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Application of Sumudu Transform in Two-Parameter Fractional Telegraph Equation

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Abstract:: In this paper we investigate the solution of two-parameter fractional telegraph equation $a(D_t^{\alpha+1}N)(t,x) + b(D_t^{\beta}N)(t,x) = (D_x^{\gamma+1}N)(t,x) + \xi N(t,x) + \psi(t,x) \ldots (1)$ with positive real parameter *a*,*b* and *c* Here $D_t^{\alpha+1}$,D_t^{β} and $D_x^{\gamma+1}$ are operator of the Riemann-Liouville fractional derivative, where $0 \le \alpha \le 1, 0 \le \beta \le 1$ and $0 \le \gamma \le 1$ by the Sumudu and Fourier integral transform. The result presented are in compact and elegant form expressed in terms of Mittag-Leffler function and generalized Mittag-Leffler function.

Keywords: Two-parameter fractional telegraph equation, Generalized Mittag-Leffler function, Sumudu transform and Fourier transform.

1 Introduction

The fractional calculus is one of the most accurate tools to refine the description of natural phenomena. Fractional differential equation have attached in recent years a considerable interest due to their frequent appearance in various field and their more accurate models of system under consideration provided by fractional derivatives.

The fractional telegraph equation has recently been considered by many authors. Cascaval et al. [14] discussed the time fractional telegraph equations, dealing with wellposedness and presenting a study of their asymptotic behavior by using the Riemann-Liouville approach. Orsingher and Bghin [4] discussed the time fractional equation and telegraph processes are governed by time fractional telegraph equations. Chen et al. [7] examined and derived a solution of the time-fractional telegraph equation with three kind of nonhomogeneous boundary conditions, by the method of separation of variables. Recently Yakubovich [16] presented a general approach to describe fundamental solution of the fractional two-parameter telegraph equation (1)

2 Mathematical Prerequisites

Ageneralization of the Mittage-Leffler function (Mittage-Leffler, 1903,1905)[12],[13]

$$E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+1)}, (\alpha \in C, Re(\alpha) > 0) \ldots\ldots (2)$$

was introduced by wiman(1905)[20] in the general form

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+\beta)}, (\alpha,\beta \in C, Re(\alpha) > 0) \ldots\ldots (3)$$

The main result of these functions are available in the handbook of Erdelyi Magnus.Oberhettinger and Tricomi(1955,Section18.1)[4]and the monographs written by Dzherbashyas (1966,1993)[10][11],Prabhakar(1971)[17] introduced a generalization of (2) in the form

$$E_{\alpha,\beta}^{\gamma}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n z^n}{\Gamma(n\alpha+\beta)n!}, (\alpha,\beta,\gamma \in C, Re(\alpha) > 0) \ldots\ldots (4)$$

Where$(\gamma)_n$ is Pochhammer's symbol.

The Riemann –Liouville fractional integral of order ν is defined by Miller and Ross [8]

$${}_0D_t^{-\nu} f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-u)^{\nu-1} f(u)du, \qquad Re(\nu) > 0 .. (5)$$

Here we define the fractional derivative for $Re(\alpha) > 0$ in the form

$${}_0D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(u)}{(t-u)^{\alpha-n+1}} du, \qquad n = [\alpha]+1 \ldots (6)$$

Where$[\alpha]$ means the integral part of the number α, In particular, if $0 < \alpha < 1$,

$${}_0D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(u)}{(t-u)^{\alpha}} du, \ldots . (7)$$

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And if $\alpha = n, \{n \in N = 1,2,3..\}$, then

$$ {}_0D_t^\alpha f(t) = D^n f(t),\ \ D = \frac{d}{dt} $$

Caputo [9] introduced fractional derivative in the following form

$$ {}_0D_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{f^m(\tau)}{(t-\tau)^{\alpha-m+1}}d\tau, \\ \quad m-1<\alpha<m, Re(\alpha)>0 \\ \frac{d^m}{dt^m},\ \ if\ \alpha = m \end{cases} ..(8) $$

We also need the Weyl fractional operator defined by

$$ {}_{-\infty}D_x^\mu f(t) = \frac{1}{\Gamma(n-\mu)}\frac{d^n}{dt^n}\int_{-\infty}^t\int_0^t \frac{f(u)}{(t-u)^{\mu-n+1}}du, \ldots.(9) $$

Where $n = [\mu]$ is an integral part of $\mu > 0$. Its Fourier transform is given by Metzler and Klafter [15]

$$ F\{{}_{-\infty}D_x^\mu f(t)\} = (ik)^\mu f^*(k) \qquad \ldots(10) $$

Where $f^*(k)$ denote the Fourier transform of the function $f(x)$.

Further modification of result is given by Metzler and Klafter [15]

$$ F\{{}_{-\infty}D_x^\mu f(x)\} = -|k|^\gamma f^*(k) \qquad \ldots(11) $$

2.1 Definition: Sumudu Transform

Over the set of function

$$ A = \{f(t)|\exists M,\ \ \tau_1,\tau_2 > 0, |f(t)| < Me^{|t|/\tau_j}, \\ if\ \ t \in (-1)^j \times [0,\infty)\} $$

The Sumudu transform defined by

$$ G(u) = S[f(t)] = \int_0^\infty f(ut)e^{-t}\,dt,\ \ u \in (\tau_1,\tau_2) \ldots.(12) $$

For Further detail and properties of this transform [3]

Some result of inverse Sumudu transform,

(i) $S^{-1}\left[\frac{1}{u(u^{-\alpha}+u^{-\beta}+b)}\right] = \sum_{r=0}^\infty(-b)^r t^{\alpha(r+1)-1}\, E_{\alpha-\beta,\alpha(r+1)}^{r+1}\left[-at^{\alpha-\beta}\right](13)$

(ii) $S^{-1}\left[\frac{u^{-\alpha}+au^{-\beta}}{(u^{-\alpha}+au^{-\beta}+b)}\right] = \sum_{r=0}^\infty(-b)^r t^{\alpha r}\, E_{\alpha-\beta,\alpha r+1}^{r}\left[-at^{\alpha-\beta}\right]..(14)$

(iii) $S^{-1}\left[\frac{u^{-2\alpha}+au^{-\alpha}}{(u^{-2\alpha}+au^{-\alpha}+b)}\right] = \frac{1}{\sqrt{a^2-4b}}[(\lambda+a)E_\alpha(\lambda t^\alpha) - (\mu+a)E_\alpha(\mu t^\alpha)](15)$

Where $a^2 - 4b > 0$ and $E_\alpha(z)$ is Mittag-Leffler function and λ, μ are the real distinct roots of the quadratic equation $x^2 + ax + b = 0$

Sumudu transform of fractional derivative is given by

$$ S[{}_0D_t^\alpha f(t)] = u^{-\alpha}F(u) - \sum_{k=0}^{n-1}\frac{{}_0D_\alpha^{n-k}f(0)}{u^{\alpha-k}} \ldots(16) $$

For using the properties of Sumudu transform [5] [6] in particular if $0 < \alpha < 1$ we have

$$ {}_0D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t \frac{f(u)du}{(t-u)^\alpha} $$

$$ S[{}_0D_t^\alpha f(t)] = S\left[\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t \frac{f(u)du}{(t-u)^\alpha}\right] \\ = u^{-\alpha}F(u) \ldots.(17) $$

Where $F(u) = S[f(t)]$

Now we derive Sumudu transform of the fractional derivative introduced by Caputo [9]

$$ S[D_t^\alpha f(t)] = S\left[\frac{1}{\Gamma(m-\alpha)}\frac{d}{dt}\int_0^t \frac{f^m(\tau)d\tau}{(t-\tau)^{\alpha-m+1}}\right] \ldots(18) $$

By using Sumudu transform of the multiple differentiation we obtain

$$ S[D_t^\alpha f(t)] = u^{m-\alpha}\left[\frac{G(u)}{u^m} - \sum_{k=0}^{m-1}\frac{f^k(0)}{u^{\alpha-k}}\right] \ldots.(19) $$

Where $G(u) = S[f(t)]$

In the next section we derive the solution of Two-parameter fractional telegraph equation with Sumudu transform

3 Solution of Two-parameter Fractional Telegraph Equation with Sumudu Transform

Theorem: Consider the two-parameter fractional telegraph equation

$$ a(D_t^{\alpha+1}N)(t,x) + b\left(D_t^\beta N\right)(t,x) \\ = \left(D_x^{\gamma+1}N\right)(t,x) + \xi N(t,x) + \psi(t,x)(20) $$

With initial condition $N(x,0) = f(x)\ for\ \ x \in R$….(21)

where ψ is constant which describes the nonlinearity in the system, and is a nonlinear function for reaction kinetics then there holds the following formula for the solution of (13) (14).

$$ N(x,t) = \sum_{r=0}^\infty \frac{(-c')^r}{\sqrt{2\pi}} \int_{-\infty}^\infty t^{(\alpha+1)r} f^*(k)\ exp(-ikx) E_{\alpha-\beta+1,1+(\alpha+1)(r+1)}^r \left(-b't^{\alpha-\beta+1}\right) dk \\ + \frac{1}{a}\sum_{r=0}^\infty \frac{(-c')^r}{\sqrt{2\pi}}\int_0^t \tau^{\alpha(r+1)-1}\int_{-\infty}^\infty \psi^*(t-\tau,k)\exp(-ikx) \\ \times E_{\alpha-\beta+1,(\alpha+1)(r+1)}^{r+1}\left(-b'\tau^{\alpha-\beta+1}\right)dkd\tau $$

Proof: Apply the Sumudu Transform with respect to the time variable t and the using the boundary condition in eq.

(1) we find

$$au^{-(\alpha+1)}\tilde{N}(u,x) - au^{-(\alpha+1)}f(x) + bu^{-\beta}\tilde{N}(u,x) - bu^{-\beta}f(x) = \left(D_x^{\gamma+1}\tilde{N}\right)(u,x) + \xi\tilde{N}(u,x) + \psi(u,x)$$

Taking the Fourier Transform of the above equation

$$au^{-(\alpha+1)}\tilde{N}^*(u,k) - au^{-(\alpha+1)}f^*(k) + bu^{-\beta}\tilde{N}^*(u,k) - bu^{-\beta}f^*(k) = |k|^{\gamma+1}\tilde{N}^*(u,x) + \xi\tilde{N}^*(u,k) + \psi^*(u,k)$$

$$\left(au^{-(\alpha+1)} + bu^{-\beta} - |k|^{\gamma+1} - \xi\right)\tilde{N}^*(u,k) = \left(au^{-(\alpha+1)} + bu^{-\beta}\right)f^*(k) + \psi^*(u,k)$$

$$\tilde{N}^*(u,k) = \frac{\left(au^{-(\alpha+1)} + bu^{-\beta}\right)f^*(k)}{\left(au^{-(\alpha+1)} + bu^{-\beta} - |k|^{\gamma+1} - \xi\right)} + \frac{u\psi^*(u,k)}{u\left(au^{-(\alpha+1)} + bu^{-\beta} - |k|^{\gamma+1} - \xi\right)}$$

$$\tilde{N}^*(u,k) = \frac{\left(u^{-(\alpha+1)} + bu^{-\beta}\right)f^*(k)}{\left(u^{-(\alpha+1)} + b'u^{-\beta} + c'\right)} + \frac{u\psi^*(u,k)}{au\left(u^{-(\alpha+1)} + b'u^{-\beta} + c'\right)}$$

Where $b' = {}^{b}/_{a}$ and $c' = -\left(\frac{|k|^{\gamma+1}+\xi}{a}\right)$, Inverting the Sumudu Transform with the help of equation (13) (14).

$$N^*(t,k) = \sum_{r=0}^{\infty}(-c')^r\, t^{(\alpha+1)r}E^{r}_{\alpha-\beta+1,1+(\alpha+1)(r+1)}\left(-b't^{\alpha-\beta+1}\right)f^*(k) + \frac{1}{a}\sum_{r=0}^{\infty}(-c')^r\int_0^t \psi^*(t-\tau,k)\,\tau^{\alpha(r+1)-1} \times E^{r+1}_{\alpha-\beta+1,(\alpha+1)(r+1)}\left(-b'\tau^{\alpha-\beta+1}\right)d\tau$$

Using the convolution theorem of sumudu transform. Now we applying the inverse Fourier transform we get the required result of telegraph equation in terms of generalized Mittag-Leffler function

$$N(x,t) = \sum_{r=0}^{\infty}\frac{(-c')^r}{\sqrt{2\pi}}\int_{-\infty}^{\infty} t^{(\alpha+1)r}f^*(k)\,\exp(-ikx)E^{r}_{\alpha-\beta+1,1+(\alpha+1)(r+1)}\left(-b't^{\alpha-\beta+1}\right)dk + \frac{1}{a}\sum_{r=0}^{\infty}\frac{(-c')^r}{\sqrt{2\pi}}\int_0^t \tau^{\alpha(r+1)-1}\int_{-\infty}^{\infty}\psi^*(t-\tau,k)\exp(-ikx) \times E^{r+1}_{\alpha-\beta+1,(\alpha+1)(r+1)}\left(-b'\tau^{\alpha-\beta+1}\right)dk\,d\tau$$

3.1 Special case

When $f(x) = \delta(x)$, where $\delta(x)$ is dirac delta function. The theorem reduce to the following

Corollary (i) consider the Two-parameter fractional telegraph equation

$$a(D_t^{\alpha+1}N)(t,x) + b\left(D_t^{\beta}N\right)(t,x) = \left(D_x^{\gamma+1}N\right)(t,x) + \xi N(t,x) + \psi(t,x)..(22)$$

Subject to the initial condition $N(x,0) = \delta(x)$ ... (23)

for $0 \le \alpha \le 1, 0 \le \beta \le 1$ and $0 \le \gamma \le 1$, where $\delta(x)$ is dirac delta function. where ψ is constant which describes the nonlinearity in the system, and is a nonlinear function for reaction kinetics then there holds the following formula for the solution of (22) subject to the initial condition (23)

$$N(x,t) = \sum_{r=0}^{\infty}\frac{(-c')^r}{\sqrt{2\pi}}\int_{-\infty}^{\infty} t^{(\alpha+1)r}exp(-ikx)E^{r}_{\alpha-\beta+1,1+(\alpha+1)(r+1)}\left(-b't^{\alpha-\beta+1}\right)dk + \frac{1}{a}\sum_{r=0}^{\infty}\frac{(-c')^r}{\sqrt{2\pi}}\int_0^t \tau^{\alpha(r+1)-1}\int_{-\infty}^{\infty}\psi^*(t-\tau,k)\exp(-ikx) \times E^{r+1}_{\alpha-\beta+1,(\alpha+1)(r+1)}\left(-b'\tau^{\alpha-\beta+1}\right)dk\,d\tau \;....(24)$$

Where $b' = {}^{b}/_{a}$ and $c' = -\left(\frac{|k|^{\gamma+1}+\xi}{a}\right)$,

Now if we set $f(x) = \delta(x), \gamma = 1, \alpha + 1$ replaced by 2α and β replaced by α in equation (20). The following result obtained.

Corollary (ii) Consider the following

$$a\frac{\partial^{2\alpha}N(x,t)}{\partial t^{2\alpha}} + b\frac{\partial^{\alpha}N(x,t)}{\partial t^{\alpha}} = \frac{\partial^2 N(x,t)}{\partial t^2} + \xi N(t,x) + \psi(t,x)..(25)$$

With the initial conditions

$$N(x,0) = \delta(x), N_t(x,0) = 0 \quad 0 \le \alpha \le 1 \quad (26)$$

Where ψ is constant which describes the nonlinearity in the system, and is a nonlinear function for reaction kinetics then there holds the following formula for the solution of (25) subject to the initial condition (26)

$$N(x,t) = \frac{1}{\sqrt{2\pi}\sqrt{(b')^2 - 4c'}}\left[\int_{-\infty}^{\infty}\exp(-ikx)\{(\lambda + b')E_\alpha(\lambda t^\alpha) - (\mu + b')E_\alpha(\mu t^\alpha)\}dk\right] - \frac{1}{a\sqrt{2\pi}}\int_0^t \tau^{\alpha-1}\int_{-\infty}^{\infty}\exp(-ikx)\,\psi^*(t-\tau,k)\left\{E_{\alpha,\alpha}(\lambda\tau^\alpha) - E_{\alpha,\alpha}(\mu\tau^\alpha)\right\}dk\,d\tau \;...(27)$$

Where λ *and* μ are real and distinct roots of quadratic equation $x^2 + b'x + c' = 0$ which are given $\lambda = \frac{-b'+\sqrt{(b')^2-4c'}}{2}$ *and* $\mu = \frac{-b'-\sqrt{(b')^2-4c'}}{2}$

Conclusion

The crux of this paper is to propose the solution of two-parameter fractional telegraph equation with positive real parameters applying operator of the Riemann-Liouville fractional derivative and the Sumudu and Fourier integral transform. The outcomes presented are in compact and elegant form expressed in terms of Mittag-Leffler function and generalized Mittag-Leffler function. New models have been proposed with great efficiency and precision as compared to the old ones.

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