure type II censoring scheme, a progressive type II censored scheme when $k = 1$. Also, it should be noted that the progressive first-failure censored scheme with distribution function $F(x)$, can be viewed as a progressive type II censored sample from a population with distribution function $1-(1-F(x))^k$. For this reason, results for the progressive type II censored scheme can be extended to progressive first-failure censored scheme easily. Therefore, progressive first-failure censoring is a generalization of progressive censoring. Obviously, although more items are used (only m of $k \times n$ items are failures) in the progressive first-failure censoring plan than in others, it has advantages in terms of reducing test cost and test time. Let $x_{1:m:n:k}, x_{2:m:n:k}, ..., x_{m:m:n:k}$ be a progressive first-failure censored sample from a population with pdf $f(.)$ and distribution function $F(.)$ with progressive censoring scheme $\underline{R}$. Let us denote $(x_{1:m:n:k}, x_{2:m:n:k}, ..., x_{m:m:n:k})$ by $\underline{x} = (x_1, x_2, ..., x_m)$. On the basis of a progressive first-failure censored sample $\underline{x}$ the likelihood function is given by [see [2] and [22]]

$$\mathrm{L}(\underline{x}) = Ak^m \prod_{i=1}^{m} \mathrm{f}(\mathrm{x_i})\left[1-\mathrm{F}(\mathrm{x_i})\right]^{(k(\mathrm{R_i}+1)-1)}, \tag{1.1}$$

where $A = n(n-1-R_1)...(n-R_1-...-R_{m-1}-m+1)$. The main aim of this paper is to approximate Bayes estimators of the scale parameter of the logistic distribution. Progressive first-failure censoring schemes using both MLEs and Bayesian approaches are obtained. The organization of the paper is as follows: Sect. 2 deals with maximum likelihood estimation of the unknown parameter, as well as the reliability and hazard rate functions. For the computation of Bayes estimates we use Lindley's approximation and importance sampling procedure in Sect. 3. In Sect. 4, a Monte Carlo simulation study is performed for comparisons of various estimates developed in this paper. Concluding remarks are given in Sect. 5.

2 Maximum Likelihood Estimators (MLEs)

In this section, we derive the MLEs of the unknown parameter β based on progressive first-failure censored samples. Assuming that the location parameter μ is known. Assume the failure time distribution to be the logistic distribution with probability density function (pdf)

$$f(x;\mu,\beta) = \frac{e^{-\frac{(x-\mu)}{\beta}}}{\beta\left(1+e^{-\frac{(x-\mu)}{\beta}}\right)^2}; -\infty < x < \infty, \quad -\infty < \mu < \infty, \beta > 0, \tag{2.1}$$

and the corresponding cumulative distribution function (cdf) is given by

$$F(x;\mu,\beta) = \frac{1}{1+e^{-\frac{(x-\mu)}{\beta}}}. \tag{2.2}$$

From (1.1), (2.1) and (2.2), the likelihood function is given by

$$\mathrm{L}(\underline{x};\mu,\beta) = Ak^m \prod_{i=1}^{m} \left(\frac{\mathrm{e}^{\frac{-(x_i-\mu)}{\beta}} \left(\frac{\mathrm{e}^{\frac{-(x_i-\mu)}{\beta}}}{1+\mathrm{e}^{\frac{-(x_i-\mu)}{\beta}}} \right)^{(k(R_i+1)-1)}}{\beta\left(1+\mathrm{e}^{\frac{-(x_i-\mu)}{\beta}}\right)^2} \right). \tag{2.3}$$

The logarithm of the likelihood function may then be written as

$$\log[L]=\ell=\log[A]+m\log[k]+\log\left[\prod_{i=1}^{m}\left(\frac{e^{\frac{-(x_i-\mu)}{\beta}}\left(\frac{e^{\frac{-(x_i-\mu)}{\beta}}}{1+e^{\frac{-(x_i-\mu)}{\beta}}}\right)^{(k(R_i+1)-1)}}{\beta\left(1+e^{\frac{-(x_i-\mu)}{\beta}}\right)^2}\right)\right].$$

$$\ell=\log[A]+m\log[k]+\sum_{i=1}^{m}\left(\begin{array}{l}-\frac{(x_i-\mu)}{\beta}+\\ (k(R_i+1)-1)\left(\begin{array}{l}-\frac{(x_i-\mu)}{\beta}-\\ \log\left[1+e^{\frac{-(x_i-\mu)}{\beta}}\right]\end{array}\right)-\\ \log[\beta]-2\log\left[1+e^{\frac{-(x_i-\mu)}{\beta}}\right]\end{array}\right). \tag{2.4}$$

Calculating the first partial derivative of (2.4) with respect to β and equating to zero, we obtain the likelihood equation

$$\sum_{i=1}^{m}\left(\begin{array}{l}-\frac{1}{\hat\beta}+\frac{(x_i-\mu)}{\hat\beta^2}-\frac{2e^{-\frac{(x_i-\mu)}{\hat\beta}}(x_i-\mu)}{\left(1+e^{-\frac{(x_i-\mu)}{\hat\beta}}\right)\hat\beta^2}+\\ (k(1+R_i)-1)\left(\frac{(x_i-\mu)}{\hat\beta^2}-\frac{e^{-\frac{(x_i-\mu)}{\hat\beta}}(x_i-\mu)}{(1+e^{-\frac{(x_i-\mu)}{\hat\beta}})\hat\beta^2}\right)\end{array}\right)=0. \tag{2.5}$$

The solution of the non-linear equation (2.5) is $\hat\beta$. The MLEs of the reliability function and the hazard rate function are given as

$$\hat R(t)=\frac{1}{1+e^{\frac{t-\mu}{\hat\beta}}}\quad,\quad \hat H(t)=\frac{1}{\hat\beta\left(1+e^{-\frac{(t-\mu)}{\hat\beta}}\right)}\quad;t>0.$$

3 Bayesian estimation

In this section, the Bayesian estimator of the unknown parameter β of the logistic distribution is obtained. Also the reliability function and hazard rate function, based on progressive first-failure censored samples, under symmetric (squared error) and asymmetric (linex and general entropy) loss functions, using Lindley?s approximation and Markov Chain Monte Carlo (MCMC) methods such as importance sampling procedure, are obtained.

3.1 Bayes Estimates Using Non-informative Prior Distribution

Assuming that μ is known (fixed) and the scale parameter β is a random variable, with a non-informative "general uniform" prior distribution in the form

$$\pi(\beta;\alpha,\gamma,\lambda)=\frac{(\lambda-1)(\alpha\gamma)^{\lambda-1}}{\beta^{\lambda}\left(\gamma^{\lambda-1}-\alpha^{\lambda-1}\right)};\ 0<\alpha\le\beta\le\gamma<\infty,\ \lambda\ge0,\ \lambda\ne1. \tag{3.1}$$

By using equations (2.3) and (3.1) we get the posterior distribution of β as follow

$$\pi(\beta|\underline{x};\mu)=\frac{\pi(\beta)\,\mathrm{L}(\underline{x};\mu|\beta)}{\int_0^\infty\pi(\beta)\,\mathrm{L}(\underline{x};\mu|\beta)\,d\beta}$$

$$= \left(\frac{1}{\beta^{\lambda}} \prod_{i=1}^{m} \left(\frac{e^{\frac{-(x_i-\mu)}{\beta}} \left(\frac{e^{\frac{-(x_i-\mu)}{\beta}}}{1+e^{\frac{-(x_i-\mu)}{\beta}}} \right)^{(k(R_i+1)-1)}}{\beta \left(1+e^{\frac{-(x_i-\mu)}{\beta}}\right)^2} \right) \right) \times \left(\int_0^{\infty} \left(\frac{1}{\beta^{\lambda}} \prod_{i=1}^{m} \left(\frac{e^{\frac{-(x_i-\mu)}{\beta}} \left(\frac{e^{\frac{-(x_i-\mu)}{\beta}}}{1+e^{\frac{-(x_i-\mu)}{\beta}}} \right)^{(k(R_i+1)-1)}}{\beta \left(1+e^{\frac{-(x_i-\mu)}{\beta}}\right)^2} \right) \right) d\beta \right)^{-1}. \quad (3.2)$$

Integration in equation (3.2) cannot be obtained in a closed form, so we solve it numerically. In the following subsections we derive Bayesian estimators for scale parameter, the reliability function and the hazard rate function under different loss functions.

3.1.1 Bayesian Estimators Under Square Error Loss Function

1. Bayesian estimator of the scale parameter β

$$\hat{\beta}_{\text{sq}} = \mathrm{E}(\beta) = \int_0^{\infty} \left(\beta \times \left(\frac{1}{\beta^{\lambda}} \prod_{i=1}^{m} \left(\frac{e^{\frac{-(x_i-\mu)}{\beta}} \left(\frac{e^{\frac{-(x_i-\mu)}{\beta}}}{1+e^{\frac{-(x_i-\mu)}{\beta}}} \right)^{(k(R_i+1)-1)}}{\beta \left(1+e^{\frac{-(x_i-\mu)}{\beta}}\right)^2} \right) \right) \times \left(\int_0^{\infty} \left(\frac{1}{\beta^{\lambda}} \prod_{i=1}^{m} \left(\frac{e^{\frac{-(x_i-\mu)}{\beta}} \left(\frac{e^{\frac{-(x_i-\mu)}{\beta}}}{1+e^{\frac{-(x_i-\mu)}{\beta}}} \right)^{(k(R_i+1)-1)}}{\beta \left(1+e^{\frac{-(x_i-\mu)}{\beta}}\right)^2} \right) \right) d\beta \right)^{-1} \right) d\beta. \quad (3.3)$$

Provided that $E(\beta)$ exists and is finite. This integration cannot be solved analytically, so we use Lindley's Bayes approximation [12]. Let $u(\beta)$ be a function of β, and we want to find Bayes estimator for it, based on $\pi(\beta)$ as a prior distribution. The log-likelihood function for the logistic distribution based on progressive first-filure censored samples is given by (2.4), Bayes estimate of $u(\beta)$ using Lindley approximation is obtained as follows:

$$E(u(\beta)|\underline{x};\mu) = \int_0^{\infty} \left(\frac{u(\beta)\pi(\beta)\mathrm{L}(\underline{x};\mu|\beta)}{\int_0^{\infty} \pi(\beta)\mathrm{L}(\underline{x};\mu|\beta)\,d\beta} \right) d\beta.$$

Let $Q(\beta) = \log[\pi(\beta)]$, then

$$E(u(\beta)|\underline{x}) \approx \left(u(\beta) + u_1 Q_1 \tau_{11} + \frac{1}{2} u_{11} \tau_{11} + \frac{1}{2} L_{111} u_1 {\tau_{11}}^2 \right)_{(\beta)_{ML}}, \quad (3.4)$$

where

$$Q_1 = \frac{\partial Q(\beta)}{\partial \beta}, u_1 = \frac{\partial u(\beta)}{\partial \beta}, u_{11} = \frac{\partial^2 u(\mu,\beta)}{\partial \beta^2}, L_{11} = \frac{\partial^2 \ell}{\partial \beta^2}, L_{111} = \frac{\partial^3 \ell}{\partial \beta^3}, \tau_{11} = -(L_{11})^{-1}.$$

Substitution in equation (3.4), $u = \beta$, the Bayesian estimator of the scale parameter β is given as

$$\hat{\beta}_{sq} \simeq \left(\beta - \frac{\lambda}{\beta} \tau_{11} + \frac{1}{2} L_{111} {\tau_{11}}^2 \right).$$

2. Bayesian estimator of the reliability function $R(t)$

Substitution in equation (3.4), $u = R(t)$, the Bayesian estimator of the reliability function $R(t)$ is given by

$$\hat{R}_{sq} \simeq \left(\frac{1}{\left(1+\mathrm{e}^{\frac{(t-\mu)}{\beta}}\right)} - \frac{\lambda}{\beta} u_1 \tau_{11} + \frac{1}{2} u_{11} \tau_{11} + \frac{1}{2} L_{111} u_1 {\tau_{11}}^2 \right).$$

3. Bayesian estimator of the hazard rate function $H(t)$

Substitution in equation (3.4), $u = H(t)$, the Bayesian estimator of the hazard rate function $H(t)$ is given by

$$\hat{H}_{sq} \simeq \left(\frac{1}{\beta\left(1+\mathrm{e}^{\frac{-(t-\mu)}{\beta}}\right)} - \frac{\lambda}{\beta} u_1 \tau_{11} + \frac{1}{2} u_{11} \tau_{11} + \frac{1}{2} L_{111} u_1 {\tau_{11}}^2 \right).$$

3.1.2 Bayesian Estimators Under Linear-Exponential Loss Function (LINEX)

1. Bayesian estimator of the scale parameter β

$\hat{\beta}_{LINEX} = -\frac{1}{c} \log\left[E\left(e^{-c\beta}\right)\right].$
Provided that $E\left(e^{-c\beta}\right)$ exists and is finite. Substituting in equation (3.4), $u = e^{-c\beta}$, the Bayesian estimator of the scale parameter β is given as

$$\hat{\beta}_{LINEX} \simeq -\frac{1}{c} \log\left[\mathrm{e}^{-c\beta} + \frac{c\lambda \tau_{11} \mathrm{e}^{-c\beta}}{\beta} + \frac{c^2 \mathrm{e}^{-c\beta} \tau_{11} - cL_{111} {\tau_{11}}^2 \mathrm{e}^{-c\beta}}{2} \right].$$

2. Bayesian estimator of the reliability function $R(t)$

Substitution in equation (3.4), $u = e^{-cR(t)}$, the Bayesian estimator of the reliability function $R(t)$ is given by

$$\hat{R}_{LINEX} \simeq -\frac{1}{c} \log\left[\mathrm{e}^{-c\frac{1}{\left(1+\mathrm{e}^{\frac{(t-\mu)}{\beta}}\right)}} - \frac{\lambda u_1 \tau_{11}}{\beta} + \frac{u_{11} \tau_{11} + L_{111} u_1 {\tau_{11}}^2}{2} \right].$$

3. Bayesian estimator of the hazard rate function $H(t)$
substitution in equation (3.4), $u = e^{-cH(t)}$, the Bayesian estimator of the hazard rate function $H(t)$ is given by

$$\hat{H}_{LINEX} \simeq -\frac{1}{c} \log\left[\mathrm{e}^{-c\frac{1}{\beta\left(1+\mathrm{e}^{\frac{-(t-\mu)}{\beta}}\right)}} - \frac{\lambda u_1 \tau_{11}}{\beta} + \frac{u_{11} \tau_{11} + L_{111} u_1 {\tau_{11}}^2}{2} \right].$$

3.1.3 Bayesian Estimators Under General Entropy Loss Function

1. Bayesian estimator of the scale parameter β

$\hat{\beta}_{Gentropy} = \left[E\left(\beta^{-q}\right)\right]^{-\frac{1}{q}}.$

Provided that $E\left(\beta^{-q}\right)$ exists and is finite. Substitution in equation (3.4), $u = \beta^{-q}$, the Bayesian estimator of the scale parameter β is given by

$$\hat{\beta}_{Gentropy} \simeq \left(\beta^{-q} + \frac{\lambda q \beta^{-1-q} \tau_{11}}{\beta} + \frac{q(1+q)\beta^{-2-q}\tau_{11} - q\beta^{-1-q} L_{111} {\tau_{11}}^2}{2} \right)^{-\frac{1}{q}}.$$

2. Bayesian estimator of the reliability function $R(t)$

Substitution in equation (3.4), $u=(R(t))^{-q}$, the Bayesian estimator of the reliability function $R(t)$ is given by

$$\hat{R}_{Gentropy} \simeq \left(\left(\frac{1}{1+e^{\frac{(t-\mu)}{\beta}}}\right)^{-q} - \frac{\lambda u_1 \tau_{11}}{\beta} + \frac{u_{11}\tau_{11} + L_{111}u_1\tau_{11}^{2}}{2}\right)^{-\frac{1}{q}}.$$

3. Bayesian estimator of the hazard rate function $H(t)$

Substitution in equation (3.4), $u=(H(t))^{-q}$, the Bayesian estimator of the hazard rate function $H(t)$ is given by

$$\hat{H}_{Gentropy} \simeq \left(\left(\frac{1}{\beta\left(1+e^{\frac{-(t-\mu)}{\beta}}\right)}\right)^{-q} - \frac{\lambda u_1 \tau_{11}}{\beta} + \frac{u_{11}\tau_{11} + L_{111}u_1\tau_{11}^{2}}{2}\right)^{-\frac{1}{q}}.$$

It is worth noting that when the value $q=-1$, the general entropy loss function is the same as the squared error loss function.

3.2 Bayes Estimates Using Informative Prior Distribution

Assuming that the informative prior distribution for the scale parameter β is a inverse gamma distribution, given by

$$\pi(\beta;\eta,\nu) = \frac{e^{-\frac{\nu}{\beta}}\left(\frac{\nu}{\beta}\right)^{\eta}}{\beta\Gamma(\eta)}, \ \beta>0, \ (\eta, \ \nu>0). \tag{3.5}$$

By using equations (2.3) and (3.5) we get the posterior distribution of β as follows

$$\pi(\beta|\underline{x},\mu) = \left(\frac{e^{-\frac{\nu}{\beta}}}{(\beta)^{\eta+1}}\prod_{i=1}^{m}\left(\frac{e^{\frac{-(x_i-\mu)}{\beta}}\left(\frac{e^{\frac{-(x_i-\mu)}{\beta}}}{1+e^{\frac{-(x_i-\mu)}{\beta}}}\right)^{(k(R_i+1)-1)}}{\beta\left(1+e^{\frac{-(x_i-\mu)}{\beta}}\right)^{2}}\right)\right)\times \left(\int_0^\infty\left(\frac{e^{-\frac{\nu}{\beta}}}{(\beta)^{\eta+1}}\prod_{i=1}^{m}\left(\frac{e^{\frac{-(x_i-\mu)}{\beta}}\left(\frac{e^{\frac{-(x_i-\mu)}{\beta}}}{1+e^{\frac{-(x_i-\mu)}{\beta}}}\right)^{(k(R_i+1)-1)}}{\beta\left(1+e^{\frac{-(x_i-\mu)}{\beta}}\right)^{2}}\right)\right)d\beta\right)^{-1}. \tag{3.6}$$

Integration in equation (3.6) cannot be obtained in a closed form, so we solve it numerically. In the following subsections we derive Bayesian estimators for the scale parameter β, the reliability function and the hazard rate function under different loss functions.

3.2.1 Bayesian Estimators Under Square Error Loss Function

1. Bayesian estimator of the scale parameter β

$$\hat{\beta}_{\text{sq}} = \mathrm{E}(\beta) = \int_0^\infty \left(\beta \times \left(\frac{e^{-\frac{\nu}{\beta}}}{(\beta)^{\eta+1}} \prod_{i=1}^{m} \left(\frac{e^{\frac{-(x_i-\mu)}{\beta}} \left(\frac{e^{\frac{-(x_i-\mu)}{\beta}}}{1+e^{\frac{-(x_i-\mu)}{\beta}}} \right)^{(k(R_i+1)-1)}}{\beta \left(1+e^{\frac{-(x_i-\mu)}{\beta}} \right)^2} \right) \right) \times \left(\int_0^\infty \left(\frac{e^{-\frac{\nu}{\beta}}}{(\beta)^{\eta+1}} \prod_{i=1}^{m} \left(\frac{e^{\frac{-(x_i-\mu)}{\beta}} \left(\frac{e^{\frac{-(x_i-\mu)}{\beta}}}{1+e^{\frac{-(x_i-\mu)}{\beta}}} \right)^{(k(R_i+1)-1)}}{\beta \left(1+e^{\frac{-(x_i-\mu)}{\beta}} \right)^2} \right) \right) d\beta \right)^{-1} \right) d\beta. \tag{3.7}$$

Provided that $E(\beta)$ exists and is finite. This integration cannot be solved analytically, so we use Lindley's Bayes approximation (3.4). Substitution in equation (3.4), $u = \beta$, the Bayesian estimator of the scale parameter β is given as

$$\hat{\beta}_{sq} \simeq \left(\beta + \left(\frac{\nu - (1+\eta)\beta}{\beta^2} \right) \tau_{11} + \frac{1}{2} L_{111} \tau_{11}{}^2 \right).$$

2. Bayesian estimator of the reliability function $R(t)$

Substitution in equation (3.4), $u = R(t)$, the Bayesian estimator of the reliability function $R(t)$ is given by

$$\hat{R}_{sq} \simeq \left(\frac{1}{\left(1+e^{\frac{(t-\mu)}{\beta}}\right)} + \left(\frac{\nu-(1+\eta)\beta}{\beta^2} \right) u_1 \tau_{11} + \frac{u_{11}\tau_{11} + L_{111} u_1 \tau_{11}{}^2}{2} \right).$$

3. Bayesian estimator of the hazard rate function $H(t)$

Substitution in equation (3.4), $u = H(t)$, the Bayesian estimator of the hazard rate function $H(t)$ is given by

$$\hat{H}_{sq} \simeq \left(\frac{1}{\beta\left(1+e^{\frac{-(t-\mu)}{\beta}}\right)} + \left(\frac{\nu-(1+\eta)\beta}{\beta^2} \right) u_1 \tau_{11} + \frac{u_{11}\tau_{11} + L_{111} u_1 \tau_{11}{}^2}{2} \right).$$

3.2.2 Bayesian Estimators Under Linear-Exponential Loss Function (LINEX)

1. Bayesian estimator of the scale parameter β

$\hat{\beta}_{LINEX} = -\frac{1}{c} \log \left[E\left(e^{-c\beta}\right) \right].$
Provided that $E\left(e^{-c\beta}\right)$ exists and is finite. Substitution in equation (3.4), $u = e^{-c\beta}$, the Bayesian estimator of the scale parameter β is given as

$$\hat{\beta}_{LINEX} \simeq -\frac{1}{c} \log \left[e^{-c\beta} - \left(\frac{\nu-(1+\eta)\beta}{\beta^2} \right) \left(c\tau_{11} e^{-c\beta} \right) + \frac{c^2 e^{-c\beta} \tau_{11} - cL_{111}\tau_{11}{}^2 e^{-c\beta}}{2} \right].$$

2. Bayesian estimator of the reliability function $R(t)$

Substitution in equation (3.4), $u = e^{-cR(t)}$, the Bayesian estimator of the reliability function $R(t)$ is given by

$$\hat{R}_{LINEX} \simeq -\frac{1}{c} \log \left[e^{-c \frac{1}{\left(1+e^{\frac{(t-\mu)}{\beta}}\right)}} + \left(\frac{\nu-(1+\eta)\beta}{\beta^2} \right) (u_1 \tau_{11}) + \frac{u_{11}\tau_{11} + L_{111} u_1 \tau_{11}{}^2}{2} \right].$$

3. Bayesian estimator of the hazard rate function $H(t)$
Substitution in equation (3.4), $u = e^{-cH(t)}$, the Bayesian estimator of the hazard rate function $H(t)$ is given by

$$\hat{H}_{LINEX} \simeq -\frac{1}{c}\log\left[\mathrm{e}^{-c\frac{1}{\beta\left(1+\mathrm{e}^{\frac{-(t-\mu)}{\beta}}\right)}} + \left(\frac{\nu-(1+\eta)\beta}{\beta^2}\right)(u_1\tau_{11}) + \frac{u_{11}\tau_{11}+L_{111}u_1\tau_{11}{}^2}{2}\right].$$

3.2.3 Bayesian Estimators Under General Entropy Loss Function

1. Bayesian estimator of the scale parameter β

$$\hat{\beta}_{Gentropy} = \left[E\left(\beta^{-q}\right)\right]^{-\frac{1}{q}}.$$

Provided that $E\left(\beta^{-q}\right)$ exists and is finite. Substitution in equation (3.4), $u=\beta^{-q}$, the Bayesian estimator of the scale parameter β is given as

$$\hat{\beta}_{Gentropy} \simeq \begin{pmatrix} \beta^{-q} - \left(\frac{\nu-(1+\eta)\beta}{\beta^2}\right)\left(q\beta^{-1-q}\tau_{11}\right) + \\ \frac{q(1+q)\beta^{-2-q}\tau_{11}-q\beta^{-1-q}L_{111}\tau_{11}{}^2}{2} \end{pmatrix}^{-\frac{1}{q}}.$$

2. Bayesian estimator of the reliability function $R(t)$

Substitution in equation (3.4), $u=(R(t))^{-q}$, the Bayesian estimator of the reliability function $R(t)$ is given by

$$\hat{R}_{Gentropy} \simeq \left(\left(\frac{1}{1+\mathrm{e}^{\frac{(t-\mu)}{\beta}}}\right)^{-q} + \left(\frac{\nu-(1+\eta)\beta}{\beta^2}\right)(u_1\tau_{11}) + \frac{u_{11}\tau_{11}+L_{111}u_1\tau_{11}{}^2}{2}\right)^{-\frac{1}{q}}.$$

3. Bayesian estimator of the hazard rate function $H(t)$

Substitution in equation (3.4), $u=(H(t))^{-q}$, the Bayesian estimator of the hazard rate function $H(t)$ is given by

$$\hat{H}_{Gentropy} \simeq \left(\left(\frac{1}{\beta\left(1+\mathrm{e}^{\frac{-(t-\mu)}{\beta}}\right)}\right)^{-q} + \left(\frac{\nu-(1+\eta)\beta}{\beta^2}\right)(u_1\tau_{11}) + \frac{u_{11}\tau_{11}+L_{111}u_1\tau_{11}{}^2}{2}\right)^{-\frac{1}{q}}.$$

It is worth noting that when the value $q=-1$, the general entropy loss function is the same as the squared error loss function.

3.3 Importance Sampling Technique

Importance sampling is the general technique of sampling from one distribution to estimate an expectation under a different distribution. In Bayesian analyses, given a likelihood $L(\theta)$ for a parameter vector θ, based on data $\underline{X}$ and a prior $\varphi(\theta)$, the posterior is given by $\varphi^*(\theta) = C^{-1}L(\theta)\varphi(\theta)$, where the normalizing constant $C = \int L(\theta)\pi(\theta)d\theta$ is determined by the constraint that the density integrate to 1. This normalizing constant often does not have an analytic expression. General problems of interest in Bayesian analyses are computing means and variances of the posterior distribution, and also finding quantities of marginal posterior distributions. In general let $g(\theta)$ be a parametric function for which

$$\tilde{g}(\theta) = \int g(\theta)\varphi^*(\theta\,|\underline{X})d\theta, \tag{3.5}$$

needs to be evaluated. In many applications,(3.5) cannot be evaluated explicitly, and it is difficult to sample directly from the posterior distribution, so importance sampling can be applied. Samples can be drawn from a distribution with density $q(\theta)$. In this case, if $\theta_1, \theta_2, ..., \theta_N$ is a random sample from $q(\theta)$ then (3.5) can be estimated with

$$\tilde{g}(\theta) = \frac{\sum_{i=1}^{N} g(\theta_i) w_i}{\sum_{i=1}^{N} w_i}, \tag{3.6}$$

where $w_i = \frac{L(\theta_i)\varphi(\theta_i)}{q(\theta_i)}$ and the sampling density $q(\theta)$ need not be normalized. This technique is described in detail by [17]. We generate a samples from inverse gamma distribution with hyper-parameters $(\eta = 3,\ \nu = 2\ (table\ 3)\ and\ \eta = 2,\ \nu = 3\ (table\ 4))$. We use the following procedure:
Step 1 Generate β from inverse gamma $(\beta; \eta, \nu)$.
Step 2 Repeat this procedure 1000 times to obtain importance sample $(\beta_1, \beta_2, \ldots, \beta_{1000})$.
The approximate value of (3.5) can be obtained by (3.6).

4 Simulation studies

In this section, we conduct a Monte Carlo simulation study to compare the performance of various estimates developed in the previous sections. A large number (1000) of progressive first-failure censored samples of varying group sizes $k = 4, 5$, number of groups in a sample $n = 40, 60$ and effective sample sizes $m = 20, 30$ out of n and different combinations of progressive censoring schemes $\underline{R}$ are generated from model (2.3). We choose true value of β equal to 0.9 (tables 1,3) and 2.0 (tables 2,4). The study includes the following steps:

1. Generate a progressive first-failure censored sample using algorithm proposed by [4] from model (2.3) for given values of $(k, n, m, \underline{R})$.
2. Calculate the maximum likelihood estimates of β, $R(t)$ and $H(t)$ according to Sect. 2.
3. According to Sect. 3, obtain the Bayes estimates of β, $R(t)$ and $H(t)$.
4. Repeat steps $(1)-(3)$, (1000) times, for different values of $(k, n, m, \underline{R})$.

Estimation average $= \frac{\sum_{i=1}^{1000} \widehat{\theta}_i}{1000}$, mean square error $= \frac{\sum_{i=1}^{1000} \left(\hat{\theta}_i - \theta\right)^2}{1000}$, where, θ is the parameter and $\widehat{\theta}$ is the estimator.
Extensive computations were performed using Mathematica 9.
Since the non-linear equations (2.5) are not solvable analytically, numerical methods can be used, as Newton Raphson method with initial values closed to real values of the parameter.

Throughout this section we will use the following abbreviations:

1. $MSEs$: The mean square errors,
2. ML : The estimate by using the (MLE),
3. B_{Sq} : The estimate under squared error loss function,
4. $B_{Lx,c=2}$: The estimate under linex loss function at $c = 2$,
5. $B_{Lx,c=4}$: The estimate under linex loss function at $c = 4$,
6. $B_{Ge,q=2}$: The estimate under general entropy loss function at $q = 2$,
7. $B_{Ge,q=4}$: The estimate under general entropy loss function at $q = 4$.

Table 1. Average values of the estimates and the corresponding MSEs, given in parentheses of the parameter β, the relaibility function and the hazard rate function when $\mu = 0.5$, $\beta = 0.9$ and $\lambda = 2$, in case non-informative prior.

$B_{Lx,c=4}$	$B_{Lx,c=2}$	$B_{Ge,q=4}$	$B_{Ge,q=2}$	B_{Sq}	ML	Scheme	(k, n, m)
The average, MSEs of the estimators of parameter β							
0.87332	0.88262	0.86546	0.873607	0.89657	0.91553	(10,18*0,10)	(4,40,20)
(0.00072)	(0.00031)	(0.00122)	(0.00072)	(0.00004)	(0.01122)		
0.88455	0.87706	0.87684	0.87026	0.89727	0.90029	(20,19*0)	
(0.00051)	(0.00053)	(0.00054)	(0.00089)	(0.00001)	(0.01346)		
0.86939	0.88643	0.86067	0.87999	0.89625	0.89386	(19*0,20)	
(0.00097)	(0.00019)	(0.00163)	(0.00046)	(0.00005)	(0.00958)		
0.88780	0.89465	0.88318	0.88920	0.90081	0.90072	(15,28*0,15)	(5,60,30)
(0.00151)	(0.00003)	(0.00028)	(0.00012)	(0.00003)	(0.00658)		
0.89417	0.88847	0.88989	0.88467	0.90334	0.90533	(30,29*0)	
(0.00034)	(0.00013)	(0.00013)	(0.00023)	(0.00001)	(0.00829)		
0.88648	0.89361	0.88083	0.89014	0.89890	0.89854	(29*0,30)	
(0.00019)	(0.00004)	(0.00038)	(0.00009)	(0.00003)	(0.00519)		
The average, MSEs of the estimators of relaibility function R(t=2)=0.15887							
0.15382	0.15399	0.14603	0.14760	0.15529	0.15648	(10,18*0,10)	(4,40,20)
(0.00002)	(0.00002)	(0.00016)	(0.00013)	(0.00001)	(0.00068)		
0.15455	0.15447	0.14896	0.14767	0.15488	0.15740	(20,19*0)	
(0.00001)	(0.00002)	(0.00010)	(0.00012)	(0.00001)	(0.00007)		
0.15311	0.15507	0.14454	0.15033	0.15568	0.15625	(19*0,20)	
(0.00003)	(0.00001)	(0.00021)	(0.00007)	(0.00011)	(0.00058)		
0.15670	0.15716	0.15196	0.15318	0.15751	0.15828	(15,28*0,15)	(5,60,30)
(0.00004)	(0.00001)	(0.00004)	(0.00003)	(0.00001)	(0.00039)		
0.15710	0.15671	0.15392	0.15277	0.15768	0.15922	(30,29*0)	
(0.00003)	(0.00001)	(0.00002)	(0.00003)	(0.00001)	(0.00049)		
0.15664	0.15703	0.15090	0.15443	0.15736	0.15790	(29*0,30)	
(0.00001)	(0.00001)	(0.00006)	(0.00001)	(0.00001)	(0.00031)		
The average, MSEs of the estimators of hazard rate function H(t=2)=0.93459							
0.93406	0.95785	0.92107	0.94155	0.97715	0.95890	(10,18*0,10)	(4,40,20)
(0.00014)	(0.00055)	(0.00038)	(0.00009)	(0.00182)	(0.02205)		
0.95568	0.93615	0.94265	0.92559	0.98354	0.95578	(20,19*0)	
(0.00049)	(0.00016)	(0.00017)	(0.00028)	(0.00242)	(0.02530)		
0.93122	0.95373	0.91479	0.94322	0.97136	0.95849	(19*0,20)	
(0.00086)	(0.00044)	(0.00052)	(0.00023)	(0.00137)	(0.01902)		
0.93160	0.94220	0.92404	0.93281	0.95472	0.94424	(15,28*0,15)	(5,60,30)
(0.00030)	(0.00001)	(0.00014)	(0.00001)	(0.00040)	(0.01211)		
0.94323	0.93437	0.93614	0.92834	0.95695	0.94062	(30,29*0)	
(0.00008)	(0.00001)	(0.00001)	(0.00005)	(0.00050)	(0.01448)		
0.92705	0.94369	0.91703	0.93806	0.95294	0.94492	(29*0,30)	
(0.00007)	(0.00008)	(0.00033)	(0.00002)	(0.00033)	(0.00956)		

Table 2. Average values of the estimates and the corresponding MSEs, given in parentheses of the parameter β, the relaibility function and the hazard rate function when $\mu = 0.5$, $\beta = 2.0$ and and $\lambda = 2$, in case non-informative prior.

$B_{Lx,c=4}$	$B_{Lx,c=2}$	$B_{Ge,q=4}$	$B_{Ge,q=2}$	B_{Sq}	ML	Scheme	(k,n,m)
The average, MSEs of the estimators of parameter β							
1.90862	1.92266	1.93826	1.93780	2.00786	2.00577	(10,18*0,10)	(4,40,20)
(0.00869)	(0.00620)	(0.00397)	(0.00401)	(0.00012)	(0.05947)		
1.95024	1.91345	1.96369	1.93926	1.99037	1.99697	(20,19*0)	
(0.00257)	(0.00765)	(0.00135)	(0.00374)	(0.00012)	(0.07077)		
1.87793	1.94977	1.90912	1.96068	1.99655	1.99189	(19*0,20)	
(0.01540)	(0.00254)	(0.00857)	(0.00155)	(0.00011)	(0.04557)		
1.93300	1.96009	1.95189	1.96979	1.99085	1.99077	(15,28*0,15)	(5,60,30)
(0.00455)	(0.00164)	(0.00234)	(0.00094)	(0.00009)	(0.03077)		
1.95926	1.94862	1.96672	1.96466	2.00110	2.00547	(30,29*0)	
(0.00167)	(0.00267)	(0.00111)	(0.00125)	(0.00001)	(0.04251)		
1.92834	1.97067	1.95124	1.97680	1.99621	1.99544	(29*0,30)	
(0.00528)	(0.00086)	(0.00245)	(0.00053)	(0.00002)	(0.02612)		
The average, MSEs of the estimators of relaibility function R(t=2)=0.32082							
0.31647	0.31442	0.31446	0.31283	0.31724	0.31921	(10,18*0,10)	(4,40,20)
(0.00001)	(0.00004)	(0.00004)	(0.00006)	(0.00001)	(0.00040)		
0.31685	0.31653	0.31550	0.31481	0.31488	0.31804	(20,19*0)	
(0.00001)	(0.00001)	(0.00002)	(0.00003)	(0.00003)	(0.00050)		
0.31397	0.31685	0.31164	0.31571	0.31717	0.31852	(19*0,20)	
(0.00004)	(0.00001)	(0.00008)	(0.00002)	(0.00001)	(0.00032)		
0.31737	0.31762	0.31618	0.31666	0.31780	0.31896	(15,28*0,15)	(5,60,30)
(0.00001)	(0.00001)	(0.00002)	(0.00001)	(0.00001)	(0.00021)		
0.31758	0.31830	0.31680	0.31734	0.31790	0.31977	(30,29*0)	
(0.00001)	(0.00001)	(0.00001)	(0.00001)	(0.00001)	(0.00028)		
0.31735	0.31847	0.31589	0.31785	0.31864	0.31951	(29*0,30)	
(0.00001)	(0.00001)	(0.00002)	(0.00001)	(0.00001)	(0.00018)		
The average, MSEs of the estimators of hazard rate function H(t=2)=0.33959							
0.34739	0.35665	0.33298	0.34501	0.35267	0.34569	(10,18*0,10)	(4,40,20)
(0.00006)	(0.00029)	(0.00006)	(0.00003)	(0.00017)	(0.00279)		
0.35009	0.34777	0.34068	0.33615	0.35976	0.34925	(20,19*0)	
(0.00011)	(0.00007)	(0.00002)	(0.00002)	(0.00041)	(0.00364)		
0.35335	0.34999	0.33532	0.34238	0.35215	0.34707	(19*0,20)	
(0.00019)	(0.00010)	(0.00003)	(0.00001)	(0.00015)	(0.00226)		
0.34656	0.34802	0.33865	0.34139	0.34950	0.34544	(15,28*0,15)	(5,60,30)
(0.00004)	(0.00007)	(0.00001)	(0.00001)	(0.00008)	(0.00151)		
0.34804	0.34458	0.34295	0.33828	0.34985	0.34367	(30,29*0)	
(0.00007)	(0.00002)	(0.00001)	(0.00001)	(0.00010)	(0.00197)		
0.34613	0.34576	0.33576	0.34171	0.34691	0.34383	(29*0,30)	
(0.00004)	(0.00003)	(0.00001)	(0.00001)	(0.00005)	(0.00125)		

Table 3. Average values of the estimates and the corresponding MSEs, given in parentheses of the parameter β, the relaibility function and the hazard rate function when $\eta = 3,\ \nu = 2,\ \mu = 0.5,\ \beta = 0.9$, in case informative prior.

$B_{Lx,c=4}$	$B_{Lx,c=2}$	$B_{Ge,q=4}$	$B_{Ge,q=2}$	B_{sq}	Technique	ML	Scheme	(k,n,m)
The average, MSEs of the estimators of parameter β								
0.87555 (0.00066)	0.88937 (0.00019)	0.86642 (0.00119)	0.88004 (0.00046)	0.90440 (0.00012)	Lindley	0.90336 (0.01494)	(20,19*0)	(4,40,20)
0.88557 (0.00078)	0.88628 (0.00078)	0.88499 (0.00078)	0.88579 (0.00078)	0.88699 (0.00078)	Importance Sampling			
0.88034 (0.00038)	0.88976 (0.00011)	0.87348 (0.00071)	0.88332 (0.00028)	0.89955 (0.00003)	Lindley	0.89487 (0.00867)	(19*0,20)	
0.89268 (0.00083)	0.89332 (0.00083)	0.89217 (0.00083)	0.89289 (0.00083)	0.89396 (0.00083)	Importance Sampling			
0.88134 (0.00035)	0.88948 (0.00011)	0.87549 (0.00061)	0.88395 (0.00026)	0.89805 (0.00001)	Lindley	0.89784 (0.00796)	(30,29*0)	(5,60,30)
0.89727 (0.00084)	0.89799 (0.00084)	0.89670 (0.00084)	0.89751 (0.00084)	0.89873 (0.00084)	Importance Sampling			
0.89139 (0.00007)	0.89657 (0.00001)	0.88755 (0.00015)	0.89309 (0.00004)	0.90188 (0.00001)	Lindley	0.90022 (0.00544)	(29*0,30)	
0.91802 (0.00081)	0.91965 (0.00081)	0.91684 (0.00081)	0.91861 (0.00081)	0.92128 (0.00081)	Importance Sampling			
The average, MSEs of the estimators of relaibility function						R(t=2)=0.15887		
0.15463 (0.00002)	0.15553 (0.00001)	0.14569 (0.00017)	0.14892 (0.00010)	0.15644 (0.00001)	Lindley	0.15800 (0.00086)	(20,19*0)	(4,40,20)
0.15545 (0.00004)	0.15550 (0.00004)	0.15483 (0.00004)	0.15511 (0.00004)	0.15554 (0.00004)	Importance Sampling			
0.15539 (0.00001)	0.15599 (0.00001)	0.14848 (0.00011)	0.15122 (0.00005)	0.15660 (0.00001)	Lindley	0.15660 (0.00053)	(19*0,20)	
0.15721 (0.00005)	0.15725 (0.00005)	0.15666 (0.00005)	0.15691 (0.00005)	0.15729 (0.00005)	Importance Sampling			
0.15539 (0.00001)	0.15591 (0.00001)	0.14946 (0.00008)	0.15183 (0.00005)	0.15643 (0.00001)	Lindley	0.15741 (0.00047)	(30,29*0)	(5,60,30)
0.15837 (0.00005)	0.15842 (0.00005)	0.15776 (0.00005)	0.15804 (0.00005)	0.15846 (0.00005)	Importance Sampling			
0.15741 (0.00001)	0.15774 (0.00001)	0.15344 (0.00002)	0.15512 (0.00001)	0.15806 (0.00001)	Lindley	0.15828 (0.00033)	(29*0,30)	
0.16369 (0.00004)	0.16379 (0.00004)	0.16241 (0.00004)	0.16300 (0.00004)	0.16388 (0.00004)	Importance Sampling			
The average, MSEs of the estimators of hazard rate function						H(t=2)=0.93459		
0.92408 (0.00032)	0.95051 (0.00038)	0.90773 (0.00097)	0.93412 (0.00017)	0.97646 (0.00184)	Lindley	0.95375 (0.02745)	(20,19*0)	(4,40,20)
0.95051 (0.00135)	0.95182 (0.00135)	0.94968 (0.00135)	0.95106 (0.00135)	0.95312 (0.00135)	Importance Sampling			
0.93069 (0.00014)	0.94785 (0.00023)	0.92042 (0.00036)	0.93751 (0.00011)	0.96532 (0.00095)	Lindley	0.95565 (0.01695)	(19*0,20)	
0.94135 (0.00143)	0.94248 (0.00143)	0.94061 (0.00143)	0.94181 (0.00143)	0.94361 (0.00143)	Importance Sampling			
0.93331 (0.00002)	0.94858 (0.00019)	0.92371 (0.00015)	0.93935 (0.00003)	0.96367 (0.00084)	Lindley	0.95011 (0.01432)	(30,29*0)	(5,60,30)
0.93488 (0.00145)	0.93616 (0.00145)	0.93400 (0.00145)	0.93538 (0.00145)	0.93743 (0.00145)	Importance Sampling			
0.93093 (0.00002)	0.94016 (0.00003)	0.92493 (0.00011)	0.93454 (0.00001)	0.94938 (0.00021)	Lindley	0.94309 (0.00996)	(29*0,30)	
0.90469 (0.00140)	0.90721 (0.00140)	0.90282 (0.00140)	0.90556 (0.00140)	0.90977 (0.00140)	Importance Sampling			

Table 4. Average values of the estimates and the corresponding MSEs, given in parentheses of the parameter β, the relaibility function and the hazard rate function when $\eta = 2,\ \nu = 3,\ \mu = 0.5,\ \beta = 2.0$, in case informative prior.

$B_{Lx,c=4}$	$B_{Lx,c=2}$	$B_{Ge,q=4}$	$B_{Ge,q=2}$	B_{sq}	Technique	ML	Scheme	(k,n,m)
The average, MSEs of the estimators of parameter β								
1.89923 (0.01055)	1.94854 (0.00276)	1.93415 (0.00453)	1.96492 (0.00128)	2.01873 (0.00044)	Lindley	2.00715 (0.06774)	(20,19*0)	(4,40,20)
1.97021 (0.00815)	1.97623 (0.00815)	1.97463 (0.00815)	1.97776 (0.00815)	1.98258 (0.00815)	Importance Sampling			
1.93314 (0.00458)	1.97208 (0.00079)	1.96124 (0.00152)	1.98372 (0.00028)	2.02017 (0.00060)	Lindley	2.00335 (0.04377)	(19*0,20)	
1.96972 (0.00847)	1.97502 (0.00847)	1.97360 (0.00847)	1.97635 (0.00847)	1.98057 (0.00847)	Importance Sampling			
1.93635 (0.00416)	1.97024 (0.00091)	1.96090 (0.00157)	1.98038 (0.00039)	2.01240 (0.00017)	Lindley	2.00586 (0.04346)	(30,29*0)	(5,60,30)
1.97753 (0.00815)	1.98042 (0.00815)	1.97966 (0.00815)	1.98116 (0.00815)	1.98348 (0.00815)	Importance Sampling			
1.95817 (0.00177)	1.98106 (0.00035)	1.97493 (0.00063)	1.98737 (0.00015)	2.00694 (0.00006)	Lindley	1.99966 (0.02543)	(29*0,30)	
1.94792 (0.00814)	1.95158 (0.00814)	1.95057 (0.00814)	1.95253 (0.00814)	1.95568 (0.00814)	Importance Sampling			
The average, MSEs of the estimators of relaibility function R(t=2)=0.32082								
0.31645 (0.00001)	0.31692 (0.00001)	0.31401 (0.00004)	0.31527 (0.00003)	0.31739 (0.00001)	Lindley	0.31904 (0.00045)	(20,19*0)	(4,40,20)
0.31906 (0.00005)	0.31911 (0.00005)	0.31881 (0.00005)	0.31895 (0.00005)	0.31915 (0.00005)	Importance Sampling			
0.318535 (0.00001)	0.31885 (0.00001)	0.31677 (0.00001)	0.31770 (0.00001)	0.31917 (0.00001)	Lindley	0.31953 (0.00030)	(19*0,20)	
0.31894 (0.00005)	0.31897 (0.00005)	0.31871 (0.00005)	0.31883 (0.00005)	0.31901 (0.00005)	Importance Sampling			
0.31821 (0.00001)	0.31849 (0.00001)	0.31669 (0.00001)	0.31749 (0.00001)	0.31877 (0.00001)	Lindley	0.31977 (0.00029)	(30,29*0)	(5,60,30)
0.31930 (0.00005)	0.31932 (0.00005)	0.31918 (0.00005)	0.31925 (0.00005)	0.31935 (0.00005)	Importance Sampling			
0.31921 (0.00001)	0.31938 (0.00001)	0.31823 (0.00001)	0.31875 (0.00001)	0.31955 (0.00001)	Lindley	0.31989 (0.00017)	(29*0,30)	
0.31691 (0.00005)	0.31694 (0.00005)	0.31675 (0.00005)	0.31683 (0.00005)	0.31697 (0.00005)	Importance Sampling			
The average, MSEs of the estimators of hazard rate function H(t=2)=0.33958								
0.34652 (0.00005)	0.34980 (0.00011)	0.32942 (0.00012)	0.33840 (0.00001)	0.35296 (0.00018)	Lindley	0.34640 (0.00321)	(20,19*0)	(4,40,20)
0.34352 (0.00034)	0.34381 (0.00034)	0.34191 (0.00034)	0.34280 (0.00034)	0.34410 (0.00034)	Importance Sampling			
0.34251 (0.00001)	0.34470 (0.00002)	0.33133 (0.00008)	0.33723 (0.00001)	0.34686 (0.00005)	Lindley	0.34438 (0.00213)	(19*0,20)	
0.34392 (0.00036)	0.34418 (0.00036)	0.34251 (0.00036)	0.34329 (0.00036)	0.34444 (0.00036)	Importance Sampling			
0.34385 (0.00001)	0.34578 (0.00003)	0.33360 (0.00004)	0.33910 (0.00001)	0.34766 (0.00006)	Lindley	0.34373 (0.00204)	(30,29*0)	(5,60,30)
0.34321 (0.00035)	0.34335 (0.00035)	0.34242 (0.00035)	0.34286 (0.00035)	0.34349 (0.00035)	Importance Sampling			
0.34218 (0.00001)	0.34336 (0.00001)	0.33599 (0.00001)	0.33934 (0.00001)	0.34452 (0.00002)	Lindley	0.34283 (0.00119)	(29*0,30)	
0.34932 (0.00034)	0.34952 (0.00034)	0.34818 (0.00034)	0.34883 (0.00034)	0.34972 (0.00034)	Importance Sampling			

From tables 1, 2, 3 and 4, we observe that the MLE and Bayes estimates of the parameter β, the reliability and the hazard rate functions are very good in terms of $MSEs$. As the number of groups n and effective sample size m increase, $MSEs$ of all estimates decrease as expected. Also, as the value of the group size k increases, $MSEs$ decrease. In general, the Bayesian estimators have $MSEs$ less than that of the MlE. Bayes estimates using inverse gamma informative prior are better as they include prior information and using non-informative prior are worst as they do not include any prior information than MLE in terms of $MSEs$. Also, Bayes estimates obtained using the importance sampling procedure is more accurate than Lindley's Bayes approximation.

5 Concluding Remarks

In this paper, assuming a good lifetime model we consider the problem of estimating the unknown parameter β, as well as the reliability and hazard rate functions, using progressive first-failure censored samples. This censoring scheme has advantages in terms of reducing test time, in which more items are used but only m of $(\mathrm{k} \times \mathrm{n})$ item are failures. We derived MLE and Bayes estimators of the parameter β, the reliability and the hazard rate functions using non-informative and inverse gamma informative priors, under both symmetric (squared error) and asymmetric (linex and general entropy) loss functions. These estimates cannot be obtained in closed form, but can be computed numerically. It is clear that the proposed Bayes estimators perform very well for different n and m. As expected, the Bayes estimators based on informative prior perform are much better than the Bayes estimators based on non-informative prior in terms of $MSEs$. Also the Bayes estimators based on non-informative prior and informative prior perform much better than the MLE in terms of $MSEs$. Also the importance sampling technique shows more accurate results than Lindley's Bayes approximation. The simulation also stresses the importance of linex and general entropy loss functions as asymmetricloss functions, in the case studied.

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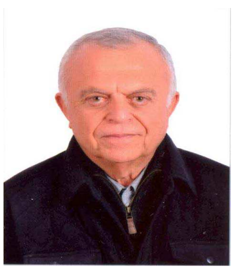

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