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Exponentiated Generalized Weibull-Gompertz Distribution with Application in Survival Analysis

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Abstract: In this paper, a new model based on exponentiated generalized Weibull-Gompertz distribution is introduced. This new model is both useful and practical in many fields such as reliability, life testing, agriculture, industry, energy, human mortality, medicine and the world of the seas and oceans. Some statistical and reliability properties of this model are presented including moments, (reversed) hazard rate, mean residual life and mean inactivity time functions, among others. Finally, applications to real data sets are given to show the flexibility based on complete sample.

Keywords: Reversed (hazard) rate function, mean residual (past) lifetime.

1 Introduction

In analyzing lifetime data, we can use Weibull, Gompertz, Weibull-Gompertz, generalized Gompertz, exponential Weibull-Gompertz distribution and etc. The Weibull distribution has been used in many different fields with many applications see [7]. For many years, researchers have been developing various extensions and modified forms of the Weibull distribution. [12] introduced Weibull extension model and a detailed statistical was given in [9]. [5] introduced exponentiated flexible Weibull extension distribution.
On the other hand, The Gompertz distribution is important in describing the pattern of adult deaths see [11]. For low levels of infant mortality, the Gompertz force of mortality extends to the whole life span see [10] of populations with no observed mortality deceleration. [4] introduced generalized Gompertz distribution.
Also, [2] introduced beta-Gompertz distribution. It includes some well-known lifetime distributions such as beta-exponential and generalized Gompertz distributions as special sub models. [3] proposed exponentiated modified Weibull extension distribution. Moreover, [8] proposed a generalized Weibull-Gompertz distribution.

2 Exponentiated Generalized Weibull-Gompertz Distribution

The random variable X is said to be has exponentiated generalized Weibull-Gompertz distribution (EGWGD) if the CDF for $a,b,c,d,\theta > 0$ as follows:

$$F_X(x;a,b,c,d,\theta) = \left[1 - e^{-ax^b(e^{cx^d}-1)}\right]^{\theta}, \tag{2.1}$$

where b,θ and d are shape parameters, a is scale parameter and c is an accelerating parameter. The pdf $f_X(x;a,b,c,d,\theta)$ of $EGWGD(a,b,c,d,\theta)$ is

$$f_X(x;a,b,c,d,\theta) = ab\theta x^{b-1} e^{-ax^b(e^{cx^d}-1)+cx^d} \left(1 + \frac{cd}{b}x^d - e^{-cx^d}\right) \left[1 - e^{-ax^b(e^{cx^d}-1)}\right]^{\theta-1}. \tag{2.2}$$

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The survival function can be obtained as follows

$$R(x;a,b,c,d,\theta) = 1 - \left[1 - e^{-ax^b(e^{cx^d}-1)}\right]^{\theta};x > 0. \tag{2.3}$$

The hazard function $h(x)$ is

$$h(x;a,b,c,d,\theta) = \frac{ab\theta x^{b-1} e^{-ax^b(e^{cx^d}-1)+cx^d}\left(1+\frac{cd}{b}x^d - e^{-cx^d}\right)\left[1-e^{-ax^b(e^{cx^d}-1)}\right]^{\theta-1}}{1-\left[1-e^{-ax^b(e^{cx^d}-1)}\right]^{\theta}}. \tag{2.4}$$

Also the reversed hazard function $r(x)$ is

$$r(x;a,b,c,d,\theta) = ab\theta x^{b-1} e^{-ax^b(e^{cx^d}-1)+cx^d}\left(1+\frac{cd}{b}x^d - e^{-cx^d}\right)\left[1-e^{-ax^b(e^{cx^d}-1)}\right]^{-1}. \tag{2.5}$$

Figures 1 and 2 provide the PDF of EGWGD(a,b,c,d,θ) for different parameter values, also Figures 3 and 4 provide the failure rate function of EGWGD(a,b,c,d,θ) for different parameter values.

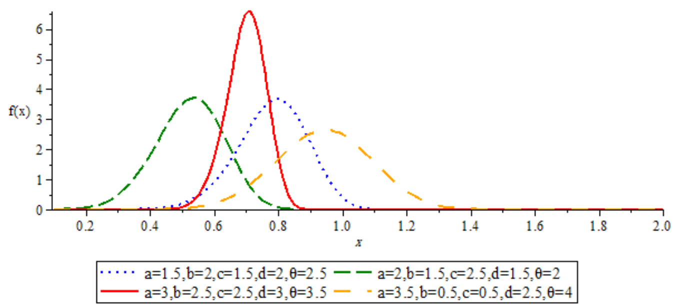

Figure 1. Plots of the pdf of the EGWG distribution for some values of the parameters.

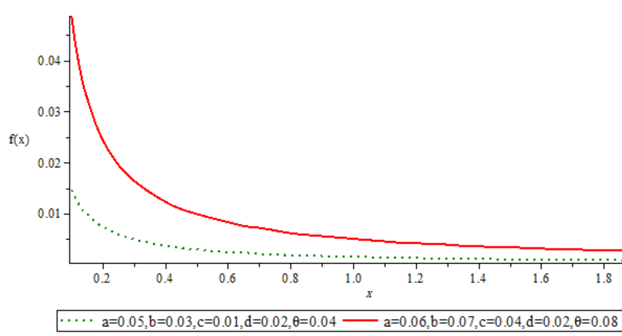

Figure 2. Plots of the pdf of the EGWG distribution for some values of the parameters.

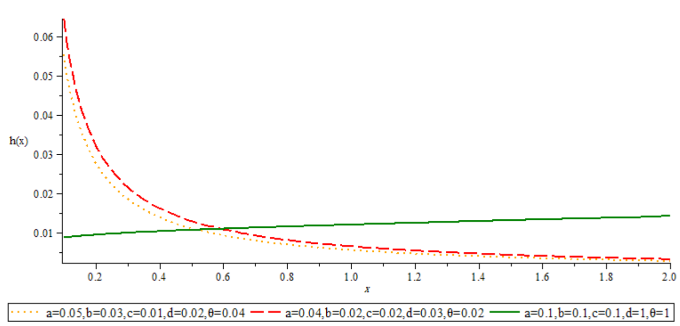

Figure 3. Plots of failure rate of the EGWG distribution for some values of the parameters.

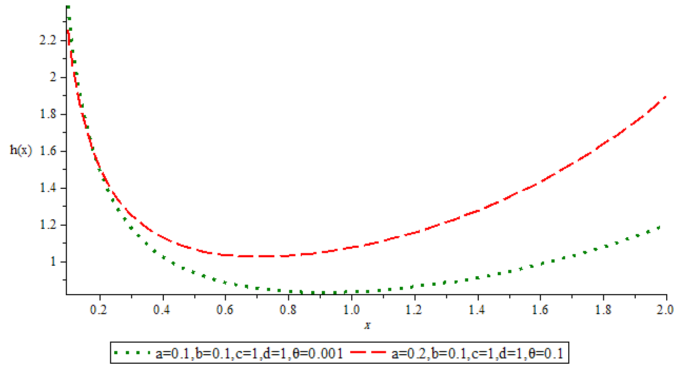

Figure 4. Plots of failure rate of the EGWG distribution for some values of the parameters.

Remark 2.1. From EGWGD(a,b,c,d,θ), we get:

1.Generalized Weibull - Gompertz distribution GWGD(a,b,c,d), when $\theta = 1$.
2.Gompertz distribution GD (a,c), when $\theta = 1, b = 0$ and $d = 1$.
3.Generalized Gompertz distribution GGD (a,c,θ), when $b = 0$ and $d = 1$.
4.Exponentiated modified Weibull extension, when $b = 0, c = (1/\alpha)^d,\ \alpha > 0$.
5.Exponential power distribution EPD(a,d,c), when $\theta = 1$ and $b = 0$.
6.Generalized exponential power distribution GEPD(a,d,c,θ), when $b = 0$.
7.Weibull extension model of Chen (2000), when $\theta = 1$, $b = 0$ and $c = 1$.
8.Weibull extension model of Xie et al. (2002), when $b = 0$.
9.Exponential distribution ED(a), when c tends to zero, $d = 1$, $\theta = 1$ and $b = 0$.
10.Generalized exponential distribution GED(a,θ), when c tends to zero, $d = 1$ and $b = 0$.

3 Statistical Properties

3.1 The median and the mode

We cannot get the quartile x_q of EGWGD(a,b,c,d,θ) in a closed form by using the equation $F_X(x_q; a, b, c, d, \theta) - q = 0$. Thus, by using Equation (2.1), we find that

$$(x_q)^b e^{c(x_q)^d} = \frac{-1}{a} \ln\left[1 - q^{\frac{1}{\theta}}\right], \quad 0 < q < 1. \tag{3.1}$$

The median $m(X)$ of EGWGD(a,b,c,d,θ) can be obtained from Equation (3.1), when $q = 0.5$, as follows

$$(x_0.5)^b e^{c(x_0.5)^d} = \frac{-1}{a} \ln\left[1 - (0.5)^{\frac{1}{\theta}}\right]. \tag{3.2}$$

Moreover, the mode of EGWGD(a,b,c,d,θ) can be obtained as a solution of the following nonlinear equation.

$$\frac{d}{dx} f_X(x; a, b, c, d, \theta) = 0$$

$$\frac{d}{dx}\left[x^{b-1} e^{-ax^b\left(e^{cx^d}-1\right)+cx^d}\left(1 + \frac{cd}{b}x^d - e^{-cx^d}\right)\left[1 - e^{-ax^b\left(e^{cx^d}-1\right)}\right]^{\theta-1}\right] = 0. \tag{3.3}$$

3.2 Moments

Lemma 3.1 If X has EGWGD(a,b,c,d,θ), the r^{th} moment of X, say $\mu_r^{\backprime}$, is given as follows for $a, b, c, d > 0, x > 0$ and θ is positive integer.

$$\mu_r^{\backprime} = \frac{ab\theta}{d}\sum_{i=0}^{\theta-1}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{\ell=0}^{\infty}\frac{(-1)^{i+j+k}c^{\ell}(a(1+i))^{j}}{j!\ell!d(ck)^{\frac{r+b(j+1)+\ell d}{d}}}\binom{j}{k}\binom{\theta-1}{i}\times \left[\left((1+j)^{\ell}-j^{\ell}\right)\Gamma\left(\frac{r+b(j+1)+d(l-1)}{d}+1\right)+\frac{d(1+j)^{\ell}}{kb}\Gamma\left(\frac{r+b(j+1)+\ell d}{d}+1\right)\right]. \quad (3.4)$$

Proof:

$$\mu_r^{\backprime} = \int_0^{\infty} x^r f_X(x;a,b,c,d,\theta)dx$$
$$= ab\theta\int_0^{\infty} x^{r+b-1}e^{-ax^b(e^{cx^d}-1)+cx^d}\left(1+\frac{cd}{b}x^d-e^{-cx^d}\right)\left[1-e^{-ax^b(e^{cx^d}-1)}\right]^{\theta-1}dx$$
$$= ab\theta I_1 - ab\theta I_2 + a\theta cd I_3, \quad (3.5)$$

where

$$I_1 = \int_0^{\infty} x^{r+b-1}e^{-ax^b(e^{cx^d}-1)+cx^d}\left[1-e^{-ax^b(e^{cx^d}-1)}\right]^{\theta-1}dx,$$
$$I_2 = \int_0^{\infty} x^{r+b-1}e^{-ax^b(e^{cx^d}-1)}\left[1-e^{-ax^b(e^{cx^d}-1)}\right]^{\theta-1}dx,$$
$$I_3 = \int_0^{\infty} x^{r+b+d-1}e^{-ax^b(e^{cx^d}-1)+cx^d}\left[1-e^{-ax^b(e^{cx^d}-1)}\right]^{\theta-1}dx,$$

since $0 < e^{-ax^b(e^{cx^d}-1)} < 1$ for $x > 0$ and θ is positive integer, then

$$I_1 = \sum_{i=0}^{\theta-1}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{\ell=0}^{\infty}\frac{(-1)^{i+j+k}(c(1+j))^{\ell}(a(1+i))^{j}}{j!\ell!}\binom{j}{k}\binom{\theta-1}{i}\int_0^{\infty} x^{r+b+bj+\ell d-1}e^{-ckx^d}dx$$
$$= \sum_{i=0}^{\theta-1}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{\ell=0}^{\infty}\frac{(-1)^{i+j+k}(c(1+j))^{\ell}(a(1+i))^{j}}{j!\ell!d(ck)^{\frac{r+b(j+1)+\ell d}{d}}}\binom{j}{k}\binom{\theta-1}{i}\Gamma\left(\frac{r+b(j+1)+d(\ell-1)}{d}+1\right). \quad (3.6)$$

Similarly, we find that

$$I_2 = \sum_{i=0}^{\theta-1}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{\ell=0}^{\infty}\frac{(-1)^{i+j+k}(cj)^{\ell}(a(1+i))^{j}}{j!\ell!d(ck)^{\frac{r+b(j+1)+\ell d}{d}}}\binom{j}{k}\binom{\theta-1}{i}\Gamma\left(\frac{r+b(j+1)+d(\ell-1)}{d}+1\right). \quad (3.7)$$

$$I_3 = \sum_{i=0}^{\theta-1}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{\ell=0}^{\infty}\frac{(-1)^{i+j+k}(c(1+j))^{\ell}(a(1+i))^{j}}{j!\ell!d(ck)^{\frac{r+b(j+1)+(\ell+1)d}{d}}}\binom{j}{k}\binom{\theta-1}{i}\Gamma\left(\frac{r+b(j+1)+\ell d}{d}+1\right). \quad (3.8)$$

Substituting from Equations (3.6), (3.7) and (3.8) into Equation (3.5), we get Equation (3.4). This completes the proof.

4 Reliability Analysis

4.1 Mean time to failure (repair)

Lemma 4.1. If T is a random variable has EGWGD(a,b,c,d,θ), then the mean time to failure (repair) is given as follows for $a,b,c,d > 0$ and θ is positive integer.

$$MTTF(MTTR) = \frac{ab\theta}{d}\sum_{i=0}^{\theta-1}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{\ell=0}^{\infty}\frac{(-1)^{i+j+k}c^{\ell}(a(1+i))^{j}}{j!\ell!d(ck)^{\frac{1+b(j+1)+ld}{d}}}\binom{j}{k}\binom{\theta-1}{i}\times \left[((1+j)^{\ell}-j^{\ell})\Gamma\left(\frac{1+b(j+1)+d(\ell-1)}{d}+1\right)+\frac{d(1+j)^{\ell}}{kb}\Gamma\left(\frac{1+b(j+1)+\ell d}{d}+1\right)\right]. \quad (4.1)$$

Proof. We have

$$MTTF(MTTR) = \int_0^{\infty} R(t)dt = \int_0^{\infty} t f(t;a,b,c,d,\theta)dt = \mu_1^{\backprime},$$

then from Equation (3.4), when $r = 1$, it is easy to prove Lemma 4.1.

4.2 Availability $A(t)$

Lemma 4.2. If the reliability function for a component is given by $R_T(t) = 1 - F_T(t)$, which T has EGWGD(a,b,c,d,θ), and the distribution of a repair time density is the PDF $g(t)$ of EGWGD(a,b,c,d,θ), then the availability $A(t)$ is given as $t = 0.5$.

Proof. It is easy to prove that.
Moreover, the mean time between failures (MTBF) is an important measure in repairable system (component). Mathematically,

$$MTBF = MTTF + MTTR. \tag{4.2}$$

4.3 Maintainability $V(t)$

If the repair time T is a random variable has a repair time density function $g(t)$ of EGWGD(a,b,c,d,θ), then the probability that the failed system will be back in service by time t is defined as

$$V(t) = P(T \leq t) = \int_0^t g(s)ds = \left[1 - e^{-at^b(e^{ct^d}-1)}\right]^\theta. \tag{4.3}$$

4.4 The Mean Residual (Past) Lifetime MRL (MPL) for EGWGD

Assuming that each component of the system has survived up to time t, the survival function of $T_i - t$ given that $T_i > t$, $i = 1, 2, \cdots, n$. This is the corresponding conditional survival function of the components at age t.

Lemma 4.3 If T is a random variable has EGWGD(a,b,c,d,θ), then the mean residual lifetime (MRL), is given as follows for $a, b, c, d > 0$ and θ is positive integer.

$$m(t) = \frac{1}{R(t)}\left[\mu_1^{\backslash} - I(t)\right], \tag{4.4}$$

where

$$R(t) = 1 - \left[1 - e^{-at^b(e^{ct^d}-1)}\right]^\theta,$$

$$\mu_1^{\backslash} = \frac{ab\theta}{d}\sum_{i=0}^{\theta-1}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{\ell=0}^{\infty}\frac{(-1)^{i+j+k}c^\ell(a(1+i))^j}{j!\ell!d(ck)^{\frac{1+b(j+1)+\ell d}{d}}}\binom{j}{k}\binom{\theta-1}{i}\times$$
$$\left[((1+j)^\ell - j^\ell)\Gamma\left(\frac{1+b(j+1)+d(\ell-1)}{d}+1\right) + \frac{d(1+j)^e ll}{kb}\Gamma\left(\frac{1+b(j+1)+\ell d}{d}+1\right)\right],$$

$$I(t) = t - \sum_{i=0}^{\theta}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{\ell=0}^{\infty}\frac{(-1)^{i+j+k+\ell}(ck)^\ell(ai)^j}{j!\ell!d(cj)^{\frac{bj+\ell d+1}{d}}}\binom{j}{k}\binom{\theta}{i}e^{cjt^d}\left[(cjt^d)^{\frac{bj+\ell d-d+1}{d}} + \right.$$
$$\left.\sum_{m=1}^{\frac{bj+\ell d-d+1}{d}}(-1)^m m!\binom{\frac{bj+\ell d-d+1}{d}}{m}(cjt^d)^{\frac{bj+\ell d-d+1}{d}-m}\right].$$

Proof. Since

$$m(t) = \frac{1}{R(t)}\int_t^\infty R(x)dx = \frac{1}{R(t)}\left[\mu_1^{\backslash} - I(t)\right],$$

where

$$\begin{aligned} I(t) &= \int_0^t R(x)dx = t - \int_0^t \left[1-e^{-ax^b(e^{cx^d}-1)}\right]^\theta dx \\ &= t - \sum_{i=0}^{\theta}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{\ell=0}^{\infty}\frac{(-1)^{i+j+k+\ell}(ck)^\ell(ai)^j}{j!\ell!}\binom{j}{k}\binom{\theta}{i}\int_0^t x^{bj+\ell d}e^{cjx^d}dx. \\ &= t - \sum_{i=0}^{\theta}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{\ell=0}^{\infty}\frac{(-1)^{i+j+k+\ell}(ck)^\ell(ai)^j}{j!\ell!d(cj)^{\frac{bj+\ell d+1}{d}}}\binom{j}{k}\binom{\theta}{i}e^{cjt^d}\left[(cjt^d)^{\frac{bj+\ell d-d+1}{d}} + \sum_{m=1}^{\frac{bj+\ell d-d+1}{d}}(-1)^m\times\right. \\ &\qquad \left. m!\binom{\frac{bj+\ell d-d+1}{d}}{m}(cjt^d)^{\frac{bj+\ell d-d+1}{d}-m}\right]. \end{aligned}$$

Lemma 4.4. If T is a random variable has EGWGD(a,b,c,d, θ), then the mean past lifetime (MPL), is given as follows for $a,b,c,d>0$ and θ is positive integer.

$$\begin{aligned} P(t) &= \left[1-e^{-at^b(e^{ct^d}-1)}\right]^{-\theta}\left(\sum_{i=0}^{\theta}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{\ell=0}^{\infty}\frac{(-1)^{i+j+k+\ell}(ck)^\ell(ai)^j}{j!\ell!d(cj)^{\frac{bj+\ell d+1}{d}}}\binom{j}{k}\binom{\theta}{i}e^{cjt^d}\times\right. \\ &\quad \left.\left[(cjt^d)^{\frac{bj+\ell d-d+1}{d}} + \sum_{m=1}^{\frac{bj+\ell d-d+1}{d}}(-1)^m m!\binom{\frac{bj+\ell d-d+1}{d}}{m}(cjt^d)^{\frac{bj+\ell d-d+1}{d}-m}\right]\right). \end{aligned} \tag{4.5}$$

Proof. Since

$$P(t) = \frac{1}{(F(t)}\int_0^t F(x)dx = \left[1-e^{-at^b(e^{ct^d}-1)}\right]^{-\theta}\int_0^t\left[1-e^{-ax^b(e^{cx^d}-1)}\right]^\theta dx. \tag{4.6}$$

Now, the value of

$$\begin{aligned} \int_0^t\left[1-e^{-ax^b(e^{cx^d}-1)}\right]^\theta dx &= \sum_{i=0}^{\theta}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{\ell=0}^{\infty}\frac{(-1)^{i+j+k+\ell}(ck)^\ell(ai)^j}{j!\ell!}\binom{j}{k}\binom{\theta}{i}\int_0^t x^{bj+\ell d}e^{cjx^d}dx, \\ &= \sum_{i=0}^{\theta}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\frac{(-1)^{i+j+k+\ell}(ck)^\ell(ai)^j}{j!\ell!d(cj)^{(bj+\ell d+1}d}\binom{j}{k}\binom{\theta}{i}e^{cjt^d}\times \\ &\quad \left[(cjt^d)^{\frac{bj+\ell d-d+1}{d}} + \sum_{m=1}^{\frac{bj+\ell d-d+1}{d}}(-1)^m m!\binom{bj+\ell d-d+1}{d}m(cjt^d)^{\frac{bj+\ell d-d+1}{d}-m}\right], \end{aligned} \tag{4.7}$$

from Equations (4.7) and (4.6), we get Equation (4.5).

5 Parameters Estimation

In this section, we derive the maximum likelihood estimates of the unknown parameters a,b,c,d and θ of EGWGD(a,b,c,d,θ), based on a complete sample. Consider a random sample $X_1, X_2, \cdots, X_n$ from EGWGD(a,b,c,d,θ). The likelihood function of this sample is

$$\ell = \prod_{i=1}^{n} f(x_i; a,b,c,d,\theta). \tag{5.1}$$

By substituting from Equation (2.2) into Equation (5.1), we get

$$\ell = \prod_{i=1}^{n} ab\theta x^{b-1}e^{-ax^b(e^{cx^d}-1)+cx^d}\left(1+\frac{cd}{b}x^d - e^{-cx^d}\right)\left[1-e^{-ax^b(e^{cx^d}-1)}\right]^{\theta-1}. \tag{5.2}$$

The log-likelihood function becomes

$$\mathscr{L} = n\ln(a)+n\ln(b)+n\ln(\theta)+c\sum_{i=1}^{n}(x_i)^d - a\sum_{i=1}^{n}(x_i)^b\left(e^{c(x_i)^d}-1\right)+(b-1)\sum_{i=1}^{n}\ln(x_i)+(\theta-1)\times$$
$$\sum_{i=1}^{n}\ln\left(1-e^{-a(x_i)^b\left(e^{c(x_i)^d}-1\right)}\right)+\sum_{i=1}^{n}\ln\left(1+\frac{cd}{b}(x_i)^d-e^{-c(x_i)^d}\right). \tag{5.3}$$

So, the normal equations are

$$\frac{\partial\mathscr{L}}{\partial a} = \frac{n}{\hat{a}}-\sum_{i=1}^{n}(x_i)^{\hat{b}}\left(e^{\hat{c}(x_i)^{\hat{d}}}-1\right)+(\theta-1)\sum_{i=1}^{n}\frac{(x_i)^{\hat{b}}\left(e^{\hat{c}(x_i)^{\hat{d}}}-1\right)e^{-\hat{a}(x_i)^{\hat{b}}\left(e^{\hat{c}(x_i)^{\hat{d}}}-1\right)}}{1-e^{-\hat{a}(x_i)^{\hat{b}}\left(e^{\hat{c}(x_i)^{\hat{}}}-1\right)}}=0. \tag{5.4}$$

$$\frac{\partial\mathscr{L}}{\partial b} = \frac{n}{\hat{b}}-\hat{a}\sum_{i=1}^{n}(x_i)^{\hat{b}}\left(e^{\hat{c}(x_i)^{\hat{d}}}-1\right)\ln(x_i)+\sum_{i=1}^{n}\ln(x_i)-\frac{\hat{c}\hat{d}}{\hat{b}^2}\sum_{i=1}^{n}\frac{(x_i)^{\hat{d}}}{\left(1+\frac{\hat{c}\hat{d}}{\hat{b}}(x_i)^{\hat{d}}-e^{-\hat{c}(x_i)^{\hat{d}}}\right)}+$$
$$\hat{a}(\hat{\theta}-1)\sum_{i=1}^{n}(x_i)^{\hat{b}}\left(e^{\hat{c}(x_i)^{\hat{d}}}-1\right)e^{-\hat{a}(x_i)^{\hat{b}}\left(e^{\hat{c}(x_i)^{\hat{d}}}-1\right)}\left(1-e^{-\hat{a}(x_i)^{\hat{b}}\left(e^{\hat{c}(x_i)^{\hat{d}}}-1\right)}\right)^{-1}=0. \tag{5.5}$$

$$\frac{\partial\mathscr{L}}{\partial c} = \sum_{i=1}^{n}(x_i)^{\hat{d}}-\hat{a}\sum_{i=1}^{n}(x_i)^{\hat{b}+\hat{d}}e^{\hat{c}(x_i)^{\hat{d}}}+\hat{a}(\hat{\theta}-1)\sum_{i=1}^{n}(x_i)^{\hat{b}+\hat{d}}e^{-\hat{a}(x_i)^{\hat{b}}\left(e^{\hat{c}(x_i)^{\hat{d}}}-1\right)+\hat{c}(x_i)^{\hat{d}}}\left(1-e^{-\hat{a}(x_i)^{\hat{b}}\left(e^{\hat{c}(x_i)^{\hat{d}}}-1\right)}\right)^{-1}$$
$$+\sum_{i=1}^{n}\frac{(x_i)^{\hat{d}}\left(\frac{\hat{d}}{\hat{b}}+e^{-\hat{c}(x_i)^{\hat{d}}}\right)}{1+\frac{\hat{c}\hat{d}}{\hat{b}}(x_i)^{\hat{d}}-e^{-\hat{c}(x_i)^{\hat{d}}}}=0. \tag{5.6}$$

$$\frac{\partial\mathscr{L}}{\partial d} = \hat{c}\sum_{i=1}^{n}(x_i)^{\hat{d}}\ln(x_i)-\hat{a}\hat{c}\sum_{i=1}^{n}(x_i)^{\hat{b}+\hat{d}}e^{\hat{c}(x_i)^{\hat{d}}}\ln(x_i)+\hat{a}\hat{c}(\hat{\theta}-1)\sum_{i=1}^{n}\frac{(x_i)^{\hat{b}+\hat{d}}e^{\hat{c}(x_i)^{\hat{d}}}\ln(x_i)}{1-e^{-\hat{a}(x_i)^{\hat{b}}\left(e^{\hat{c}(x_i)^{\hat{d}}}-1\right)}}+$$
$$\frac{\hat{c}}{\hat{b}}\sum_{i=1}^{n}\frac{(x_i)^{\hat{d}}\left(1+\hat{d}\ln(x_i)+\hat{b}e^{-\hat{c}(x_i)^{\hat{d}}}\ln(x_i)\right)}{1+\frac{\hat{c}\hat{d}}{\hat{b}}(x_i)^{\hat{d}}-e^{-\hat{c}(x_i)^{\hat{d}}}}=0. \tag{5.7}$$

The MLE of θ, say $\hat{\theta}(\hat{a},\hat{b},\hat{c},\hat{d})$ can be obtained as

$$\hat{\theta}(\hat{a},\hat{b},\hat{c},\hat{d}) = -n\left(\sum_{i=1}^{n}\ln\left[1-e^{-\hat{a}(x_i)^{\hat{b}}\left(e^{\hat{c}(x_i)^{\hat{d}}-1}\right)}\right]\right)^{-1}. \tag{5.8}$$

So, the MLEs of $\hat{a},\hat{b},\hat{c}$ and $\hat{d}$ can be obtained by solving four nonlinear Equations (**??**)-(5.4) by using Equation (5.8).

5.1 Asymptotic confidence bounds

We derive it for these parameters when $a,b,c,d>0$ and $\theta>0$ as the MLEs of the unknown parameters a,b,c,d can not be obtained in closed forms, by using variance covariance matrix see (Lawless(2003)), where

$$I_0^{-1}=\begin{bmatrix} -\frac{\partial^2\mathscr{L}}{\partial a^2} & -\frac{\partial^2\mathscr{L}}{\partial a\partial b} & -\frac{\partial^2\mathscr{L}}{\partial a\partial c} & -\frac{\partial^2\mathscr{L}}{\partial a\partial d} & -\frac{\partial^2\mathscr{L}}{\partial a\partial\theta}\\ -\frac{\partial^2\mathscr{L}}{\partial b\partial a} & -\frac{\partial^2\mathscr{L}}{\partial b^2} & -\frac{\partial^2\mathscr{L}}{\partial b\partial c} & -\frac{\partial^2\mathscr{L}}{\partial b\partial d} & -\frac{\partial^2\mathscr{L}}{\partial b\partial\theta}\\ -\frac{\partial^2\mathscr{L}}{\partial c\partial a} & -\frac{\partial^2\mathscr{L}}{\partial c\partial b} & -\frac{\partial^2\mathscr{L}}{\partial c^2} & -\frac{\partial^2\mathscr{L}}{\partial c\partial d} & -\frac{\partial^2\mathscr{L}}{\partial c\partial\theta}\\ -\frac{\partial^2\mathscr{L}}{\partial d\partial a} & -\frac{\partial^2\mathscr{L}}{\partial d\partial b} & -\frac{\partial^2\mathscr{L}}{\partial d\partial c} & -\frac{\partial^2\mathscr{L}}{\partial d^2} & -\frac{\partial^2\mathscr{L}}{\partial d\partial\theta}\\ -\frac{\partial^2\mathscr{L}}{\partial\theta\partial a} & -\frac{\partial^2\mathscr{L}}{\partial\theta\partial b} & -\frac{\partial^2\mathscr{L}}{\partial\theta\partial c} & -\frac{\partial^2\mathscr{L}}{\partial\theta\partial d} & -\frac{\partial^2\mathscr{L}}{\partial\theta^2} \end{bmatrix}^{-1}, \tag{5.9}$$

thus

$$I_0^{-1}=\begin{bmatrix} var(\hat{a}) & cov(\hat{a},\hat{b}) & cov(\hat{a},\hat{c}) & cov(\hat{a},\hat{d}) & cov(\hat{a},\hat{\theta}))\\ cov(\hat{b},\hat{a}) & var(\hat{b}) & cov(\hat{b},\hat{c}) & cov(\hat{b},\hat{d}) & cov(\hat{b},\hat{\theta}))\\ cov(\hat{c},\hat{a}) & cov(\hat{c},\hat{b}) & var(\hat{c}) & cov(\hat{c},\hat{d}) & cov(\hat{c},\hat{\theta}))\\ cov(\hat{d},\hat{a}) & cov(\hat{d},\hat{b}) & cov(\hat{d},\hat{c}) & var(\hat{d}) & cov(\hat{d},\hat{\theta}))\\ cov(\hat{\theta},\hat{a}) & cov(\hat{\theta},\hat{b}) & cov(\hat{\theta},\hat{c}) & cov(\hat{\theta},\hat{d}) & var(\hat{\theta})) \end{bmatrix}. \tag{5.10}$$

The elements in I_0 are given as follows:

$$
\begin{aligned}
\frac{\partial^2 \mathscr{L}}{\partial \theta^2} &= -\frac{n}{\theta^2},\\
\frac{\partial^2 \mathscr{L}}{\partial \theta \partial a} &= \sum_{i=1}^{n} \frac{x_i^b (e^{cx_i^d}-1) e^{-ax_i^b (e^{cx_i^d}-1)}}{1-e^{-ax_i^b (e^{cx_i^d}-1)}},\\
\frac{\partial^2 \mathscr{L}}{\partial \theta \partial b} &= \sum_{i=1}^{n} \frac{a x_i^b (e^{cx_i^d}-1) e^{-ax_i^b (e^{cx_i^d}-1)} \ln(x_i)}{1-e^{-ax_i^b (e^{cx_i^d}-1)}},\\
\frac{\partial^2 \mathscr{L}}{\partial \theta \partial c} &= \sum_{i=1}^{n} \frac{a x_i^{b+d} e^{-ax_i^b (e^{cx_i^d}-1)+cx_i^d}}{1-e^{-ax_i^b (e^{cx_i^d}-1)}},\\
\frac{\partial^2 \mathscr{L}}{\partial \theta \partial d} &= \sum_{i=1}^{n} \frac{ac x_i^{b+d} e^{-ax_i^b (e^{cx_i^d}-1)+cx_i^d} \ln(x_i)}{1-e^{-ax_i^b (e^{cx_i^d}-1)}},\\
\frac{\partial^2 \mathscr{L}}{\partial b^2} &= -\frac{n}{b^2} - \sum_{i=1}^{n} x_i^b (e^{cx_i^d}-1)(\ln(x_i))^2 - \frac{cd}{b^2} \sum_{i=1}^{n} \frac{2b^{-1} x_i^b (1+\frac{cd}{b} x_i^d - e^{-cx_i^d}) - (\frac{cd}{b^2} x_i^d - e^{-cx_i^d}) x_i^d}{\left(1+\frac{cd}{b} x_i^d - e^{-cx_i^d}\right)^2}\\
&\quad + a(\theta-1) \frac{\partial}{\partial b} \sum_{i=1}^{n} \frac{x_i^b (e^{cx_i^d}-1) e^{-ax_i^b (e^{cx_i^d}-1)} \ln(x_i)}{1-e^{-ax_i^b (e^{cx_i^d}-1)}},\\
\frac{\partial^2 \mathscr{L}}{\partial b \partial a} &= -\sum_{i=1}^{n} x_i^b (e^(cx_i^d)-1)\ln(x_i) + a(\theta-1) \frac{\partial}{\partial a} \sum_{i=1)} n \frac{x_i^b (e^{cx_i^d}-1) e^{-ax_i^b (e^{cx_i^d}-1)} \ln(x_i)}{1-e^{-ax_i^b (e^{cx_i^d}-1)}},\\
\frac{\partial^2 \mathscr{L}}{\partial b \partial c} &= -a \sum_{i=1}^{n} x_i^{b+d} e^{cx_i^d} \ln(x_i) - \frac{d}{b^2} \sum_{i=1}^{n} x_i^d \left(1+\frac{cd}{b} x_i^d - e^{-cx_i^d}\right) - \frac{cx^{2d}\left(\frac{d}{b} - e^{-cx_i^d}\right)}{\left(1+\frac{cd}{b} x_i^d - e^{-cx_i^d}\right)^2}\\
&\quad a(\theta-1) \frac{\partial}{\partial c} \sum_{i=1}^{n} x_i^b (e^{cx_i^d}-1) e^{-ax_i^b (e^{cx_i^d}-1)} \ln(x_i) \left(1-e^{-ax_i^b (e^{cx_i^d}-1)}\right),\\
\frac{\partial^2 \mathscr{L}}{\partial b \partial d} &= -ac \sum_{i=1}^{n} x_i^{b+d} e^{cx_i^d} (\ln(x_i))^2 + a(\theta-1) \frac{\partial}{\partial d} \sum_{i=1}^{n} \frac{x_i^b (e^{cx_i^d}-1) e^{-ax_i^b (e^{cx_i^d}-1)} \ln(x_i)}{1-e^{-ax_i^b (e^{cx_i^d}-1)}}\\
&\quad -\frac{c}{b^2} \sum_{i=1}^{n} \frac{x_i^d (1+\frac{cd}{b} x_i^d - e^{-cx_i d})(1+d\ln(x_i)) - dx_i^2 d[\frac{c}{b}(1+d\ln(x_i)) + ce^{-cx_i^d} \ln(x_i)]}{\left(1+\frac{cd}{b} x_i^d - e^{-cx_i^d}\right)^2},\\
\frac{\partial^2 \mathscr{L}}{\partial a^2} &= -\frac{n}{a^2} - (\theta-1) \sum_{i=1}^{n} \frac{x_i^2 b (e^{cx_i^d}-1)^2 e^{-ax_i^b (e^{cx_i^d}-1)}}{\left(1-e^{-ax_i^b (e^{cx_i^d}-1)}\right)^2},\\
\frac{\partial^2 \mathscr{L}}{\partial a \partial c} &= -\sum_{i=1}^{n} x_i^{b+d} e^{cx_i^d} + (\theta-1) \frac{\partial}{\partial c} \sum_{i=1}^{n} \frac{x_i^b (e^{cx_i^d}-1) e^{-ax_i^b (e^{cx_i^d}-1)}}{1-e^{-ax_i^b (e^{cx_i^d}-1)}},\\
\frac{\partial^2 \mathscr{L}}{\partial a \partial d} &= -c \sum_{i=1}^{n} x_i^{b+d} e^{cx_i^d} \ln(x_i) + (\theta-1) \frac{\partial}{\partial d} \sum_{i=1}^{n} \frac{x_i^b (e^{cx_i^d}-1) e^{-ax_i^b (e^{cx_i^d}-1)}}{1-e^{-ax_i^b (e^{cx_i^d}-1)}},\\
\frac{\partial^2 \mathscr{L}}{\partial c^2} &= -a \sum_{i=1}^{n} x_i^{b+2d} e^{cx_i^d} - \sum_{i=1}^{n} \frac{x_i^2 d e^{-cx_i^d} (1+\frac{cd}{b} x_i^d - e^{-cx_i^d}) + x_i^{2d} (\frac{d}{b} + e^{-cx_i^d})^2}{\left(1+\frac{cd}{b} x_i^d - e^{-cx_i^d}\right)^2} + a(\theta-1) \frac{\partial}{\partial c} \sum_{i=1}^{n} \frac{x_i^{b+d} e^{-ax_i^b (e^{cx_i^d}-1)+cx_i^d}}{1-e^{-ax_i^b (e^{cx_i^d}-1)}},\\
\frac{\partial^2 \mathscr{L}}{\partial c \partial d} &= \sum_{i=1}^{n} x_i^d \ln(x_i) - a \sum_{i=1}^{n} x_i^{b+d} e^{cx_i^d} (1+cx_i^d) \ln(x_i) + a(\theta-1) \frac{\partial}{\partial d} \sum_{i=1}^{n} \left(\frac{x_i^{b+d} e^{-ax_i^b (e^{cx_i^d}-1)+cx_i^d}}{1-e^{-ax_i^b (e^{cx_i^d}-1)}} + \frac{x_i^d (\frac{d}{b} + e^{-cx_i^d})}{1+\frac{cd}{b} x_i^d - e^{-cx_i^d}} \right),\\
\frac{\partial^2 \mathscr{L}}{\partial d^2} &= c \sum_{i=1}^{n} x_i^d (\ln(x_i))^2 - ac \sum_{i=1}^{n} x_i^{b+d} e^{cx_i^d} (1+cx_i^d (\ln(x_i))^2 +\\
&\quad ac(\theta-1) \frac{\partial}{\partial d} \sum_{i=1}^{n} \left(\frac{x_i^{b+d} e^{cx_i^d} \ln(x_i)}{1-e^{-ax_i^b (e^{cx_i^d}-1)}} + \frac{cx_i^d (1+d\ln(x_i) + be^{-cx_i^d} \ln(x_i))}{b\left(1+\frac{cd}{b} x_i^d - e^{-cx_i^d}\right)} \right).
\end{aligned}
$$

So, we can get the (1-δ)100% confidence intervals of the parameters a,b,c,d,θ as $\hat{a} \pm Z_{\frac{\delta}{2}} \sqrt{var(\hat{a})}$, $\hat{b} \pm Z_{\frac{\delta}{2}} \sqrt{var(\hat{b})}$, $\hat{c} \pm Z_{\frac{\delta}{2}} \sqrt{var(\hat{c})}$, $\hat{d} \pm Z_{\frac{\delta}{2}} \sqrt{var(\hat{d})}$ and $\hat{\theta} \pm Z_{\frac{\delta}{2}} \sqrt{var(\hat{\theta})}$.

6 Data Analysis and Discussion

We present the analysis of a real data set using the EGWGD(a,b,c,d,θ) model and compare it with the other fitted models like: GED, ED, GD and GWGD. The data have been obtained from [1]. It represents the lifetimes of 50 devices. The EGWGD model is used to fit this data set. The MLE(s) of the unknown parameter(s), the value of log – likelihood (L), K-S, AIC, CAIC and BIC value and its respective p-values for five different models are given in Tables 1 and 2.

Table 1: The MLE(s) of the parameter(s), (K-S) values and P-values.

	MLEs						
The Model	$\hat{a}$	$\hat{\theta}$	$\hat{b}$	$\hat{c}$	$\hat{d}$	$K-S$	$p-Value$
GED	0.021	0.902	-	-	-	0.194	0.0502
ED	0.022	-	-	-	-	0.191	0.0554
GD	0.011	-	-	0.018	-	0.157	0.1650
GWGD	0.0025	-	0.001	0.187	0.869	0.151	0.1804
EGWGD	0.000085	0.246	0.128	0.401	0.699	0.125	0.371

Table 2: The log-likelihood, AIC, CAIC and BIC values.

The Model	- L	AIC	CAIC	BIC
GED	241.36	484.72	484.96	488.54
ED	240.09	484.18	484.26	486.09
GD	235.39	474.78	475.024	478.60
GWGD	234.81	470.73	471.62	473.38
EGWGD	220.76	451.52	452.88	461.08

Tables 1 and 2 show that EGWGD is the best among those distributions because it has the smallest value of (K-S), AIC, CAIC and BIC . Figure 5 obtains the empirical and estimated reliability functions of the EGWG model for devices data.

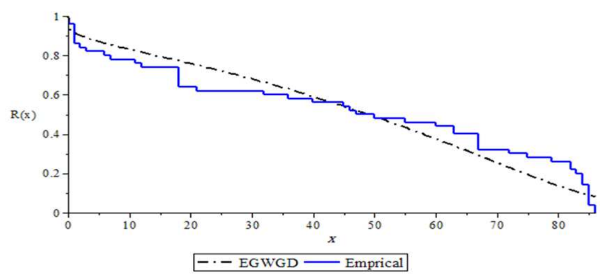

Fig. 5: The Empirical and estimated reliability functions of the

The variance covariance matrix is

$$I_0^{-1} = \begin{bmatrix} 5.854\times10^{-10} & -1.581\times10^{-4} & 8.574\times10^{-6} & 3.987\times10^{-8} & 4.158\times10^{-4} \\ -1.581\times10^{-4} & 3.101\times10^{-5} & -7.004\times10^{-4} & -1.012\times10^{-5} & 2.175\times10^{-4} \\ 8.574\times10^{-6} & -7.004\times10^{-4} & 3.215\times10^{-7} & -5.254\times10^{-5} & -2.158\times10^{-6} \\ 3.987\times10^{-8} & -1.012\times10^{-5} & -5.274\times10^{-4} & 9.257\times10^{-9} & -5.254\times10^{-5} \\ 4.158\times10^{-4} & 2.175\times10^{-4} & -2.158\times10^{-6} & -5.274\times10^{-4} & 8.154\times10^{-5} \\ \end{bmatrix}.$$

The approximate 95% two sided confidence interval of the parameters a,b,c,d and θ are [0.000037, 0.000132], [0.11708, 0.13891], [0.39988, 0.40211], [0.69881, 0.69918] and [0.2285, 0.26389] respectively.

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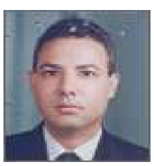

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