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Approximation Properties by Generalized-Baskakov-Kantorovich-Stancu Type Operators

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Abstract: In this paper, we introduce generalized-Baskakov-Kantorovich operators based on two parameters and investigate universal Korovkin type theorem and weighted Korovkin type theorem and also give quantitative estimates in term of classical modulus of continuity for continuous functions and first order differentiable functions. Further, we present order of approximation using second modulus of continuity, Ditzian-Totik modulus of smoothness, Peeter's K-functional and Lipschitz class.

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1 Introduction

For $f \in C[0,1]$, Bernstein [1] defined the linear positive operators which are the classical example of linear approximation as

$$B_n(f;x) = \sum_{k=0}^{n} \binom{n}{k} P_{n,k}(x) f\left(\frac{k}{n}\right), \tag{1}$$

where $P_{n,k}(x) = x^k(1-x)^{n-k}$, $x \in [0,1]$ and $n \in \mathbb{N}$. In this paper, Bernstein showed that these operators approximate uniformly on [0,1] to every continuous function $f \in C[0,1]$. But these operators are not suitable for discontinuous functions. Later on, Kantorovich [2] generalized the operators (1) to approximate the measurable functions (see Lorentz [3]). For $n \in \mathbb{N}$ and $f \in L_p[0,1]$, $1 \le p < \infty$, the Kantorovich operators $K_n : L_p([0,1]) \to L_p([0,1])$ defined by

$$K_n(f;x) = (n+1) \sum_{k=0}^{n} P_{n,k}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt. \tag{2}$$

In 1957, Baskakov [4] introduced a sequence of positive linear operators for continuous functions which is known as Baskakov operators now a days. Mihesan [5] gave an important generalization of Baskakov operators depending on a constant $a \ge 0$, independent on n as

$$B_n^a(f;x) = \sum_{k=0}^{\infty} W_{n,k}^a(x) f\left(\frac{k}{n}\right), \tag{3}$$

where

$$W_{n,k}^a(x) = e^{-\frac{ax}{1+x}} \frac{p_k(n,a)}{k!} \frac{x^k}{(1+x)^{k+n}}, \tag{4}$$

such that $\sum_{k=0}^{\infty} W_{n,k}^a(x) = 1$ and $p_k(n,a) = \sum_{i=0}^{\infty} \binom{n}{k} (n)_i a^{k-i}$, with $(n)_0 = 1, (n)_i = n(n+1)...(n+i-1)$. Wafi and Khatoon [6] defined a generalized Baskakov-Kantorovich operators on $[0,\infty)$ as follows

$$V_n^a(f;x) = n \sum_{k=0}^{\infty} W_{n,k}^a(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)dt. \tag{5}$$

Several authors (see [7]-[11]) have studied the approximation properties for the operators defined by (3) and (5) in different type of spaces. The Stancu operators [12] and its Kantorovich version [13] are respectively

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given by

$$B_n^{\alpha,\beta}(f;x)=(n+\beta+1)\sum_{k=0}^{n}x^k(1-x)^{n-k}f\left(\frac{k+\alpha}{n+\beta}\right), \quad (6)$$

$$K_n^{\alpha,\beta}(f;x)=(n+\beta+1)\sum_{k=0}^{n}x^k(1-x)^{n-k}\int_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+\alpha+1}{n+\beta+1}}f(t)dt, \quad (7)$$

where α,β are any two non negative real numbers such that $0\leq\alpha\leq\beta$. If $\alpha=\beta=0$, the operators (6) and (7) reduce to operators (1) and (2) respectively. Recently, many authors have studied in this direction for instance ([14], [15], [16], [17], [18] & [19]). In this article, we define generalized Baskakov-Kantorovich-Stancu type opeators as follows

$$T_{n,a}^{\alpha,\beta}(f;x)=(n+\beta)\sum_{k=0}^{\infty}W_{n,k}^{a}(x)\int_{\frac{k+\alpha}{n+\beta}}^{\frac{k+\alpha+1}{n+\beta}}f(t)dt, \quad (8)$$

where $W_{n,k}^{a}(x)$ is defined in (4). For $\alpha=\beta=0$, we get the operators (5). In this paper, we study the universal Korovkin type theorem, order of approximation using Ditzian modulus of smoothness, Peeter's K-functional, second modulus of continuity, Lipschitz class and weighted Korovkin type theorem.

2 Basic results

Lemma 2.1 [9] Let $a,x\geq 0$ and $n=1,2,3\ldots.$ Then, we have

$$V_n^a(1;x)=1,$$
$$V_n^a(t;x)=x+\frac{ax}{n(1+x)}+\frac{1}{2n},$$
$$V_n^a(t^2;x)=\frac{1+n}{n}x^2+\frac{1}{3n^2}+\frac{2x}{n}+\frac{a^2x^2}{n^2(1+x)^2}+\frac{2ax^2}{n(1+x)}+\frac{2ax}{n^2(1+x)}.$$

Lemma 2.2 For all $a\geq 0$, $x\in[0,\infty)$ and $0\leq\alpha\leq\beta$, we have the following recursive relation between $T_{n,a}^{\alpha,\beta}(t^m;x)$, $m=0,1,2,\ldots$ and $V_n^a(t^i;x)$, $i=0,1,2,\ldots$ where $f(t)=t^i$ is the test function as

$$T_{n,a}^{\alpha,\beta}(t^m;x)=\sum_{i=0}^{m}\binom{m}{i}\left(\frac{n}{n+\beta}\right)^i\left(\frac{\alpha}{n+\beta}\right)^{m-i}V_n^a(t^i;x).$$

Proof. From equation (8), we have

$$T_{n,a}^{\alpha,\beta}(f;x)=(n+\beta)\sum_{k=0}^{\infty}W_{n,k}^{a}(x)\int_{\frac{k+\alpha}{n+\beta}}^{\frac{k+\alpha+1}{n+\beta}}f(t)dt.$$

We can rewrite this equation as

$$\begin{aligned}T_{n,a}^{\alpha,\beta}(f;x)&=(n+\beta)\sum_{k=0}^{\infty}W_{n,k}^{a}(x)\int_{\frac{k}{n}}^{\frac{k+1}{n}}f\left(\frac{nt+\alpha}{n+\beta}\right)\frac{n}{n+\beta}dt\\&=n\sum_{k=0}^{\infty}W_{n,k}^{a}(x)\int_{\frac{k}{n}}^{\frac{k+1}{n}}f\left(\frac{nt+\alpha}{n+\beta}\right)dt\\&=n\sum_{k=0}^{\infty}W_{n,k}^{a}(x)\int_{\frac{k}{n}}^{\frac{k+1}{n}}\sum_{i=0}^{m}\binom{m}{i}\left(\frac{nt}{n+\beta}\right)^i\\&\times\left(\frac{\alpha}{n+\beta}\right)^{m-i}dt\\&=\sum_{i=0}^{m}\binom{m}{i}\left(\frac{n}{n+\beta}\right)^i\left(\frac{\alpha}{n+\beta}\right)^{m-i}\\&\times n\sum_{k=0}^{\infty}W_{n,k}^{a}(x)\int_{\frac{k}{n}}^{\frac{k+1}{n}}t^i dt\\&=\sum_{i=0}^{m}\binom{m}{i}\left(\frac{n}{n+\beta}\right)^i\left(\frac{\alpha}{n+\beta}\right)^{m-i}V_n^a(t^i;x).\end{aligned}$$

Lemma 2.3 Let $a,x\geq 0$. Then

$$T_{n,a}^{\alpha,\beta}(1;x)=1,$$
$$T_{n,a}^{\alpha,\beta}(t;x)=\frac{n}{n+\beta}x+\frac{a}{n+\beta}\frac{x}{1+x}+\frac{2\alpha+1}{2(n+\beta)},$$
$$T_{n,a}^{\alpha,\beta}(t^2;x)=\frac{n(1+n)}{(n+\beta)^2}x^2+\frac{2n(1+\alpha)}{(n+\beta)^2}x+\frac{a^2}{(n+\beta)^2}\frac{x^2}{(1+x)^2}+\frac{2an}{(n+\beta)^2}\frac{x^2}{(1+x)}+\frac{2a(1+\alpha)}{(n+\beta)^2}\frac{x}{1+x}+\frac{3\alpha^2++6\alpha+1}{3(n+\beta)^2}.$$

Proof. From Lemma 2.1 and recursive relation in Lemma 2.2, we prove Lemma 2.3.

Lemma 2.4 Let $\psi_x^i(t)=(t-x)^i, i=1,2,3,\ldots.$ Then, we have

$$T_{n,a}^{\alpha,\beta}(\psi_x^0(t);x)=1,$$
$$T_{n,a}^{\alpha,\beta}(\psi_x^1(t);x)=\left(\frac{n}{n+\beta}-1\right)x+\frac{a}{n+\beta}\frac{x}{1+x}+\frac{2\alpha+1}{2(n+\beta)},$$
$$T_{n,a}^{\alpha,\beta}(\psi_x^2(t);x)=\frac{n+\beta^2}{(n+\beta)^2}x^2+\frac{n-(2\alpha+1)\beta}{(n+\beta)^2}x+\frac{a^2}{(n+\beta)^2}\times\frac{x^2}{(1+x)^2}-\frac{2a\beta}{(n+\beta)^2}\frac{x^2}{(1+x)}+\frac{a(2+2\alpha)}{(n+\beta)^2}\times\frac{x}{1+x}+\frac{3\alpha^2+6\alpha+1}{3(n+\beta)^2}.$$

Proof. By using the linearity property and Lemma 2.3, we can easily prove Lemma 2.4.

3 Rate of convergence

Let $E = \left\{ f : x \in [0,\infty), \frac{f(x)}{1+x^2} \text{ is convergent as } x \to \infty \right\}$, and modulus of continuity of $f \in C[0,\infty)$ is defined by

$$\omega(f;\delta) = \sup_{x,y \in [0,\infty), |x-y| \leq \delta} |f(x) - f(y)|,$$

where $C[0,\infty)$ denotes the space of uniformly continuous functions on $[0,\infty)$. For any $\delta > 0$ and $x \in [0,\infty)$, we know that

$$|f(x) - f(y)| \leq \omega(f;\delta) \left(\frac{|x-y|}{\delta} + 1 \right). \tag{9}$$

Theorem 3.1 For $f \in C[0,\infty) \bigcap E$, we have

$$\lim_{n\to\infty} T_{n,a}^{\alpha,\beta}(f;x) = f(x),$$

uniformly in $[0,\infty)$.

Proof. Using Lemma 2.3, we find

$$\lim_{n\to\infty} T_{n,a}^{\alpha,\beta}(t^i;x) = x^i, \qquad i = 0,1,2,$$

uniformly in $[0,\infty)$. Applying universal Korovkin-type property (vi) of Theorem 4.1.4 in [20], we get the required result.

Theorem 3.2 If $f \in C[0,\infty) \bigcap E$, then, we have

$$|T_{n,a}^{\alpha,\beta}(f;x) - f(x)| \leq 2\omega(f;\delta_{n,a}^{\alpha,\beta}),$$

where $\delta_{n,a}^{\alpha,\beta} = \sqrt{T_{n,a}^{\alpha,\beta}(\psi_x^2(t);x)}$.

Proof. Using (9), we have

$$|T_{n,a}^{\alpha,\beta}(f;x) - f(x)| \leq (n+\beta) \sum_{k=0}^{\infty} W_{n,k}^a(x) \times \int_{\frac{k+\alpha}{n+\beta}}^{\frac{k+\alpha+1}{n+\beta}} |f(t) - f(x)| dt \tag{10}$$

$$|T_{n,a}^{\alpha,\beta}(f;x) - f(x)| \leq \left\{ 1 + \frac{(n+\beta)}{\delta} \sum_{k=0}^{\infty} W_{n,k}^a(x) \times \int_{\frac{k+\alpha}{n+\beta}}^{\frac{k+\alpha+1}{n+\beta}} |t-x| dt \right\} \omega(f;\delta_{n,a}^{\alpha,\beta}). \tag{11}$$

Applying the Cauchy-Schwarz inequality, we have

$$\int_{\frac{k+\alpha}{n+\beta}}^{\frac{k+\alpha+1}{n+\beta}} |t-x| dt \leq \frac{1}{\sqrt{n+\beta}} \left(\int_{\frac{k+\alpha}{n+\beta}}^{\frac{k+\alpha+1}{n+\beta}} |t-x|^2 dt \right)^{1/2},$$

this implies that

$$\sum_{k=0}^{\infty} W_{n,k}^a(x) \int_{\frac{k+\alpha}{n+\beta}}^{\frac{k+\alpha+1}{n+\beta}} |t-x| dt \leq \frac{1}{\sqrt{n+\beta}} \sum_{k=0}^{\infty} W_{n,k}^a(x) \times \left(\int_{\frac{k+\alpha}{n+\beta}}^{\frac{k+\alpha+1}{n+\beta}} |t-x|^2 dt \right)^{1/2}, \tag{12}$$

using (12) in (11), we get

$$|T_{n,a}^{\alpha,\beta}(f;x) - f(x)| \leq \left\{ 1 + \frac{\sqrt{T_{n,a}^{\alpha,\beta}(\psi_x^2(t);x)}}{\delta} \right\} \omega(f;\delta_{n,a}^{\alpha,\beta}).$$

Now, we prove the following Theorem for the first order differentiable functions as

Theorem 3.3 If $f'(x)$ has continuous derivative over $[0,\infty)$ and $\omega_1(f';\delta)$ is the modulus of continuity of $f'(x)$, then, for $0 \leq \alpha \leq \beta$ and $x \in [0,b], b < \infty$, we have

$$|T_{n,a}^{\alpha,\beta}(f;x) - f(x)| \leq \omega_1\left((n+\beta)^{-1}\right) \sqrt{T_{n,a}^{\alpha,\beta}(\psi_x^2(t);x)} \times \left\{ 1 + \sqrt{(n+\beta)} \sqrt{T_{n,a}^{\alpha,\beta}(\psi_x^2(t);x)} \right\}.$$

Proof. It is known that

$$\begin{aligned} f(x_1) - f(x_2) &= (x_1 - x_2) f'(\xi), \\ &= (x_1 - x_2) f'(x_1) \\ &\quad + (x_1 - x_2)[f'(\xi) - f'(x_1)], \end{aligned} \tag{13}$$

for $x_1, x_2 \in [0,b]$ and $x_1 < \xi < x_2$. Also, we have (see [3], Theorem 1.6.2, pp. 21)

$$|(x_1 - x_2)[f'(\xi) - f'(x_1)]| \leq |x_1 - x_2|(\lambda + 1)\omega_1(\delta), \tag{14}$$

where $\lambda = \lambda(x_1, x_2; \delta)$. Next, we find

$$|T_{n,a}^{\alpha,\beta}(f;x) - f(x)| = \left| (n+\beta) \sum_{k=0}^{\infty} W_{n,k}^a(x) \times \int_{\frac{k+\alpha}{n+\beta}}^{\frac{k+\alpha+1}{n+\beta}} f(t) - f(x) dt \right|. \tag{15}$$

Using (13) and (14), we get

$$|T_{n,a}^{\alpha,\beta}(f;x)-f(x)| \leq \left|(n+\beta)\sum_{k=0}^{\infty}W_{n,k}^{a}(x) \times \int_{\frac{k+\alpha}{n+\beta}}^{\frac{k+\alpha+1}{n+\beta}}(x-t)f'(x)dt\right| + \omega_1(\delta_n^{\beta})(n+\beta) \times \sum_{k=0}^{\infty}W_{n,k}^{a}(x)\int_{\frac{k+\alpha}{n+\beta}}^{\frac{k+\alpha+1}{n+\beta}}(\lambda+1)|t-x|dt$$

$$\leq \omega_1(\delta_n^{\beta})\left\{(n+\beta)\sum_{k=0}^{\infty}W_{n,k}^{a}(x) \times \int_{\frac{k+\alpha}{n+\beta}}^{\frac{k+\alpha+1}{n+\beta}}|t-x|dt + (n+\beta)\sum_{k=0}^{\infty}W_{n,k}^{a}(x) \times \int_{\frac{k+\alpha}{n+\beta}}^{\frac{k+\alpha+1}{n+\beta}}|t-x|\lambda(x,t;\delta)dt\right\}$$

$$\leq \omega_1(\delta_n^{\beta})\left\{(n+\beta)\sum_{k=0}^{\infty}W_{n,k}^{a}(x) \times \int_{\frac{k+\alpha}{n+\beta}}^{\frac{k+\alpha+1}{n+\beta}}|t-x|dt + (n+\beta)\sum_{k=0}^{\infty}W_{n,k}^{a}(x) \times \int_{\frac{k+\alpha}{n+\beta}}^{\frac{k+\alpha+1}{n+\beta}}(t-x)^2dt\right\}$$

$$\leq \omega_1(\delta_n^{\beta})\left(\sqrt{T_{n,a}^{\alpha,\beta}(\psi_x^2;x)} + \frac{T_{n,a}^{\alpha,\beta}(\psi_x^2;x)}{\delta_n^{\beta}}\right)$$

$$= \omega_1(\delta_n^{\beta})\sqrt{T_{n,a}^{\alpha,\beta}(\psi_x^2;x)} \times \left\{1+\frac{\sqrt{T_{n,a}^{\alpha,\beta}(\psi_x^2;x)}}{\delta_n^{\beta}}\right\}.$$

Taking $\delta_n^{\beta}=(n+\beta)^{-1}$, we get

$$|T_{n,a}^{\alpha,\beta}(f;x)-f(x)| \leq \omega_1((n+\beta)^{-1})\sqrt{T_{n,a}^{\alpha,\beta}(\psi_x^2(t);x)} \times \left\{1+\sqrt{(n+\beta)}\sqrt{T_{n,a}^{\alpha,\beta}(\psi_x^2(t);x)}\right\}.$$

4 Direct Estimates

Ditzian-Totik Modulus of smoothness [21] is defined as:

$$\omega_{\varphi^{\lambda}}^2(f;\delta) = \sup_{0<h\leq\delta} \| \Delta_{h\varphi(x)}^2 f(x) \|, = \sup_{0<h\leq\delta} \sup_{x\pm h\varphi^{\lambda}\in[0,\infty)} |f(x-h\varphi^{\lambda}(x))-2f(x) + f(x+h\varphi^{\lambda}(x))|,$$

where $\varphi^2(x)=x(x+1)$. And, Peetre's K-functional [21] is given by

$$K_{\varphi^{\lambda}}(f,\delta^2) = \inf_{g}\left(\|f-g\|_{C[0,\infty)}+\delta^2\|\varphi^2\lambda g''\|_{C[0,\infty)}\right), \quad g,g'\in AC_{loc}. \tag{16}$$

The K-functional is equivalent to the modulus of smoothness, i.e.,

$$C^{-1}K_{\varphi^{\lambda}}(f,\delta^2)\leq\omega_{\varphi^{\lambda}}^2(f,\delta)\leq CK_{\varphi^{\lambda}}(f,\delta^2), C>0. \tag{17}$$

First result based on Ditziaz-Totik modulus of smoothness was given by Ditzian [22] for the Bernstein polynomials as:

$$|B_n(f;x)-f(x)|\leq C\omega_{\varphi^{\lambda}}^2(f,n^{-\frac{1}{2}}\varphi(x)^{1-\lambda}).$$

Here, we give the similar result for the operator $T_{n,a}^{\alpha,\beta}$.

Theorem 4.1 Let $f\in C[0,\infty)$. Then, we have

$$|T_{n,a}^{\alpha,\beta}(f;x)-f(x)|\leq C\omega_{\varphi^{\lambda}}^2\left(f,(n+\beta)^{-\frac{1}{2}}\varphi(x)^{1-\lambda}\right),$$

for large value of n,where $0\leq\lambda\leq1$, $\varphi^2(x)=x(x+1)$.
Proof. Using (16) and (17), we have

$$\| f-g \|_{C[0,\infty)}\leq A\omega_{\varphi^{\lambda}}^2\left(f,(n+\beta)^{-\frac{1}{2}}\varphi(x)^{1-\lambda}\right), \tag{18}$$

$$(n+\beta)^{-1} \varphi(x)^{2-2\lambda}\|\varphi^{2\lambda}g''\|_{C[0,\infty)} \leq B\omega_{\varphi^{\lambda}}^2\left(f,(n+\beta)^{-\frac{1}{2}}\varphi(x)^{1-\lambda}\right). \tag{19}$$

Choosing $g_n\equiv g_{n,x,\lambda}$ for fixed x and λ such that

$$|T_{n,a}^{\alpha,\beta}(f;x)-f(x)| \leq |T_{n,a}^{\alpha,\beta}(f-g_n;x)-(f-g_n)(x)| + |T_{n,a}^{\alpha,\beta}(g_n;x)-g_n(x)|, \leq 2\| f-g_n \|_{C[0,\infty)} + |T_{n,a}^{\alpha,\beta}(g_n;x)-g_n(x)|.$$

From (18), we get

$$|T_{n,a}^{\alpha,\beta}(f;x)-f(x)| \leq 2A\omega_{\varphi^{\lambda}}^2\left(f,(n+\beta)^{-\frac{1}{2}}\varphi(x)^{1-\lambda}\right) + |T_{n,a}^{\alpha,\beta}(g_n;x)-g_n(x)|. \tag{20}$$

Now, the last term can be calculated by using Taylor's formula

$$|T_{n,a}^{\alpha,\beta}(g_n(t)-g_n(x);x)|$$
$$\leq |g_n'(x)T_{n,a}^{\alpha,\beta}((t-x);x)| + \left|T_{n,a}^{\alpha,\beta}\left(\int_t^x (x-u)g_n''(u)du;x\right)\right|$$
$$\leq T_{n,a}^{\alpha,\beta}\left(\frac{|x-\frac{k}{n}|}{\varphi^{2\lambda}(x)}\int_{\frac{k}{n}}^x \varphi^{2\lambda}(u)|g_n''(u)du|;x\right)$$
$$\leq \| \varphi^{2\lambda}g_n'' \|_{C[0,\infty)} \frac{1}{\varphi^{2\lambda}(x)} T_{n,a}^{\alpha,\beta}((t-x)^2;x)$$
$$\leq \| \varphi^{2\lambda}g_n'' \|_{C[0,\infty)} \frac{1}{\varphi^{2\lambda}(x)}$$
$$\times \frac{x(x+1)}{n+\beta}\frac{(n+\beta)T_{n,a}^{\alpha,\beta}((t-x)^2;x)}{x(x+1)}$$
$$\leq \| \varphi^{2\lambda}g_n'' \|_{C[0,\infty)} \frac{\varphi^2(x)(n+\beta)^{-1}}{\varphi^{2\lambda}(x)}$$
$$\times \frac{(n+\beta)T_{n,a}^{\alpha,\beta}((t-x)^2;x)}{x(x+1)}.$$
$$\leq \| \varphi^{2\lambda}g_n'' \|_{C[0,\infty)} \varphi^{2(1-\lambda)}(x)(n+\beta)^{-1}$$
$$\times \frac{(n+\beta)T_{n,a}^{\alpha,\beta}((t-x)^2;x)}{x(x+1)}.$$

For the large value of n, we get

$$\frac{(n+\beta)T_{n,a}^{\alpha,\beta}((t-x)^2;x)}{x(1+x)} \leq 1. \tag{21}$$

From (20) and (21), we have

$$|T_{n,a}^{\alpha,\beta}(g_n(t)-g_n(x);x)| \leq B\omega_{\varphi^\lambda}^2\left(f,(n+\beta)^{-\frac{1}{2}}\varphi(x)^{1-\lambda}\right). \tag{22}$$

Using (21) and (22), we get

$$|T_{n,a}^{\alpha,\beta}(f(t)-f(x);x)| \leq M\omega_\lambda^2\left(f,(n+\beta)^{-1/2}\varphi(x)^{1-\lambda}\right),$$

where $M = max(2A,B)$.

Let $C_B[0,\infty)$ denote the space of real valued continuous and bounded functions f on $[0,\infty)$ endowed with the norm $\|f\| = \sup_{0\leq x<\infty}|f(x)|$. Then, for any $\delta > 0$, Peeter's K-functional is defined as

$$K_2(f,\delta) = inf\{\|f-g\| + \delta\|g''\| : g \in C_B^2[0,\infty)\},$$

where $C_B^2[0,\infty) = \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}$. By Devore and Lorentz [[23], p.177, Theorem 2.4], there exits an absolute constant $C > 0$ such that

$$K_2(f;\delta) \leq C\omega_2(f;\sqrt{\delta}),$$

where $\omega_2(f;\delta)$ is the second order modulus of continuity is defined as

$$\omega_2(f,\sqrt{\delta}) = \sup_{0<h\leq\sqrt{\delta}}\sup_{x\in[0,\infty)}|f(x+2h)-2f(x+h)+f(x)|.$$

Theorem 4.2 Let $f \in C_B^2[0,\infty)$. Then for all $x \in [0,\infty)$ there exist a constant $C > 0$ such that

$$|\, T_{n,a}^{\alpha,\beta}(f;x) - f(x)\,| \leq C\omega_2(f;\sqrt{\gamma_{n,a}^{\alpha,\beta}(x)}) + \omega\left(f;T_{n,a}^{\alpha,\beta}(\psi_x;x)\right),$$

where

$$\gamma_{n,a}^{\alpha,\beta}(x) = \left\{\frac{n+2\beta^2}{(n+\beta)^2}x^2 + \frac{n-\beta}{(n+\beta)^2}x + \frac{2a^2}{(n+\beta)^2}\frac{x^2}{(1+x)^2} + \frac{a(3+4\alpha)}{(n+\beta)^2}\frac{x}{1+x} + \frac{7\alpha^2+4\alpha+2}{3(n+\beta)^2}\right\}.$$

Proof. First, we define the auxiliary operators

$$\hat{T}_n^{\alpha,\beta}(f;x) = T_{n,a}^{\alpha,\beta}(f;x) + f(x) - f\left(\Lambda_{n,a}^{\alpha,\beta}(x)\right),$$

where

$$\Lambda_{n,a}^{\alpha,\beta}(x) = \left(\frac{\beta}{n+\beta}\right)x + \frac{a}{n+\beta}\frac{x}{1+x} + \frac{2\alpha+1}{2(n+\beta)} + x.$$

We find that

$$\hat{T}_n^{\alpha,\beta}(1;x) = 1,$$

$$\hat{T}_n^{\alpha,\beta}(\psi_x(t);x) = 0,$$

$$|\hat{T}_n^{\alpha,\beta}(f;x)| \leq 3\|f\|. \tag{23}$$

Let $g \in C_B^2[0,\infty)$. By the Taylor's theorem

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-v)g''(v)dv. \tag{24}$$

Now, using (24) and the auxiliary operators (23) is given

$$\hat{T}_n^{\alpha,\beta}(g;x) - g(x) = g'(x)\hat{T}_n^{\alpha,\beta}(t-x;x) + \hat{T}_n^{\alpha,\beta}\left(\int_x^t (t-v)g''(v)dv;x\right)$$
$$= \hat{T}_n^{\alpha,\beta}\left(\int_x^t (t-v)g''(v)dv;x\right)$$
$$= T_{n,a}^{\alpha,\beta}\left(\int_x^t (t-v)g''(v)dv;x\right)$$
$$- \int_x^{\Lambda_{n,a}^{\alpha,\beta}(x)} \left(\Lambda_{n,a}^{\alpha,\beta}(x) - v\right)g''(v)dv.$$

Therefore

$$|\hat{T}_n^{\alpha,\beta}(g;x)-g(x)| \leq \left|T_{n,a}^{\alpha,\beta}\Big(\int_x^t (t-v)g''(v)dv;x\Big)\right| + \left|\int_x^{\Lambda_{n,a}^{\alpha,\beta}(x)} \Big(\Lambda_{n,a}^{\alpha,\beta}(x)-v\Big)g''(v)dv\right|. \quad (25)$$

Since

$$\left|\int_x^t (t-v)g''(v)dv\right| \leq (t-x)^2 \parallel g'' \parallel \quad (26)$$

and

$$\left|\int_x^{\Lambda_{n,a}^{\alpha,\beta}(x)} \Big(\Lambda_{n,a}^{\alpha,\beta}(x)-v\Big)g''(v)dv\right| \leq \Big(\Lambda_{n,a}^{\alpha,\beta}(x)-x\Big)^2 \parallel g'' \parallel, \quad (27)$$

then from (25), (26) and (27), we have

$$\begin{aligned}|\hat{T}_n^{\alpha,\beta}(g;x)-g(x)| &\leq \Big\{T_{n,a}^{\alpha,\beta}((t-x)^2;x) + \Big(\Lambda_{n,a}^{\alpha,\beta}(x)-x\Big)^2\Big\}\|g''\| \\ &= \Big\{T_{n,a}^{\alpha,\beta}((t-x)^2;x) + \Big(T_{n,a}^{\alpha,\beta}(\psi_x;x)\Big)^2\Big\}\|g''\| \\ &= \gamma_{n,a}^{\alpha,\beta}(x)\|g''\|. \quad (28)\end{aligned}$$

Next

$$\begin{aligned}|T_{n,a}^{\alpha,\beta}(f;x)-f(x)| &\leq |\hat{T}_n^{\alpha,\beta}(f-g;x)|+|(f-g)(x)| \\ &+ |\hat{T}_n^{\alpha,\beta}(g;x)-g(x)| \\ &+ \big|f(\Lambda_{n,a}^{\alpha,\beta}(x))-f(x)\big|,\end{aligned}$$

using (28), we have

$$\begin{aligned}|T_{n,a}^{\alpha,\beta}(f;x)-f(x)| &\leq 4\|f-g\|+\gamma_{n,a}^{\alpha,\beta}(x)\|g''\| \\ &+ \omega\Big(f;T_{n,a}^{\alpha,\beta}(\psi_x;x)\Big).\end{aligned}$$

By the definition of Peetre's K-functional

$$\begin{aligned}|T_{n,a}^{\alpha,\beta}(f;x)-f(x)| &\leq C\omega_2\big(f;\sqrt{\gamma_{n,a}^{\alpha,\beta}(x)}\big) \\ &+ \omega\Big(f;T_{n,a}^{\alpha,\beta}(\psi_x;x)\Big).\end{aligned}$$

Now, we use the Lipschitz type space

$$Lip_M^* = \left\{f\in C[0,\infty): |f(t)-f(x)| \leq M\frac{|t-x|^\alpha}{(t+x)^{\frac{\alpha}{2}}}\right\},$$

where $x,t\in(0,\infty)$, M is a constant and $0<\alpha\leq 1$, to prove the following theorem:

Theorem 4.3 Let $f\in Lip_M^*(\alpha)$ and $x\in(0,\infty)$. Then, we have

$$|T_{n,a}^{\alpha,\beta}(f;x)-f(x)| \leq M\left[\frac{\Lambda_n^{\alpha,\beta}(x)}{x}\right]^{\frac{\alpha}{2}},$$

where $\Lambda_n^{\alpha,\beta}(x) = T_n^{\alpha,\beta}((t-x)^2;x)$.

Proof Let $\alpha=1$ and $x\in(0,\infty)$. Then, for $f\in Lip_M^*(1)$, we have

$$\begin{aligned}|T_{n,a}^{\alpha,\beta}(f;x)-f(x)| &\leq (n+\beta)\sum_{k=0}^\infty W_{n,k}^a(x)\int_{\frac{k+\alpha}{n+\beta}}^{\frac{k+\alpha+1}{n+\beta}} |f(t)-f(x)|dt \\ &\leq M(n+\beta)\sum_{k=0}^\infty W_{n,k}^a(x)\int_{\frac{k+\alpha}{n+\beta}}^{\frac{k+\alpha+1}{n+\beta}} \frac{|t-x|}{\sqrt{t+x}}dt. \\ &\leq \frac{M}{\sqrt{x}}(n+\beta)\sum_{k=0}^\infty W_{n,k}^a(x)\int_{\frac{k+\alpha}{n+\beta}}^{\frac{k+\alpha+1}{n+\beta}} |t-x|dt \\ &\leq \frac{M}{\sqrt{x}}T_{n,a}^{\alpha,\beta}(|t-x|;x) \\ &\leq M\frac{\sqrt{T_{n,a}^{\alpha,\beta}((t-x)^2;x)}}{\sqrt{x}} \\ &= M\left(\frac{\Lambda_n^{\alpha,\beta}(x)}{x}\right)^{\frac{1}{2}}.\end{aligned}$$

Thus, the assertion hold for $\alpha=1$. Now, we will prove for $\alpha\in(0,1)$. From the Holder inequality with $p=\frac{1}{\alpha}, q=\frac{1}{1-\alpha}$, we have

$$\begin{aligned}&|T_{n,a}^{\alpha,\beta}(f;x)-f(x)| = \left(\sum_{k=0}^\infty W_{n,k}^a(x)\right. \\ &\left.\times\left((n+\beta)\int_{\frac{k+\alpha}{n+\beta}}^{\frac{k+\alpha+1}{n+\beta}} |f(t)-f(x)|dt\right)^{\frac{1}{\alpha}}\right)^\alpha \big(W_{n,k}^a(x)\big)^{1-\alpha} \\ &\leq \left(\sum_{k=0}^\infty W_{n,k}^a(x)\left((n+\beta)\int_{\frac{k+\alpha}{n+\beta}}^{\frac{k+\alpha+1}{n+\beta}} |f(t)-f(x)|dt\right)^{\frac{1}{\alpha}}\right)^\alpha.\end{aligned}$$

Since $f\in Lip_M^*(\alpha)$, we obtain

$$\begin{aligned}&|T_{n,a}^{\alpha,\beta}(f;x)-f(x)| \\ &\leq M\left((n+\beta)\sum_{k=0}^\infty W_{n,k}^a(x)\int_{\frac{k+\alpha}{n+\beta}}^{\frac{k+\alpha+1}{n+\beta}} \frac{|t-x|}{\sqrt{t+x}}dt\right)^\alpha \\ &\leq \frac{M}{x^{\frac{\alpha}{2}}}\left((n+\beta)\sum_{k=0}^\infty W_{n,k}^a(x)\int_{\frac{k+\alpha}{n+\beta}}^{\frac{k+\alpha+1}{n+\beta}} |t-x|dt\right)^\alpha \\ &= \frac{M}{x^{\frac{\alpha}{2}}}\big(T_{n,a}^{\alpha,\beta}(|t-x|;x)\big)^\alpha \\ &\leq M\left(\frac{\Lambda_n^{\alpha,\beta}(x)}{x}\right)^{\frac{\alpha}{2}}.\end{aligned}$$

5 Weighted approximation

Here we investigate the weighted approximation theorem on positive semi axes($[0,\infty)$). We recall the weighted spaces of the functions ([24], [25]) as follows
$B_\rho[0,\infty) = \{f(x) : |f(x)| \leq M_f\rho(x), \rho(x)$ is weight function, M_f is a constant depending on f and $x \in [0,\infty)\}$, $C_\rho[0,\infty)$ is the space of continuous function in $B_\rho[0,\infty)$ with the norm $\|f(x)\|_\rho = \sup\limits_{x\in[0,\infty)} \frac{|f(x)|}{\rho(x)}$ and $C_\rho^K = \{f \in C_\rho : \lim\limits_{|x|\to\infty} \frac{f(x)}{\rho(x)} = K$, where K is a constant depending on $f\}$.

Theorem 5.1 Let $T_{n,a}^{\alpha,\beta}$ be the sequence of linear positive operators defined by (8). Then for $f \in C_\rho^K$,
$\lim\limits_{n\to\infty} \|T_{n,a}^{\alpha,\beta}(f;x) - f(x)\|_\rho = 0.$

Proof To prove the theorem, it is sufficient to show that

$$\lim_{n\to\infty} \|T_{n,a}^{\alpha,\beta}(t^i;x) - x^i\|_\rho = 0, \qquad for \quad i = 0,1,2.$$

It is obvious that $\lim\limits_{n\to\infty} \|T_{n,a}^{\alpha,\beta}(1;x) - 1\|_\rho = 0$. Now,from the Lemma 2.3, we have

$$\begin{aligned}\|T_{n,a}^{\alpha,\beta}(t;x) - x\| &= \sup_{x\in[0,\infty)} \frac{\left|\left(T_{n,a}^{\alpha,\beta}(t;x) - x\right|\right.}{1+x^2}\\ &\leq \left(\frac{\beta}{n+\beta}\right) \sup_{x\in[0,\infty)} \frac{x}{1+x^2}\\ &+ \frac{a}{n+\beta} \sup_{x\in[0,\infty)} \frac{x}{(1+x)(1+x)^2}\\ &+ \frac{2\alpha+1}{2(n+\beta)} \sup_{x\in[0,\infty)} \frac{1}{(1+x)^2}\end{aligned}$$

$$\| T_n^{\alpha,\beta}(t;x) - x \|_\rho \quad \to 0 \qquad as \;\; n \to \infty.$$

Also, we can write

$$\begin{aligned}\|T_{n,a}^{\alpha,\beta}(t^2;x) - x^2\| &= \sup_{x\in[0,\infty)} \frac{|T_{n,a}^{\alpha,\beta}(t^2;x) - x^2|}{1+x^2}\\ &\leq \frac{n(1-2\beta)+\beta^2}{(n+\beta)^2} \sup_{x\in[0,\infty)} \frac{x^2}{1+x^2}\\ &+ \frac{n(2+2\alpha)}{(n+\beta)^2} \sup_{x\in[0,\infty)} \frac{x}{1+x^2}\\ &+ \frac{a^2}{(n+\beta)^2} \sup_{x\in[0,\infty)} \frac{x^2}{(1+x)^2(1+x^2)}\\ &+ \frac{2an}{(n+\beta)^2} \sup_{x\in[0,\infty)} \frac{x^2}{(1+x)(1+x^2)}\\ &+ \frac{a(2+2\alpha)}{(n+\beta)^2} \sup_{x\in[0,\infty)} \frac{x}{(1+x)(1+x^2)}\\ &+ \frac{3\alpha^2+1}{3(n+\beta)^2} \sup_{x\in[0,\infty)} \frac{1}{(1+x^2)}.\end{aligned}$$

Which implies that $\|T_{n,a}^{\alpha,\beta}(t^2;x) - x^2\|_\rho \to 0$ as $n \to \infty$. Hence, proof is completed.

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