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On Chebyshev Type Inequalities Using Generalized k-Fractional Integral Operator

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Abstract: In this paper, using generalized k-fractional integral operator (in terms of the Gauss hypergeometric function), we establish new results on generalized k-fractional integral inequalities by considering the extended Chebyshev functional in case of synchronous function and some other inequalities.

Keywords: Chebyshev inequality, generalized k-fractional integral, Gauss hypergeometric function.

1 Introduction

Recently many authors have studied on fractional integral inequalities by using different fractional integral operator such as Riemann-Liouville, Hadamard, Saigo and Erdelyi-Kober, see [1,2,3,4,5,6,7,8,9,10,11,12]. In [13] S. Kilinc and H. Yildirim establish new generalized k-fractional integral inequalities involving Gauss hypergeometric function related to Chebyshev functional. In [5,14] authors gave the following fractional integral inequalities, using the Hadamard and Riemann-Liouville fractional integral for extended Chebyshev functional.

Theorem 1.*Let f and g be two synchronous function on $[0,\infty[$, and $r,p,q:[0,\infty)\to[0,\infty)$. Then for all $t>0$, $\alpha>0$, we have*

$$\begin{aligned}
&2\,{}_HD_{1,t}^{-\alpha}r(t)\left[{}_HD_{1,t}^{-\alpha}p(t)\,{}_HD_{1,t}^{-\alpha}(qfg)(t)+{}_HD_{1,t}^{-\alpha}q(t)\,{}_HD_{1,t}^{-\alpha}(pfg)(t)\right]+\\
&2\,{}_HD_{1,t}^{-\alpha}p(t)\,{}_HD_{1,t}^{-\alpha}q(t)\,{}_HD_{1,t}^{-\alpha}(rfg)(t)\geq\\
&{}_HD_{1,t}^{-\alpha}r(t)\left[{}_HD_{1,t}^{-\alpha}(pf)(t)\,{}_HD_{1,t}^{-\alpha}(qg)(t)+{}_HD_{1,t}^{-\alpha}(qf)(t)\,{}_HD_{1,t}^{-\alpha}(pg)(t)\right]+\\
&{}_HD_{1,t}^{-\alpha}p(t)\left[{}_HD_{1,t}^{-\alpha}(rf)(t)\,{}_HD_{1,t}^{-\alpha}(qg)(t)+{}_HD_{1,t}^{-\alpha}(qf)(t)\,{}_HD_{1,t}^{-\alpha}(rg)(t)\right]+\\
&{}_HD_{1,t}^{-\alpha}q(t)\left[{}_HD_{1,t}^{-\alpha}(rf)(t)\,{}_HD_{1,t}^{-\alpha}(pg)(t)+{}_HD_{1,t}^{-\alpha}(pf)(t)\,{}_HD_{1,t}^{-\alpha}(rg)(t)\right].
\end{aligned}\tag{1}$$

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Theorem 2. *Let f and g be two synchronous function on $[0,\infty[$, and $r,p,q:[0,\infty)\rightarrow[0,\infty)$. Then for all $t>0$, $\alpha>0$, we have:*

$$
\begin{aligned}
&{}_HD_{1,t}^{-\alpha}r(t)\times\\
&\left[{}_HD_{1,t}^{-\alpha}q(t){}_HD_{1,t}^{-\beta}(pfg)(t)+2{}_HD_{1,t}^{-\alpha}p(t){}_HD_{1,t}^{-\beta}(qfg)(t)+{}_HD_{1,t}^{-\beta}q(t){}_HD_{1,t}^{-\alpha}(pfg)(t)\right]\\
&+\left[{}_HD_{1,t}^{-\alpha}p(t){}_HD_{1,t}^{-\beta}q(t)+{}_HD_{1,t}^{-\beta}p(t){}_HD_{1,t}^{-\alpha}q(t)\right]{}_HD_{1,t}^{-\alpha}(rfg)(t)\geq\\
&{}_HD_{1,t}^{-\alpha}r(t)\left[{}_HD_{1,t}^{-\alpha}(pf)(t){}_HD_{1,t}^{-\beta}(qg)(t)+{}_HD_{1,t}^{-\beta}(qf)(t){}_HD_{1,t}^{-\alpha}(pg)(t)\right]+\\
&{}_HD_{1,t}^{-\alpha}p(t)\left[{}_HD_{1,t}^{-\alpha}(rf)(t){}_HD_{1,t}^{-\beta}(qg)(t)+{}_HD_{1,t}^{-\beta}(qf)(t){}_HD_{1,t}^{-\alpha}(rg)(t)\right]+\\
&{}_HD_{1,t}^{-\alpha}q(t)\left[{}_HD_{1,t}^{-\alpha}(rf)(t){}_HD_{1,t}^{-\beta}(pg)(t)+{}_HD_{1,t}^{-\beta}(pf)(t){}_HD_{1,t}^{-\alpha}(rg)(t)\right].
\end{aligned}
\tag{2}
$$

The main objective of this paper is to establish some Chebyshev type inequalities and some other inequalities using generalized k-fractional integral operator. The paper has been organized as follows. In Section 2, we define basic definitions related to generalized k-fractional integral operator. In section 3, we obtain Chebyshev type inequalities using generalized k-fractional. In Section 4 , we prove some inequalities for positive continuous functions.

2 Preliminaries

In this section, we recall the notation and preliminaries and which will be used to obtain the main result.

Definition 1. *Two function f and g are said to synchronous (asynchronous) on $[a,b]$, if*

$$((f(u)-f(v))(g(u)-g(v)))\geq(\leq)0, \tag{3}$$

for all $u,v\in[0,\infty)$.

Definition 2. *[13,12] The function $f(x)$, for all $x>0$ is said to be in the $L_{p,k}[0,\infty)$, if*

$$L_{p,k}[0,\infty)=\left\{f:\|f\|_{L_{p,k}[0,\infty)}=\left(\int_0^\infty|f(x)|^px^kdx\right)^{\frac{1}{p}}<\infty\ 1\leq p<\infty k\geq 0\right\}. \tag{4}$$

Definition 3. *[13,15,12] Let $f\in L_{1,k}[0,\infty)$,. The generalized Riemann-Liouville fractional integral $I^{\alpha,k}f(x)$ of order $\alpha,k\geq 0$ is defined by*

$$I^{\alpha,k}f(x)=\frac{(k+1)^{1-\alpha}}{\Gamma(\alpha)}\int_0^x(x^{k+1}-t^{k+1})^{\alpha-1}t^kf(t)dt. \tag{5}$$

Definition 4. *[13,12] Let $k\geq 0,\xi>0,\omega>-1$ and $\phi,\psi\in R$. The generalized k-fractional integral $I_{t,k}^{\xi,\phi,\psi,\omega}$ (in terms of the Gauss hypergeometric function)of order ξ for real-valued continuous function $f(t)$ is defined by*

$$
\begin{aligned}
I_{t,k}^{\xi,\phi,\psi,\omega}[f(t)]=&\frac{(k+1)^{\omega+\phi+1}t^{(k+1)(-\xi-\phi-2\omega)}}{\Gamma(\xi)}\int_0^t\sigma^{(k+1)\omega}(t^{k+1}-\sigma^{k+1})^{\xi-1}\times\\
&{}_2F_1(\xi+\phi+\omega,-\psi;\xi;1-(\frac{\sigma}{t})^{k+1})\sigma^kf(\sigma)d\sigma.
\end{aligned}
\tag{6}
$$

where, the function ${}_2F_1(-)$ in the right-hand side of (6) is the Gaussian hypergeometric function defined by

$${}_2F_1(a,b;c;t)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{t^n}{n!}, \tag{7}$$

and $(a)_n$ is the Pochhammer symbol

$$(a)_n=a(a+1)...(a+n-1)=\frac{\Gamma(a+n)}{\Gamma(a)},\ (a)_0=1.$$

Consider the function

$$F(t,\sigma)=\frac{(k+1)^{\omega+\phi+1}t^{(k+1)(-\xi-\phi-2\omega)}}{\Gamma(\xi)}\sigma^{(k+1)\omega}$$
$$(t^{k+1}-\sigma^{k+1})^{\xi-1}\times{}_2F_1(\xi+\phi+\omega,-\psi;\xi;1-(\frac{\sigma}{t})^{k+1})$$
$$=\sum_{n=0}^{\infty}\frac{(\xi+\phi+\omega)_n(-n)_n}{\Gamma(\xi+n)n!}t^{(k+1)(-\xi-\phi-2\omega-\psi)}\sigma^{(k+1)\omega}(t^{k+1}-\sigma^{k+1})^{\xi-1+n}(k+1)^{\omega+\phi+1}$$
$$=\frac{\sigma^{(k+1)\omega}(t^{k+1}-\sigma^{k+1})^{\xi-1}(k+1)^{\omega+\phi+1}}{t^{k+1}(\xi+\phi+2\omega)\Gamma(\xi)}+ \tag{8}$$
$$\frac{\sigma^{(k+1)\omega}(t^{k+1}-\sigma^{k+1})^{\xi}(k+1)^{\omega+\phi+1}(\xi+\phi+\omega)(-n)}{t^{k+1}(\xi+\phi+2\omega+1)\Gamma(\xi+1)}+$$
$$\frac{\sigma^{(k+1)\omega}(t^{k+1}-\sigma^{k+1})^{\xi+1}(k+1)^{\omega+\phi+1}(\xi+\phi+\omega)(\xi+\omega+1)(-n)(-n+1)}{t^{k+1}(\xi+\phi+2\omega+1)\Gamma(\xi+2)2!}+....$$

It is clear that $F(t,\sigma)$ is positive because for all $\sigma\in(0,t)$, $(t>0)$ since each term of the (8) is positive.

3 Fractional Integral Inequalities for Extended Chebyshev Functional

Here, we establish some Chebyshev type fractional integral inequalities by using the generalized k-fractional integral (in terms of the Gauss hypergeometric function) operator.

Lemma 1. *Let f and g be two synchronous function on $[0,\infty[$, and $x,y:[0,\infty)\to[0,\infty)$ be two nonnegative functions. Then for all $k\geq 0$, $t>0$, $\xi>max\{0,-\phi-\omega\}$, $\phi<1$, $\omega>-1$, $\phi-1<\psi<0$, we have,*

$$\begin{aligned}&I_{t,k}^{\xi,\phi,\psi,\omega}x(t)I_{t,k}^{\xi,\phi,\psi,\omega}(yfg)(t)+I_{t,k}^{\xi,\phi,\psi,\omega}y(t)I_{t,k}^{\xi,\phi,\psi,\omega}(xfg)(t)\geq\\&I_{t,k}^{\xi,\phi,\psi,\omega}(xf)(t)I_{t,k}^{\xi,\phi,\psi,\omega}(yg)(t)+I_{t,k}^{\xi,\phi,\psi,\omega}(yf)(t)I_{t,k}^{\xi,\phi,\psi,\omega}(xg)(t).\end{aligned} \tag{9}$$

Proof: Since f and g are synchronous on $[0,\infty[$ for all $\sigma\geq 0$, $\rho\geq 0$, we have

$$(f(\sigma)-f(\rho))(g(\sigma)-g(\rho))\geq 0. \tag{10}$$

From (10),

$$f(\sigma)g(\sigma)+f(\rho)g(\rho)\geq f(\sigma)g(\rho)+f(\rho)g(\sigma). \tag{11}$$

Now, multiplying (11) by $\sigma^k x(\sigma)F(t,\sigma)$, $\sigma\in(0,t)$, $t>0$. Then the integrate resulting identity with respect to σ from 0 to t, we get

$$\begin{aligned}&I_{t,k}^{\xi,\phi,\psi,\omega}(xfg)(t)+f(\rho)g(\rho)I_{t,k}^{\xi,\phi,\psi,\omega}(x)(t)\geq\\&g(\rho)I_{t,k}^{\xi,\phi,\psi,\omega}(xf)(t)+f(\rho)I_{t,k}^{\xi,\phi,\psi,\omega}(xg)(t).\end{aligned} \tag{12}$$

Now, multiplying (12) by $\rho^k y(\rho)F(t,\rho)$, $\rho\in(0,t)$, $t>0$, where $F(t,\rho)$ defined in view of (8). Then the integrate resulting identity with respect to ρ from 0 to t and using definition (4), we have

$$\begin{aligned}&I_{t,k}^{\xi,\phi,\psi,\omega}y(t)I_{t,k}^{\xi,\phi,\psi,\omega}(xfg)(t)+I_{t,k}^{\xi,\phi,\psi,\omega}(yfg)(t)I_{t,k}^{\xi,\phi,\psi,\omega}(x)(t)\\&\geq I_{t,k}^{\xi,\phi,\psi,\omega}(yg)(t)I_{t,k}^{\xi,\phi,\psi,\omega}(xf)(t)+I_{t,k}^{\xi,\phi,\psi,\omega}(yf)(t)I_{t,k}^{\xi,\phi,\psi,\omega}(xg)(t).\end{aligned} \tag{13}$$

This complete the proof of (9)
Further we have the following main theorem.

Theorem 3. *Let f and g be two synchronous function on $[0,\infty[$, and $r,p,q:[0,\infty)\rightarrow[0,\infty)$. Then for all $k\geq 0$, $t>0$, $\xi>max\{0,-\phi-\omega\}$, $\phi<1$, $\omega>-1$, $\phi-1<\psi<0$, we have,*

$$\begin{aligned}&2I^{\xi,\phi,\psi,\omega}_{t,k}r(t)\left[I^{\xi,\phi,\psi,\omega}_{t,k}p(t)I^{\xi,\phi,\psi,\omega}_{t,k}(qfg)(t)+I^{\xi,\phi,\psi,\omega}_{t,k}q(t)I^{\xi,\phi,\psi,\omega}_{t,k}(pfg)(t)\right]+\\&2I^{\xi,\phi,\psi,\omega}_{t,k}p(t)I^{\xi,\phi,\psi,\omega}_{t,k}q(t)I^{\xi,\phi,\psi,\omega}_{t,k}(rfg)(t)\geq\\&I^{\xi,\phi,\psi,\omega}_{t,k}r(t)\left[I^{\xi,\phi,\psi,\omega}_{t,k}(pf)(t)I^{\xi,\phi,\psi,\omega}_{t,k}(qg)(t)+I^{\xi,\phi,\psi,\omega}_{t,k}(qf)(t)I^{\xi,\phi,\psi,\omega}_{t,k}(pg)(t)\right]+\\&I^{\xi,\phi,\psi,\omega}_{t,k}p(t)\left[I^{\xi,\phi,\psi,\omega}_{t,k}(rf)(t)I^{\xi,\phi,\psi,\omega}_{t,k}(qg)(t)+I^{\xi,\phi,\psi,\omega}_{t,k}(qf)(t)I^{\xi,\phi,\psi,\omega}_{t,k}(rg)(t)\right]+\\&I^{\xi,\phi,\psi,\omega}_{t,k}q(t)\left[I^{\xi,\phi,\psi,\omega}_{t,k}(rf)(t)I^{\xi,\phi,\psi,\omega}_{t,k}(pg)(t)+I^{\xi,\phi,\psi,\omega}_{t,k}(pf)(t)I^{\xi,\phi,\psi,\omega}_{t,k}(rg)(t)\right].\end{aligned}\tag{14}$$

Proof: To prove above theorem, putting $x=p,\ y=q$, and using lemma 1, we get

$$\begin{aligned}&I^{\xi,\phi,\psi,\omega}_{t,k}p(t)I^{\xi,\phi,\psi,\omega}_{t,k}(qfg)(t)+I^{\xi,\phi,\psi,\omega}_{t,k}q(t)I^{\xi,\phi,\psi,\omega}_{t,k}(pfg)(t)\geq,\\&I^{\xi,\phi,\psi,\omega}_{t,k}(pf)(t)I^{\xi,\phi,\psi,\omega}_{t,k}(qg)(t)+I^{\xi,\phi,\psi,\omega}_{t,k}(qf)(t)I^{\xi,\phi,\psi,\omega}_{t,k}(pg)(t).\end{aligned}\tag{15}$$

Now, multiplying both side by (15) $I^{\xi,\phi,\psi,\omega}_{t,k}r(t)$, we have

$$\begin{aligned}&I^{\xi,\phi,\psi,\omega}_{t,k}r(t)\left[I^{\xi,\phi,\psi,\omega}_{t,k}p(t)I^{\xi,\phi,\psi,\omega}_{t,k}(qfg)(t)+I^{\xi,\phi,\psi,\omega}_{t,k}q(t)I^{\xi,\phi,\psi,\omega}_{t,k}(pfg)(t)\right]\geq,\\&I^{\xi,\phi,\psi,\omega}_{t,k}r(t)\left[I^{\xi,\phi,\psi,\omega}_{t,k}(pf)(t)I^{\xi,\phi,\psi,\omega}_{t,k}(qg)(t)+I^{\xi,\phi,\psi,\omega}_{t,k}(qf)(t)I^{\xi,\phi,\psi,\omega}_{t,k}(pg)(t)\right],\end{aligned}\tag{16}$$

putting $x=r,y=q$, and using lemma 1, we get

$$\begin{aligned}&I^{\xi,\phi,\psi,\omega}_{t,k}r(t)I^{\xi,\phi,\psi,\omega}_{t,k}(qfg)(t)+I^{\xi,\phi,\psi,\omega}_{t,k}q(t)I^{\xi,\phi,\psi,\omega}_{t,k}(rfg)(t)\geq,\\&I^{\xi,\phi,\psi,\omega}_{t,k}(rf)(t)I^{\xi,\phi,\psi,\omega}_{t,k}(qg)(t)+I^{\xi,\phi,\psi,\omega}_{t,k}(qf)(t)I^{\xi,\phi,\psi,\omega}_{t,k}(rg)(t),\end{aligned}\tag{17}$$

multiplying both side by (17) $I^{\xi,\phi,\psi,\omega}_{t,k}p(t)$, we have

$$\begin{aligned}&I^{\xi,\phi,\psi,\omega}_{t,k}p(t)\left[I^{\xi,\phi,\psi,\omega}_{t,k}r(t)I^{\xi,\phi,\psi,\omega}_{t,k}(qfg)(t)+I^{\xi,\phi,\psi,\omega}_{t,k}q(t)I^{\xi,\phi,\psi,\omega}_{t,k}(rfg)(t)\right]\geq\\&I^{\xi,\phi,\psi,\omega}_{t,k}p(t)\left[I^{\xi,\phi,\psi,\omega}_{t,k}(rf)(t)I^{\xi,\phi,\psi,\omega}_{t,k}(qg)(t)+I^{\xi,\phi,\psi,\omega}_{t,k}(qf)(t)I^{\xi,\phi,\psi,\omega}_{t,k}(rg)(t)\right].\end{aligned}\tag{18}$$

With the same arguments as before, we can write

$$\begin{aligned}&I^{\xi,\phi,\psi,\omega}_{t,k}q(t)\left[I^{\xi,\phi,\psi,\omega}_{t,k}r(t)I^{\xi,\phi,\psi,\omega}_{t,k}(pfg)(t)+I^{\xi,\phi,\psi,\omega}_{t,k}p(t)I^{\xi,\phi,\psi,\omega}_{t,k}(rfg)(t)\right]\geq\\&I^{\xi,\phi,\psi,\omega}_{t,k}q(t)\left[I^{\xi,\phi,\psi,\omega}_{t,k}(rf)(t)I^{\xi,\phi,\psi,\omega}_{t,k}(pg)(t)+I^{\xi,\phi,\psi,\omega}_{t,k}(pf)(t)I^{\xi,\phi,\psi,\omega}_{t,k}(rg)(t)\right].\end{aligned}\tag{19}$$

Adding the inequalities (16), (18) and (19), we get required inequality (14).
Here, we give the lemma which is useful to prove our second main result.

Lemma 2. *Let f and g be two synchronous function on $[0,\infty[$. and $x,y:[0,\infty[\rightarrow[0,\infty[$. Then for all $k\geq 0$, $t>0$, $\xi>max\{0,-\phi-\omega\}$, $\gamma>max\{0,-\delta-\upsilon\}$ $\phi,\delta<1$, $\upsilon,\omega>-1$, $\phi-1<\psi<0$, $\delta-1<\zeta<0$, we have,*

$$\begin{aligned}&I^{\xi,\phi,\psi,\omega}_{t,k}x(t)I^{\gamma,\delta,\zeta,\upsilon}_{t,k}(yfg)(t)+I^{\gamma,\delta,\zeta,\upsilon}_{t,k}y(t)I^{\xi,\phi,\psi,\omega}_{t,k}(xfg)(t)\geq\\&I^{\xi,\phi,\psi,\omega}_{t,k}(xf)(t)I^{\gamma,\delta,\zeta,\upsilon}_{t,k}(yg)(t)+I^{\gamma,\delta,\zeta,\upsilon}_{t,k}(yf)(t)I^{\xi,\phi,\psi,\omega}_{t,k}(xg)(t).\end{aligned}\tag{20}$$

Proof: Now multiplying both side of (12) by

$$\begin{aligned}&\frac{(k+1)^{\upsilon+\delta+1}t^{(k+1)(-\delta-\gamma-2\upsilon)}}{\Gamma(\gamma)}\rho^{(k+1)\upsilon}y(\rho)\\&(t^{k+1}-\rho^{k+1})^{\gamma-1}\times{}_2F_1(\gamma+\delta+\upsilon,-\zeta;\gamma;1-(\frac{\rho}{t})^{k+1})\rho^{k},\end{aligned}\tag{21}$$

which remains positive in view of the condition stated in lemma 2, $\rho \in (0,t)$, $t>0$, we obtain

$$\begin{aligned}
&\frac{(k+1)^{\upsilon+\delta+1}t^{(k+1)(-\delta-\gamma-2\upsilon)}}{\Gamma(\gamma)}\rho^{(k+1)\upsilon}y(\rho)\\
&(t^{k+1}-\rho^{k+1})^{\gamma-1}\times{}_2F_1(\gamma+\delta+\upsilon,-\zeta;\gamma;1-(\frac{\rho}{t})^{k+1})\rho^k I_{t,k}^{\xi,\phi,\psi,\omega}(xfg)(t)\\
&+\frac{(k+1)^{\upsilon+\delta+1}t^{(k+1)(-\delta-\gamma-2\upsilon)}}{\Gamma(\gamma)}\rho^{(k+1)\upsilon}y(\rho)f(\rho)g(\rho)\\
&(t^{k+1}-\rho^{k+1})^{\gamma-1}\times{}_2F_1(\gamma+\delta+\upsilon,-\zeta;\gamma;1-(\frac{\rho}{t})^{k+1})\rho^k I_{t,k}^{\xi,\phi,\psi,\omega}x(t)\geq\\
&\frac{(k+1)^{\upsilon+\delta+1}t^{(k+1)(-\delta-\gamma-2\upsilon)}}{\Gamma(\gamma)}\rho^{(k+1)\upsilon}y(\rho)g(\rho)\\
&(t^{k+1}-\rho^{k+1})^{\gamma-1}\times{}_2F_1(\gamma+\delta+\upsilon,-\zeta;\gamma;1-(\frac{\rho}{t})^{k+1})\rho^k I_{t,k}^{\xi,\phi,\psi,\omega}(xf)(t)\\
&+\frac{(k+1)^{\upsilon+\delta+1}t^{(k+1)(-\delta-\gamma-2\upsilon)}}{\Gamma(\gamma)}\rho^{(k+1)\upsilon}y(\rho)f(\rho)\\
&(t^{k+1}-\rho^{k+1})^{\gamma-1}\times{}_2F_1(\gamma+\delta+\upsilon,-\zeta;\gamma;1-(\frac{\rho}{t})^{k+1})\rho^k I_{t,k}^{\xi,\phi,\psi,\omega}(xg)(t),
\end{aligned}\tag{22}$$

then integrating (22) over (0,t), we obtain

$$\begin{aligned}
&I_{t,k}^{\xi,\phi,\psi,\omega}(xfg)(t)I_{t,k}^{\gamma,\delta,\zeta,\upsilon}y(t)+I_{t,k}^{\xi,\phi,\psi,\omega}(x)(t)I_{t,k}^{\gamma,\delta,\zeta,\upsilon}(yfg)(t)\\
&\geq I_{t,k}^{\xi,\phi,\psi,\omega}(xf)(t)I_{t,k}^{\gamma,\delta,\zeta,\upsilon}yg(t)+I_{t,k}^{\xi,\phi,\psi,\omega}(xg)(t)I_{t,k}^{\gamma,\delta,\zeta,\upsilon}yf(t),
\end{aligned}\tag{23}$$

this ends the proof of inequality (20).

Theorem 4.*Let f and g be two synchronous function on $[0,\infty[$, and $r,p,q:[0,\infty)\rightarrow[0,\infty)$. Then for all $k\geq 0$, $t>0$, $\xi>max\{0,-\phi-\omega\}$,$\gamma>max\{0,-\delta-\upsilon\}$ $\phi,\delta<1$, $\upsilon,\omega>-1$, $\phi-1<\psi<0$, $\delta-1<\zeta<0$, we have*

$$\begin{aligned}
&I_{t,k}^{\xi,\phi,\psi,\omega}r(t)\times\\
&\left[I_{t,k}^{\xi,\phi,\psi,\omega}q(t)I_{t,k}^{\gamma,\delta,\zeta,\upsilon}(pfg)(t)+2I_{t,k}^{\xi,\phi,\psi,\omega}p(t)I_{t,k}^{\gamma,\delta,\zeta,\upsilon}(qfg)(t)+I_{t,k}^{\gamma,\delta,\zeta,\upsilon}q(t)I_{t,k}^{\xi,\phi,\psi,\omega}(pfg)(t)\right],\\
&+\left[I_{t,k}^{\xi,\phi,\psi,\omega}p(t)I_{t,k}^{\gamma,\delta,\zeta,\upsilon}q(t)+I_{t,k}^{\gamma,\delta,\zeta,\upsilon}p(t)I_{t,k}^{\xi,\phi,\psi,\omega}q(t)\right]I_{t,k}^{\xi,\phi,\psi,\omega}(rfg)(t)\geq,\\
&I_{t,k}^{\xi,\phi,\psi,\omega}r(t)\left[I_{t,k}^{\xi,\phi,\psi,\omega}(pf)(t)I_{t,k}^{\gamma,\delta,\zeta,\upsilon}(qg)(t)+I_{t,k}^{\gamma,\delta,\zeta,\upsilon}(qf)(t)I_{t,k}^{\xi,\phi,\psi,\omega}(pg)(t)\right]+\\
&I_{t,k}^{\xi,\phi,\psi,\omega}p(t)\left[I_{t,k}^{\xi,\phi,\psi,\omega}(rf)(t)I_{t,k}^{\gamma,\delta,\zeta,\upsilon}(qg)(t)+I_{t,k}^{\gamma,\delta,\zeta,\upsilon}(qf)(t)I_{t,k}^{\xi,\phi,\psi,\omega}(rg)(t)\right]+\\
&I_{t,k}^{\xi,\phi,\psi,\omega}q(t)\left[I_{t,k}^{\xi,\phi,\psi,\omega}(rf)(t)I_{t,k}^{\gamma,\delta,\zeta,\upsilon}(pg)(t)+I_{t,k}^{\gamma,\delta,\zeta,\upsilon}(pf)(t)I_{t,k}^{\xi,\phi,\psi,\omega}(rg)(t)\right].
\end{aligned}\tag{24}$$

Proof: To prove above theorem, putting $x=p$, $y=q$, and using Lemma 2 we get

$$\begin{aligned}
&I_{t,k}^{\xi,\phi,\psi,\omega}p(t)I_{t,k}^{\gamma,\delta,\zeta,\upsilon}(qfg)(t)+I_{t,k}^{\gamma,\delta,\zeta,\upsilon}q(t)I_{t,k}^{\xi,\phi,\psi,\omega}(pfg)(t)\geq,\\
&I_{t,k}^{\xi,\phi,\psi,\omega}(pf)(t)I_{t,k}^{\gamma,\delta,\zeta,\upsilon}(qg)(t)+I_{t,k}^{\gamma,\delta,\zeta,\upsilon}(qf)(t)I_{t,k}^{\xi,\phi,\psi,\omega}(pg)(t).
\end{aligned}\tag{25}$$

Now, multiplying both side by (25) $I_{t,k}^{\xi,\phi,\psi,\omega}r(t)$, we have

$$\begin{aligned}
&I_{t,k}^{\xi,\phi,\psi,\omega}r(t)\left[I_{t,k}^{\xi,\phi,\psi,\omega}p(t)I_{t,k}^{\gamma,\delta,\zeta,\upsilon}(qfg)(t)+I_{t,k}^{\gamma,\delta,\zeta,\upsilon}q(t)I_{t,k}^{\xi,\phi,\psi,\omega}(pfg)(t)\right]\geq,\\
&I_{t,k}^{\xi,\phi,\psi,\omega}r(t)\left[I_{t,k}^{\xi,\phi,\psi,\omega}(pf)(t)I_{t,k}^{\gamma,\delta,\zeta,\upsilon}(qg)(t)+I_{t,k}^{\gamma,\delta,\zeta,\upsilon}(qf)(t)I_{t,k}^{\xi,\phi,\psi,\omega}(pg)(t)\right],
\end{aligned}\tag{26}$$

putting $x=r$, $y=q$, and using lemma 2, we get

$$\begin{aligned}
&I_{t,k}^{\xi,\phi,\psi,\omega}r(t)I_{t,k}^{\gamma,\delta,\zeta,\upsilon}(qfg)(t)+I_{t,k}^{\gamma,\delta,\zeta,\upsilon}q(t)I_{t,k}^{\xi,\phi,\psi,\omega}(rfg)(t)\geq,\\
&I_{t,k}^{\xi,\phi,\psi,\omega}(rf)(t)I_{t,k}^{\gamma,\delta,\zeta,\upsilon}(qg)(t)+I_{t,k}^{\gamma,\delta,\zeta,\upsilon}(qf)(t)I_{t,k}^{\xi,\phi,\psi,\omega}(rg)(t),
\end{aligned}\tag{27}$$

multiplying both side by (27) $I_{t,k}^{\xi,\phi,\psi,\omega}p(t)$, we have

$$\begin{aligned} &I_{t,k}^{\xi,\phi,\psi,\omega}p(t)\left[I_{t,k}^{\xi,\phi,\psi,\omega}r(t)I_{t,k}^{\gamma,\delta,\zeta,\upsilon}(qfg)(t)+I_{t,k}^{\gamma,\delta,\zeta,\upsilon}q(t)I_{t,k}^{\xi,\phi,\psi,\omega}(rfg)(t)\geq\right], \\ &I_{t,k}^{\xi,\phi,\psi,\omega}p(t)\left[I_{t,k}^{\xi,\phi,\psi,\omega}(rf)(t)I_{t,k}^{\gamma,\delta,\zeta,\upsilon}(qg)(t)+I_{t,k}^{\gamma,\delta,\zeta,\upsilon}(qf)(t)I_{t,k}^{\xi,\phi,\psi,\omega}(rg)(t)\right]. \end{aligned} \tag{28}$$

With the same argument as before, we obtain

$$\begin{aligned} &I_{t,k}^{\xi,\phi,\psi,\omega}q(t)\left[I_{t,k}^{\xi,\phi,\psi,\omega}r(t)I_{t,k}^{\gamma,\delta,\zeta,\upsilon}(pfg)(t)+I_{t,k}^{\gamma,\delta,\zeta,\upsilon}p(t)I_{t,k}^{\xi,\phi,\psi,\omega}(rfg)(t)\right]\geq, \\ &I_{t,k}^{\xi,\phi,\psi,\omega}q(t)\left[I_{t,k}^{\xi,\phi,\psi,\omega}(rf)(t)I_{t,k}^{\gamma,\delta,\zeta,\upsilon}(pg)(t)+I_{t,k}^{\gamma,\delta,\zeta,\upsilon}(pf)(t)I_{t,k}^{\xi,\phi,\psi,\omega}(rg)(t)\right]. \end{aligned} \tag{29}$$

Adding the inequalities (26), (28) and (29), we follows the inequality (24).

*Remark.*If f,g,r,p *and* q satisfies the following condition,

1.The function f and g is asynchronous on $[0,\infty)$.
2.The function r,p,q are negative on $[0,\infty)$.
3.Two of the function r,p,q are positive and the third is negative on $[0,\infty)$.

then the inequality 14 and 24 are reversed.

4 Other Fractional Integral Inequalities

In this section, we proved some fractional integral inequalities for positive and continuous functions which as follows:

Theorem 5.*Suppose that f, g and h be three positive and continuous functions on $[0,\infty[$, such that*

$$(f(\sigma)-f(\rho))(g(\sigma)-g(\sigma))(h(\sigma)+h(\rho))\geq 0;\ \sigma,\rho\in(0,t)\ \ t>0, \tag{30}$$

and x be a nonnegative function on $[0,\infty)$. Then for all $k\geq 0$, $t>0$, $\xi>max\{0,-\phi-\omega\}$,$\gamma>max\{0,-\delta-\upsilon\}$ $\phi,\delta<1$, $\upsilon,\omega>-1$, $\phi-1<\psi<0$, $\delta-1<\zeta<0$, we have,

$$\begin{aligned} &I_{t,k}^{\xi,\phi,\psi,\omega}(x)(t)I_{t,k}^{\gamma,\delta,\zeta,\upsilon}(xfgh)(t)+I_{t,k}^{\xi,\phi,\psi,\omega}(xh)(t)I_{t,k}^{\gamma,\delta,\zeta,\upsilon}(xfg)(t), \\ &+I_{t,k}^{\xi,\phi,\psi,\omega}(xfg)(t)I_{t,k}^{\gamma,\delta,\zeta,\upsilon}(xh)(t)+I_{t,k}^{\xi,\phi,\psi,\omega}(xfgh)(t)I_{t,k}^{\gamma,\delta,\zeta,\upsilon}(x)(t), \\ &\geq I_{t,k}^{\xi,\phi,\psi,\omega}(xf)(t)I_{t,k}^{\gamma,\delta,\zeta,\upsilon}(xgh)(t)+I_{t,k}^{\xi,\phi,\psi,\omega}(xg)(t)I_{t,k}^{\gamma,\delta,\zeta,\upsilon}(xfh)(t), \\ &+I_{t,k}^{\xi,\phi,\psi,\omega}(xgh)(t)I_{t,k}^{\gamma,\delta,\zeta,\upsilon}(xf)(t)+I_{t,k}^{\xi,\phi,\psi,\omega}(xfh)(t)I_{t,k}^{\gamma,\delta,\zeta,\upsilon}(xg)(t). \end{aligned} \tag{31}$$

Proof: Since f, g and h be three positive and continuous functions on $[0,\infty[$ by (30), we can write

$$\begin{aligned} &f(\sigma)g(\sigma)h(\sigma)+f(\rho)g(\rho)h(\rho)+f(\sigma)g(\sigma)h(\rho)+f(\rho)g(\rho)h(\sigma), \\ &\geq f(\sigma)g(\rho)h(\sigma)+f(\sigma)g(\rho)h(\rho)+f(\rho)g(\sigma)h(\sigma)+f(\rho)g(\sigma)h(\rho). \end{aligned} \tag{32}$$

Now, multiplying both side of (32) by $\sigma^k x(\sigma)F(t,\sigma)$, $\sigma\in(0,t)$, $t>0$. Then the integrating resulting identity with respect to σ from 0 to t, we obtain by definition (4)

$$\begin{aligned} &I_{t,k}^{\xi,\phi,\psi,\omega}(xfgh)(t)+f(\rho)g(\rho)h(\rho)I_{t,k}^{\xi,\phi,\psi,\omega}x(t)+g(\tau)h(\rho)I_{t,k}^{\xi,\phi,\psi,\omega}(xf)(t), \\ &+f(\rho)g(\rho)I_{t,k}^{\xi,\phi,\psi,\omega}(xh)(t)\geq g(\rho)I_{t,k}^{\xi,\phi,\psi,\omega}(xfh)(t)+g(\rho)h(\rho)I_{t,k}^{\xi,\phi,\psi,\omega}(xf)(t), \\ &+f(\rho)I_{t,k}^{\xi,\phi,\psi,\omega}(xgh)(t)+f(\rho)h(\rho)I_{t,k}^{\xi,\phi,\psi,\omega}(xg)(t). \end{aligned} \tag{33}$$

Now multiplying both side of (33) by

$$\begin{aligned} &\frac{(k+1)^{\upsilon+\delta+1}t^{(k+1)(-\delta-\gamma-2\upsilon)}}{\Gamma(\gamma)}\rho^{(k+1)\upsilon}x(\rho), \\ &(t^{k+1}-\rho^{k+1})^{\gamma-1}\times{}_2F_1(\gamma+\delta+\upsilon,-\zeta;\gamma;1-(\frac{\rho}{t})^{k+1})\rho^k. \end{aligned} \tag{34}$$

which remains positive in view of the condition stated in Theorem 5, $\rho \in (0,t)$, $t > 0$ and integrating resulting identity with respective ρ from 0 to t, we obtain

$$\begin{aligned} & I_{t,k}^{\xi,\phi,\psi,\omega}(xfgh)(t)I_{t,k}^{\gamma,\delta,\zeta,\upsilon}x(t)+I_{t,k}^{\gamma,\delta,\zeta,\upsilon}(xfgh)(t)I_{t,k}^{\xi,\phi,\psi,\omega}x(t), \\ & +I_{t,k}^{\gamma,\delta,\zeta,\upsilon}(xh)(t)I_{t,k}^{\xi,\phi,\psi,\omega}(xgf)(t)+I_{t,k}^{\gamma,\delta,\zeta,\upsilon}(xfg)(t)I_{t,k}^{\xi,\phi,\psi,\omega}(xh)(t), \\ & \geq I_{t,k}^{\gamma,\delta,\zeta,\upsilon}xg(t)I_{t,k}^{\xi,\phi,\psi,\omega}(xfh)(t)+I_{t,k}^{\gamma,\delta,\zeta,\upsilon}(xgh)(t)I_{t,k}^{\xi,\phi,\psi,\omega}(xf)(t), \\ & +I_{t,k}^{\gamma,\delta,\zeta,\upsilon}(xf)(t)I_{t,k}^{\xi,\phi,\psi,\omega}(xgh)(t)+I_{t,k}^{\gamma,\delta,\zeta,\upsilon}(xfh)(t)I_{t,k}^{\xi,\phi,\psi,\omega}(xg)(t). \end{aligned} \tag{35}$$

which implies the proof inequality 31.
Here, we give another inequality which is as follows.

Theorem 6. *Let f, g and h be three positive and continuous functions on $[0,\infty[$, which satisfying the condition (30) and x and y be two nonnegative functions on $[0,\infty)$. Then for all $k \geq 0$, $t > 0$, $\xi > max\{0,-\phi-\omega\}$,$\gamma > max\{0,-\delta-\upsilon\}$ $\phi,\delta < 1$, $\upsilon,\omega > -1$, $\phi-1 < \psi < 0$, $\delta-1 < \zeta < 0$, we have,*

$$\begin{aligned} & I_{t,k}^{\xi,\phi,\psi,\omega}(x)(t)I_{t,k}^{\gamma,\delta,\zeta,\upsilon}(yfgh)(t)+I_{t,k}^{\xi,\phi,\psi,\omega}(xh)(t)I_{t,k}^{\gamma,\delta,\zeta,\upsilon}(yfg)(t), \\ & +I_{t,k}^{\xi,\phi,\psi,\omega}(xfg)(t)I_{t,k}^{\gamma,\delta,\zeta,\upsilon}(yh)(t)+I_{t,k}^{\xi,\phi,\psi,\omega}(xfgh)(t)I_{t,k}^{\gamma,\delta,\zeta,\upsilon}y(t), \\ & \geq I_{t,k}^{\xi,\phi,\psi,\omega}(xf)(t)I_{t,k}^{\gamma,\delta,\zeta,\upsilon}(ygh)(t)+I_{t,k}^{\xi,\phi,\psi,\omega}(xg)(t)I_{t,k}^{\gamma,\delta,\zeta,\upsilon}(yfh)(t), \\ & +I_{t,k}^{\xi,\phi,\psi,\omega}(xgh)(t)I_{t,k}^{\gamma,\delta,\zeta,\upsilon}(yf)(t)+I_{t,k}^{\xi,\phi,\psi,\omega}(xfh)(t)I_{t,k}^{\gamma,\delta,\zeta,\upsilon}(yg)(t). \end{aligned} \tag{36}$$

Proof: Multiplying both side of (32) by $\sigma^k x(\sigma)F(t,\sigma)$, $\sigma \in (0,t)$, $t > 0$, where $F(t,\sigma)$ defined by (8). Then the integrating resulting identity with respect to τ from 0 to t, we obtain by definition (4)

$$\begin{aligned} & I_{t,k}^{\xi,\phi,\psi,\omega}(xfgh)(t)+f(\rho)g(\rho)h(\rho)I_{t,k}^{\xi,\phi,\psi,\omega}x(t)+g(\tau)h(\rho)I_{t,k}^{\xi,\phi,\psi,\omega}(xf)(t), \\ & +f(\rho)g(\rho)I_{t,k}^{\xi,\phi,\psi,\omega}(xh)(t) \geq g(\rho)I_{t,k}^{\xi,\phi,\psi,\omega}(xfh)(t)+g(\rho)h(\rho)I_{t,k}^{\xi,\phi,\psi,\omega}(xf)(t), \\ & +f(\rho)I_{t,k}^{\xi,\phi,\psi,\omega}(xgh)(t)+f(\rho)h(\rho)I_{t,k}^{\xi,\phi,\psi,\omega}(xg)(t). \end{aligned} \tag{37}$$

Now multiplying both side of (37) by

$$\begin{aligned} & \frac{(k+1)^{\upsilon+\delta+1}t^{(k+1)(-\delta-\gamma-2\upsilon)}}{\Gamma(\gamma)}\rho^{(k+1)\upsilon}y(\rho), \\ & (t^{k+1}-\rho^{k+1})^{\gamma-1} \times_2 F_1(\gamma+\delta+\upsilon,-\zeta;\gamma;1-(\frac{\rho}{t})^{k+1})\rho^k. \end{aligned} \tag{38}$$

which remains positive in view of the condition stated in Theorem 6, $\rho \in (0,t)$, $t > 0$ and integrating resulting identity with respective ρ from 0 to t, we obtain

$$\begin{aligned} & I_{t,k}^{\xi,\phi,\psi,\omega}(xfgh)(t)I_{t,k}^{\gamma,\delta,\zeta,\upsilon}y(t)+I_{t,k}^{\gamma,\delta,\zeta,\upsilon}(yfgh)(t)I_{t,k}^{\xi,\phi,\psi,\omega}x(t) \\ & +I_{t,k}^{\gamma,\delta,\zeta,\upsilon}(yh)(t)I_{t,k}^{\xi,\phi,\psi,\omega}(xgf)(t)+I_{t,k}^{\gamma,\delta,\zeta,\upsilon}(yfg)(t)I_{t,k}^{\xi,\phi,\psi,\omega}(xh)(t) \\ & \geq I_{t,k}^{\gamma,\delta,\zeta,\upsilon}(yg)(t)I_{t,k}^{\xi,\phi,\psi,\omega}(xfh)(t)+I_{t,k}^{\gamma,\delta,\zeta,\upsilon}(ygh)(t)I_{t,k}^{\xi,\phi,\psi,\omega}(xf)(t) \\ & +I_{t,k}^{\gamma,\delta,\zeta,\upsilon}(yf)(t)I_{t,k}^{\xi,\phi,\psi,\omega}(xgh)(t)+I_{t,k}^{\gamma,\delta,\zeta,\upsilon}(yfh)(t)I_{t,k}^{\xi,\phi,\psi,\omega}(xg)(t). \end{aligned} \tag{39}$$

which implies the proof inequality 36.

5 Conclusion

Finally, we conclude this paper by remarking that we established some new Chebyshev type fractional integral inequalities for extended Chebyshev functional. Also, in this paper we established some fractional inequalities for positive and continuous functions. Note that, these established results give some contribution to theory of fractional calculus and inequalities.

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