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Existence Results of Solution to a Coupled System of Multi-Point Boundary Value Problems

Hammad Khalil[1,*], *Riffat Rashim*[1], *Kamal Shah*[2] *and Rahmat Ali Khan*[2]

[1] Department of Mathematics, Faculty of Basic and applied Sciences, University of Poonch Rawalakot, AJK, Pakistan
[2] Department of Mathematics, University of Malakand, KPK, Pakistan

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Abstract: In this paper we studied the existence and uniqueness properties of solution of a fractional order differential equation subject to nonlocal boundary constrains in the form of multi-point boundary conditions. The problems are highly nonlinear fractional order system of differential equations. The system under consideration is a more general form and many systems of the aforesaid area are a special cases. By using the classical fixed point theorems and contraction mappings, we develop sufficient conditions which guarantees existence unique solutions of the system. Finally, we demonstrate our main results by providing two examples.

Keywords: Multi-point boundary value problems, coupled system, fractional differential equations, existence and uniqueness of solutions, fixed point theorems.

1 Introduction

In the last few decades the fractional calculus attracted the attentions of many researchers of various disciplines such as mathematics, physics computer science as well as engineering. The reason behind this popularity is the wide range of applications in many real world problems. The studies of fractional calculus devoted to the analysis of fractional differential equations (FDEs) is of special importance due to its applications in many scientific and engineering disciplines. For detail studied we refer the reader to study [1,2,3,4,5] and the references there in. In last few years many authors have studied the existence of unique of solutions of initial and terminal value problem for FDEs, see for example [6,7,8,9, 10]. FDEs subject to nonlocal conditions are recently being investigated by many authors, for example Shah et al.[11], investigated the existence property of solutions nonlocal FDEs. Mathematical model of FDEs plays important rolls in modeling system having hereditary properties. FDEs related to modeling Memory characteristic of various materials and genetical problem in biological studies are more reliable as compare to integer order differential equations, for detail see [12,13,14,15,16] and the reference there in. It has been investigated that system of boundary value problems for FDEs are involved in numerous phenomena and models of physics, biology and psychology. Due to these reasons, researchers are taking interest in the study of FDEs especially there coupled systems. For example, when we mobilized human behavior and nature in the form of mathematical model will led us to a coupled system of FDEs. For such type of application see [17,18,19,20,21].
The above applications motivated our interest to the study of FDEs and we consider coupled system FDEs subject to of

* Corresponding author e-mail: hammadk310@gmail.com

m-points boundary conditions of the form

$$
\begin{cases}
D_{0+}^{\alpha}u(t) = f(t, v(t), D^{p}v(t)), \\
D_{0+}^{\beta}v(t) = g(t, u(t), D^{q}u(t)), \\
u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0,\ D^{\delta}u(1) = \sum_{i=1}^{m-2}\lambda_i D^{\delta}u(\eta_i), \\
v(0) = v'(0) = \cdots = v^{(n-2)}(0) = 0,\ D^{\gamma}v(1) = \sum_{i=1}^{m-2}\delta_i D^{\gamma}v(\xi_i), \\
\text{where } 0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1,\ 0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1,
\end{cases} \tag{1}
$$

where $\alpha, \beta \in (n-1, n],\ \gamma, \delta \in (0,1]$ and $D_{0+}^{\alpha}, D_{0+}^{\beta}$ represents the Riemann-Liouville fractional derivative of order α, β. $\lambda_i,\ \delta_i \in (0,\infty)$ and are defined such that $\sum_{i=1}^{m-2}\lambda_i\eta_i^{\alpha-\delta-1} < 1,\ \sum_{i=1}^{m-2}\delta_i\xi_i^{\beta-\gamma-1} < 1$. Further $f, g : I \times R \times R \to R$ are assumed non linear functions. Existence and uniqueness of solutions are established by using classical theorems like Leray-Schauder and Banach contraction mapping.

The rest of the article is organized as follows; In Section 2, some basic notations and definitions are presented which are necessary for our further investigation. In Section 3, the main finding of the research is presented. In Section 4, some test problems are investigated, and the last Section is devoted to a short conclusion.

2 Preliminaries

In this section we recall some basic definitions and results from fractional calculus and fixed point theory and functional analysis [1].

Definition 1. *The Riemann-Liouville fractional integral of order $\alpha \in R^{+}$ of a function $y \in c((0,\infty), R)$, is defined as*

$$I_{0+}^{\alpha}y(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)\,ds,$$

where $\alpha > 0$ and Γ is Gamma function.

Definition 2. *The fractional derivative of a continuous function $y : (0,\infty) \to R$ in Riemann-Liouville is defined as*

$$D_{0+}^{\alpha}y(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^{n}\int_0^t (t-s)^{n-\alpha-1}y(s)\,ds,$$

where $n = [\alpha]+1$ and $[\alpha]$ represents the integer part of α.

Lemma 1. *The FDE of order $\alpha > 0$*

$$D^{\alpha}y(t) = 0,\ n-1 \leq \alpha < n,$$

have a unique solution of the form $y(t) = \sum_{i=1}^{n} c_i t^{\alpha-n}$, where $c_i \in R,\ i = 1,2,3,\cdots,n, n = [\alpha]+1$.

Lemma 2. *The following relation holds true for FDEs*

$$I^{\alpha}D^{\alpha}y(t) = y(t) + \sum_{i=1}^{n} c_i t^{\alpha-n},$$

for arbitrary $c_i \in R$, $\quad i = 1,2,...,n, n = [\alpha]+1$.

Let us introduce the spaces $E_1 = \{u(t) : \ u(t),\ D^{q}u(t) \in C^{n}([0,1]), 0 < q < 1\}$ and $E_2 = \{v(t) : \ v(t),\ D^{p}v(t) \in C^{n}([0,1]), 0 < p < 1\}$ whose norm are defined by $\|u\| = \max_{t\in[0,1]}|u(t)| + \max_{t\in[0,1]}|D^{q}u(t)|$, $\|v\| = \max_{t\in[0,1]}|v(t)| + \max_{t\in[0,1]}|D^{p}v(t)|$ respectively. Then obviously $(E_1, \|u\|)$ and $(E_2, \|v\|)$ are Banach spaces. Also the product space $(E_1 \times E_2, \|(u,v)\|)$ is Banach space, whose norm is defined by $\|(u,v)\| = \max\{\|u\|, \|v\|\}$.

3 Main Results

In this section our aim is to establish the existence criteria of a unique solution of the coupled system (1). We start our analysis by establishing various important concepts.

Lemma 3. *Let $\Phi(t) \in C([0,1],R)$, then the unique solution of the problem*

$$\begin{cases} D^{\alpha}u(t) = f(t,v(t),D^{p}v(t)) = \Phi(t), 0<t<1, n-1<\alpha\leq n, \\ u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0,\ D^{\delta}u(1)=\sum_{i=1}^{m-2}\lambda_i D^{\delta}u(\eta_i), 0<\delta<1, \end{cases} \tag{2}$$

where $0<\eta_1<\eta_2<\cdots<\eta_{m-2}<1$ *and* $0<\lambda_i<1, i=1,2\ldots m-2, \sum_{i=1}^{m-2}\lambda_i\eta_i^{\alpha-\delta-1}<1$ *is given by* $u(t)=\int_0^1 G_{\alpha}(t,s)\Phi(s)ds$, *where* $G_{\alpha}(t,s)$ *is defined as*

$$G_{\alpha}(t,s)=\begin{cases} \frac{(t-s)^{\alpha-1}}{\Gamma\alpha}+\frac{t^{\alpha-1}}{\Gamma(\alpha+1)(\alpha-s)\Delta_1}\left[\sum_{i=1}^{m-2}\lambda_i(\eta_i-s)^{\alpha-\delta-1}-(1-s)^{\alpha-\delta-1}\right], \\ s\leq t,\ \eta_i<s<\eta_{i+1}, i=1,2...m-2, \\ \frac{t^{\alpha-1}}{\Gamma(\alpha+1)(\alpha-\delta)\Delta_1}\left[\sum_{i=1}^{m-2}\lambda_i(\eta_i-s)^{\alpha-\delta-1}-(1-s)^{\alpha-\delta-1}\right], \\ t\leq s,\ \eta_i<s<\eta_{i+1}, i=1,2,\ldots,m-2. \end{cases} \tag{3}$$

Proof. From Lemma 2.4 and from (2), we conclude that

$$u(t)=c_1t^{\alpha-1}+c_2t^{\alpha-2}+\cdots+c_nt^{\alpha-n}+I^{\alpha}\Phi(t), \tag{4}$$

using $u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0$, we get $c_2=c_3=\cdots=c_n=0$. Then equation (4) may have the following form

$$u(t)=c_1t^{\alpha-1}+I^{\alpha}\Phi(t), \tag{5}$$

from which we can write

$$D^{\delta}u(t)=c_1\frac{\Gamma\alpha}{\Gamma(\alpha-\delta)}t^{\alpha-\delta-1}+I^{\alpha-\delta}\Phi(t),$$

$$D^{\delta}u(1)=\sum_{i=1}^{m-2}\lambda_i\left[c_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-\delta)}\eta_i^{\alpha-\delta-1}+I^{\alpha-\delta}\phi(\eta_i)\right]$$

$$c_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-\delta)}+I^{\alpha-\delta}\Phi(1)=c_1\sum_{i=1}^{m-2}\lambda_i\eta_i^{\alpha_{\delta}-1}+\sum_{i=1}^{m-2}\lambda_iI^{\alpha-\delta}\phi(\eta_i)$$

$$c_1=\frac{1}{(\alpha-\delta)\Gamma(\alpha)\Delta_1}\left[\sum_{i=1}^{m-2}\lambda_i\int_0^{\eta_i}(\eta_i-s)^{\alpha-\delta-1}\Phi(s)-\int_0^1(1-s)^{\alpha-\delta-1}\Phi(s)ds\right],$$

where $\Delta_1=1-\sum_{i=1}^{m-2}\lambda_i\eta_i^{\alpha-\delta-1}$.

Thus (5) becomes

$$\begin{aligned} u(t)&=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\Phi(s)ds+\frac{t^{\alpha-1}}{(\alpha-\delta)\Gamma(\alpha)\Delta_1}\left[\sum_{i=1}^{m-2}\lambda_i\int_0^{\eta_i}(\eta_i-s)^{\alpha-\delta-1}\Phi(s)ds\right] \\ &=\int_0^1 G_{\alpha}(t,s)\Phi(s)ds. \end{aligned} \tag{6}$$

This completes the proof.

In view of Lemma 3, our considered coupled system (1) is written as a coupled system of Fredholm integral equations as

$$\begin{cases} u(t) = \int_0^1 G_\alpha(t,s) f(s, v(s), D^p v(s))ds, \\ v(t) = \int_0^1 G_\beta(t,s) g(s, u(s), D^q u(s)). \end{cases} \tag{7}$$

Further, we use these notations
$G_\alpha^* = \sup_{t\in[0,1]} \int_0^1 |G_\alpha(t,s)|ds,\ G_\beta^* = \sup_{t\in[0,1]} \int_0^1 |G_\beta(t,s)|ds.$

Lemma 4. *Assume that $f, g : [0,1]\times R^2 \to R$ are continuous functions. Then $(u,v) \in E_1 \times E_2$ is a solution of BVP (1) if $(u,v) \in E_1 \times E_2$ is a solution of system (7).*

Proof. If $(u,v) \in E_1 \times E_2$ is a solution of BVP (1), then in view of Lemma 3 (u,v) it is also a solution of the integral equations (7). Conversely let (u,v) satisfies (7), using $D^\alpha t^{\alpha-k} = 0, k = 1,2,...,N$, N is integer part of α as in[8], we have

$$D^\alpha u(t) = D^\alpha \Big[\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, v(s), D^p v(s))ds\Big]$$
$$+ D^\alpha \Big[\frac{t^{\alpha-1}}{(\alpha-\delta)\Gamma(\alpha+1)\Delta_1} \Big(\sum_{i=1}^{m-2} \lambda_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-\delta-1} f(s, v(s), D^p v(s))ds$$
$$- \int_0^1 (1-s)^{\alpha-\delta-1} f(s, v(s), D^p v(s))ds\Big)\Big]$$
$$\Rightarrow D^\alpha u(t) = D^\alpha \Big[I^\alpha \Phi(t) + \frac{t^{\alpha-1}}{(\alpha-\delta)\Gamma(\alpha+1)\Delta_1} \Big(\sum_{i=1}^{m-2} \lambda_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-\delta-1} f(s, v(s), D^p v(s))ds$$
$$- \int_0^1 (1-s)^{\alpha-\delta-1} f(s, v(s), D^p v(s))ds\Big)\Big]$$
$$\Rightarrow D^\alpha u(t) = f(t, v(t), D^p v(t)),$$

and similarly $D^\beta v(t) = g(t, u(t), D^q u(t))$.
Further it is easy to verify that $u(0) = 0, u'(0) = 0, \ldots, u^{n-2}(0) = 0, D^\delta u(1) = \sum_{i=1}^{m-2} \lambda_i D^\delta u(\eta_i)$,
$v(0) = 0, v'(0) = 0, \ldots, v^{n-2}(0) = 0, D^\gamma v(1) = \sum_{i=1}^{m-2} \delta_i D^\gamma v(\xi_i)$.

Define $T : E_1 \times E_2 \to E_1 \times E_2$ by $T(u(t), v(t)) = (T_1 v(t), T_2 u(t))$, where

$$\begin{cases} T_2 v(t) = u(t) = \int_0^1 G_\alpha(t,s) f(s, v(s), D^p v(s))ds, \\ T_1 u(t) = v(t) = \int_0^1 G_\beta(t,s) g(s, u(s), D^q u(s))ds. \end{cases} \tag{8}$$

Then by Lemma 3 solution of the BVP (1) are the fixed points of the coupled system (8) of operator equations.
Let one of the following hypothesis holds:

(A_1) There exist $a_i, b_i \in R^+ \cup \{0\}$ and $0 \le \theta_i, \nu_i < 1, i = 0,1,2$ such that
$|f(t,u,v)| \le a_0(t) + a_1|u|^{\theta_1} + a_2|v|^{\theta_2}$ and $|g(t,u,v)| \le b_0(t) + b_1|u|^{\nu_1} + b_2|v|^{\nu_2}$.

(A_2)There exist $c_i, d_i \in R^+ \cup \{0\}$ and $\theta_i, \nu_i > 1 (i = 1, 2)$ such that

$$|f(t,u,v)| \le c_1|u|^{\theta_1} + c_2|v|^{\theta_2}, \ \ |g(t,u,v)| \le d_1|u|^{\nu_1} + d_2|v|^{\nu_2}.$$

For convenience we use the following notations:

$$K_1 = \int_0^1 |G_\alpha(t,s)a_0(s)|ds + \frac{1}{\Gamma(\alpha-q)}\int_0^1 (1-s)^{\alpha-q-1}a_0(s)ds + \sum_{i=1}^{m-2}\lambda_i \int_0^{\eta_i} \frac{(\eta_i-s)^{\alpha-\delta-1}a_0(s)ds}{\Delta_1\alpha(\alpha-\delta)\Gamma(\alpha-q)}$$
$$+ \int_0^1 \frac{(1-s)^{\alpha-\delta-1}a_0(s)ds}{\alpha(\alpha-\delta)\Gamma(\alpha-q)\Delta_1},$$
$$K_2 = \int_0^1 |G_\beta(t,s)a_0(s)|ds + \frac{1}{\Gamma(\beta-p)}\int_0^1 (1-s)^{\beta-p-1}b_0(s)ds + \sum_{i=1}^{m-2}\delta_i \int_0^{\xi_i} \frac{(\xi_i-s)^{\beta-\gamma-1}b_0(s)ds}{\Delta_2\beta(\beta-\gamma)}$$
$$+ \int_0^1 \frac{(1-s)^{\beta-\gamma-1}a_0(s)ds}{\beta(\beta-\gamma)\Gamma(\beta-p)\Delta_2},$$
$$A = \frac{1}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)(\alpha-\delta)^2\Delta_1} + \frac{1}{\alpha-q+1} + \frac{2}{\Delta_1\alpha(\alpha-\delta)^2\Gamma(\alpha-q)},$$
$$B = \frac{1}{\Gamma(\beta+1)} + \frac{1}{\Gamma(\beta)(\beta-\gamma)^2\Delta_2} + \frac{1}{\beta-p+1} + \frac{2}{\Delta_2\beta(\beta-\gamma)^2\Gamma(\beta-p)}.$$

$(A_3)|G_\alpha(t,s) - G_\alpha(\tau,s)| \le G_{\alpha,c,d,e}(t,\tau)$, where

$$G_{\alpha,c,d,e}(s) = \frac{(\alpha-1)(c-s)^{\alpha-2}}{\Gamma(\alpha)} - (\alpha-1)d^{\alpha-2}\left(\frac{\sum_{i=1}^{m-2}\lambda_i(\eta_i-s)^{\alpha-\delta-1} - (1-s)^{\alpha-\delta-1}}{\Gamma(\alpha)(\alpha-\delta)\Delta_1}\right)$$
$$+ \frac{(\alpha-1)e^{\alpha-2}}{(\alpha-\delta)\Gamma(\alpha)\Delta_1}\left(\sum_{i=1}^{m-2}\lambda_i(\eta_i-s)^{\alpha-\delta-1} - (1-s)^{\alpha-\delta-1}\right)$$

and $|G_\beta(t,s) - G_\beta(\tau,s)| \le G_{\beta,c,d,e}(t,\tau)$, where

$$G_{\beta,c,d,e}(s) = \frac{(\beta-1)(c-s)^{\beta-2}}{\Gamma(\beta)} - (\beta-1)d^{\beta-2}\left(\frac{\sum_{i=1}^{m-2}\delta_i(\eta_i-s)^{\beta-\gamma-1} - (1-s)^{\beta-\gamma-1}}{\Gamma(\beta)(\beta-\delta)\Delta_1}\right)$$
$$+ \frac{(\beta-1)e^{\beta-2}}{(\beta-\delta)\Gamma(\beta)\Delta_1}\left(\sum_{i=1}^{m-2}\delta_i(\eta_i-s)^{\beta-\gamma-1} - (1-s)^{\beta-\gamma-1}\right).$$

Theorem 1*Assume that $f, g : I \times R \times R \to R$ are continuous and the assumption (A_1) holds. Then BVP (1) has at least one solution.*

*Proof.*Assume that $f, g : [0,1] \times R \times R \to R$ are continuous and defined

$$B = \{(u,v)|(u,v) \in E_1 \times E_2, \| (u,v) \| \le \mathscr{R}, t \in [0,1]\},$$

where

$$\max\left\{(3Aa_1)^{\frac{1}{1-\theta_1}}, (3Aa_2)^{\frac{1}{1-\theta_2}}, (3Bb_1)^{\frac{1}{1-\nu_1}}, (3Bb_2)^{\frac{1}{1-\nu_2}}, 3K_1, 3K_2\right\} \le \mathscr{R}.$$

Obviously B is the ball in Banach space $E_1 \times E_2$. Now we will prove that $T : B \to B$, for this we consider

$$
\begin{aligned}
|T_1 v(t)| &= \left| \int_0^1 G_\alpha(t,s) f(s, v(s), D^p v(s)) ds \right| \leq \int_0^1 |G_\alpha(t,s)||f(s, v(s), D^p v(s))| ds \\
&\leq \int_0^1 |G_\alpha(t,s)|[a_0(s) + a_1 \mathscr{R}^{\theta_1} + a_2 \mathscr{R}^{\theta_2}] ds \\
&= \int_0^1 |G_\alpha(t,s) a_0(s)| ds + (a_1 \mathscr{R}^{\theta_1} + a_2 \mathscr{R}^{\theta_2}) \int_0^1 |G_\alpha(t,s)| ds \\
&\leq \int_0^1 |G_\alpha(t,s) a_0(s)| ds + (a_1 \mathscr{R}^{\theta_1} + a_2 \mathscr{R}_2^{\theta}) \left(\frac{1}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha+1)(\alpha-\delta)^2 \Delta_1} \right).
\end{aligned}
$$

Also, we have

$$
\begin{aligned}
D^q u(t) &= D^q \left[I^\alpha f(t, v(t), D^p v(t)) \right] + D^q \left[\frac{t^{\alpha-1}}{(\alpha-\delta)\Gamma(\alpha+1)\Delta_1} \right] \times \\
&\left(\sum_{i=1}^{m-2} \lambda_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-\delta-1} f(s, v(s), D^p v(s)) ds - \int_0^1 (1-s)^{\alpha-\delta-1} f(s, v(s), D^p v(s)) ds \right) \\
&= I^{\alpha-q} f(t, v(t), D^p v(t)) + \frac{\Gamma(\alpha) t^{\alpha-q-1}}{\Gamma(\alpha-q)(\alpha-\delta)\alpha(\Gamma)\alpha\Delta_1} \times \\
&\left(\sum_{i=1}^{m-2} \lambda_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-\delta-1} f(s, v(s), D^p v(s)) ds - \int_0^1 (1-s)^{\alpha-\delta-1} f(s, v(s), D^p v(s)) ds \right).
\end{aligned}
$$

Thus, one can get

$$
\begin{aligned}
T_1(D^q v(t)) &= \frac{1}{\Gamma(\alpha-q)} \int_0^t (t-s)^{\alpha-q-1} f(s, v(s), D^p v(s)) ds + \frac{t^{\alpha-q-1}}{\alpha\Gamma(\alpha-q)(\alpha-\delta)\Delta_1} \times \\
&\left[\sum_{i=1}^{m-2} \lambda_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-\delta-1} f(s, v(s), D^p v(s)) ds - \int_0^1 (1-s)^{\alpha-\delta-1} f(s, v(s), D^p v(s)) ds \right].
\end{aligned}
$$

Which implies

$$|D^qT_1v(t)| = \Big|\frac{1}{\Gamma(\alpha-q)}\int_0^t (t-s)^{\alpha-q-1}f(s,v(s),D^pv(s))ds$$
$$+\frac{t^{\alpha-q-1}}{\alpha\Gamma(\alpha-q)(\alpha-\delta)\Delta_1}\Big[\sum_{i=1}^{m-2}\lambda_i\int_0^{\eta_i}(\eta_i-s)^{\alpha-\delta-1}f(s,v(s),D^pv(s))ds$$
$$-\int_0^1(1-s)^{\alpha-\delta-1}f(s,v(s),D^pv(s))ds\Big]\Big|$$
$$\leq \frac{1}{\Gamma(\alpha-q)}\int_0^t (t-s)^{\alpha-q-1}(a_0(s)+a_1\mathscr{R}^{\theta_1}+a_2\mathscr{R}^{\theta_2})$$
$$+\frac{1}{\alpha\Gamma(\alpha-q)(\alpha-\delta)\Delta_1}\Big[\sum_{i=1}^{m-2}\lambda_i\int_0^{\eta_i}(\eta_i-s)^{\alpha-\delta-1}(a_0(s)+a_1\mathscr{R}^{\theta_1}+a_2\mathscr{R}^{\theta_2})ds$$
$$+\int_0^1(1-s)^{\alpha-\delta-1}(a_0(s)+a_1\mathscr{R}^{\theta_1}+a_2\mathscr{R}^{\theta_2})ds\Big]$$
$$\Rightarrow |D^qT_1v(t)| \leq \frac{1}{\Gamma(\alpha-q)}\int_0^1(1-s)^{\alpha-q-1}a_0(s)ds+\frac{1}{\alpha\Gamma(\alpha-q)(\alpha-\delta)\Delta_1}\Big[\sum_{i=1}^{m-2}\lambda_i\int_0^{\eta_i}(\eta_i-s)^{\alpha-\delta-1}a_0(s)ds$$
$$+\int_0^1(1-s)^{\alpha-\delta-1}a_0(s)ds\Big]+\frac{(a_1\mathscr{R}^{\theta_1}+a_2\mathscr{R}^{\theta_2})}{\Gamma(\alpha-q+1)}+\frac{(a_1\mathscr{R}^{\theta_1}+a_2\mathscr{R}^{\theta_2})(2)}{\alpha\Gamma(\alpha-q)(\alpha-\delta)^2\Delta_1}$$
$$\Rightarrow |D^qT_1v(t)| \leq \frac{1}{\Gamma(\alpha-q)}\int_0^1(1-s)^{\alpha-q-1}a_0(s)ds+\frac{1}{\alpha\Gamma(\alpha-q)(\alpha-\delta)\Delta_1}\Big[\sum_{i=1}^{m-2}\lambda_i\int_0^{\eta_i}(\eta_i-s)^{\alpha-\delta-1}a_0(s)ds$$
$$+\int_0^1(1-s)^{\alpha-\delta-1}a_0(s)ds\Big]+(a_1\mathscr{R}^{\theta_1}+a_2\mathscr{R}^{\theta_2})\left[\frac{1}{\Gamma(\alpha-q+1)}+\frac{2}{\alpha\Gamma(\alpha-q)(\alpha-\delta)^2\Delta_1}\right].$$

Thus, we have

$$\| T_1v \| \leq K_1+(a_1\mathscr{R}^{\theta_1}+a_2\mathscr{R}^{\theta_2})A \leq \frac{\mathscr{R}}{3}+\frac{\mathscr{R}}{3}+\frac{\mathscr{R}}{3}=\mathscr{R}. \tag{9}$$

Similarly, one can get

$$\|T_2u\| \leq \mathscr{R}. \tag{10}$$

Therefore $\|T(u,v)\| \leq \mathscr{R}$. Since $T_1v(t), T_2u(t), T_1D^qv(t), T_2D^pu(t)$ are continuous on $[0,1]$. Thus, we have $T: B \to B$ is also continuous as f, g $G_\alpha(t,s)$, $G_\beta(t,s)$ are continuous.
Now for any $(u,v) \in B$, let t,τ be such that $t \leq \tau$ and let us take $s,d \in (t,\tau)$ when $t < s$ and $e \in (t,\tau)$ when $s \leq t$. Then in view of Mean value theorem, we have

$$|T_1v(t)-T_1v(\tau)| = \left|\int_0^1 |G_\alpha(t,s)-G_\alpha(\tau,s)||f(s,v(s),D^pv(s))|ds\right|$$
$$\leq \int_0^1 |G_\alpha(t,s)-G_\alpha(\tau,s)||f(s,v(s),D^pv(s))|ds$$
$$\leq \int_0^1 G_{\alpha,c,d,e}(t-\tau)(a_0(s)+a_1\mathscr{R}^{\theta_1}+a_2\mathscr{R}^{\theta_2})ds,$$

where $G_{\alpha,c,d,e}(s)$ is already defined in (A_3). So, we have

$$|T_1v(t)-T_1v(\tau)|\leq(t-\tau)\int_0^1 G_{\alpha,c,d,e}(a_0(s)+a_1\mathscr{R}^{\theta_1}+a_2\mathscr{R}^{\theta_2})ds$$

$$\begin{aligned}|D^qT_1v(t)-D^qT_1v(\tau)|\leq{}&\frac{1}{\Gamma(\alpha-q)}\left[\int_0^t(1-s)^{\alpha-q-1}a_0(s)ds-\int_0^\tau(\tau-s)^{\alpha-q-1}a_0(s)ds\right]\\&+\frac{(a_1\mathscr{R}^{\theta_1}+a_2\mathscr{R}^{\theta_2})}{\alpha-q+1}(t^{\alpha-q}-\tau^{\alpha-q})+\frac{(t^{\alpha-q-1}-\tau^{\alpha-q-1})}{\alpha\Gamma(\alpha-q)(\alpha-\delta)\Delta_1}\times\\&\Big[\sum_{i=1}^{m-2}\lambda_i\int_0^{\eta_i}(\eta_i-s)^{\alpha-\delta-1}a_0(s)ds-\int_0^1(1-s)^{\alpha-\delta-1}a_0(s)ds\Big].\end{aligned}$$

Similarly

$$|T_2u(t)-T_2u(\tau)|\leq(t-\tau)\int_0^1 G_{\beta,c,d,e}(b_0(s)+b_1\mathscr{R}^{\nu 1}+b_2\mathscr{R}^{\nu_2})ds$$

and

$$\begin{aligned}|D^pT_2u(t)-D^pT_2u(\tau)|\leq{}&\frac{1}{\Gamma(\beta-p)}\left[\int_0^t(t-s)^{\beta-p-1}b_0(s)ds-\int_0^\tau(\tau-s)^{\alpha-q-1}b_0(s)ds\right]\\&+\frac{(b_1R^{\nu_1}+b_2R^{\nu_2})}{\alpha-q+1}(t^{\beta-p}-\tau^{\beta-p})+\frac{(t^{\beta-p-1}-\tau^{\beta-p-1})}{\beta\Gamma(\beta-p)(\beta-\gamma)\Delta_1}\times\\&\Big[\sum_{i=1}^{m-2}\delta_i\int_0^{\xi_i}(\xi_i-s)^{\beta-\gamma-1}b_0(s)ds-\int_0^1(1-s)^{\beta-\gamma-1}b_0(s)ds\Big].\end{aligned}$$

Clearly when $t\to\tau$ then $|T_1v(t)-T_1v(\tau)|\to 0$, $|D^qT_1v(t)-D^qT_1v(\tau)|\to 0$ and $|T_2u(t)-T_2u(\tau)|$, $|D^pT_2u(t)-D^pT_2u(\tau)|\to 0$. By Arzelá Ascoli's theorem, it follows that $T:E_1\times E_2\to E_1\times E_2$ is completely continuous operator. Thus by Schauder fixed point theorem T has at least one fixed point in B which is the corresponding solution of Coupled system (1).

Theorem 2*Assume that $f,g:I\times R\times R\to R$ are continuous and if (A_2) holds. Then BVP (1) has at least one solution.*

*Proof.*Proof is similar to Theorem 1, so we omit it.

Theorem 3*Under the continuity of f,g and if the following assumptions hold:*

(A_4)*There exist constant K,L such that for each $t\in[0,1]$ and for all $u,\bar{u},v,\bar{v}\in R$*

$$|f(t,u,v)-f(t,\bar{u},\bar{v}]\leq K[|u-\bar{u}|+|v-\bar{v}|]$$

and

$$|g(t,u,v)-g(t,\bar{u},\bar{v}]\leq L[|u-\bar{u}|+|v-\bar{v}|];$$

(A_5)*If* $\max\{\rho_1,\rho_2\}<1$, *where*

$$\rho_1=\max\{2G^*_\alpha K,(\frac{1}{\Gamma(\alpha-q+1)}+\frac{1}{\Gamma(\alpha-q)\alpha(\alpha-\delta)(\alpha-\delta-1)})2K\}$$

$$\rho_2=\max\{2G^*_\beta L,(\frac{1}{\Gamma(\beta-p+1)}+\frac{2L}{\Gamma(\beta-p)\beta(\beta-\gamma)(\beta-\gamma-1)})\}.$$

Then BVP (1) has unique solution.

Proof. Assume that $(A_4),(A_5)$ hold and let us consider

$$|T_1v(t)-T_1\bar{v}(t)| \le \int_0^1 |G_\alpha(t,s)||f(s,v(s),D^pv(s)) - \int_0^1 f(s,\bar{v}(s),D^p\bar{v}(s))|ds$$
$$\le \sup_{t\in J} K\int_0^1 |G_\alpha(t,s)|[|v-\bar{v}|+|D^pv-D^p\bar{v}|]ds$$
$$\le 2KG_\alpha^* \parallel v-\bar{v}\parallel .$$

$$|D^qT_1v(t)-D^qT_1\bar{v}(t)| \le \frac{1}{\Gamma(\alpha-q)}\int_0^t (t-s)^{\alpha-q-1}|f(s,v(s),D^pv(s))-f(s,\bar{v}(s),D^p\bar{v}(s))|ds$$
$$+\frac{1}{\Delta_1\alpha\Gamma(\alpha-q)(\alpha-\delta)}\left(\sum_{i=1}^{m-2}\lambda_i\int_0^{\eta_i}(\eta_i-s)^{\alpha-\delta-1}|f(s,v(s),D^pv(s))-f(s,\bar{v}(s),D^p\bar{v}(s))|ds\right)$$
$$\le \frac{2K}{\Gamma(\alpha-q+1)}\parallel v-\bar{v}\parallel + \frac{2K\parallel v-\bar{v}\parallel}{\Delta_1\alpha\Gamma(\alpha-q)(\alpha-\delta)^2}$$
$$\Rightarrow |D^qT_1v(t)-D^qT_1\bar{v}(t)| \le \left(\frac{1}{\Gamma(\alpha-q+1)}+\frac{1}{\Delta_1\alpha\Gamma(\alpha-q)(\alpha-\delta)^2}\right)2K\parallel v-\bar{v}\parallel .$$

Now

$$\parallel T_1v-T_1\bar{v}\parallel = \max\{\parallel T_1v-T_1\bar{v}\parallel, \parallel D^qT_1v(t)-D^qT_1\bar{v}(t)\parallel\}$$
$$\le \max\{2G_\alpha^*K, \left(\frac{1}{\Gamma(\alpha-q+1)}+\frac{1}{\Gamma(\alpha-q)\alpha(\alpha-\delta)(\alpha-\delta-1)}\right)2K\}\parallel v-\bar{v}\parallel$$

$\Rightarrow \parallel T_1v-T_1\bar{v}\parallel \le \rho_1 \parallel v-\bar{v}\parallel$
and similarly
$\parallel T_2u-T_2\bar{u}\parallel \le \rho_2\parallel u-\bar{u}\parallel .$

Now

$$\parallel T(u,v)-T(\bar{u},\bar{v})\parallel \le \rho \parallel (u,v)-(\bar{u},\bar{v})\parallel$$

where $\rho=\max\{\rho_1,\rho_2\}<1$
Thus T is contraction. Hence by Banach Contraction principle T has a unique fixed point which is the unique solution of BVP (1).

4 Illustrative Example

Example 1. Consider the problem

$$\begin{cases} D^{\frac{3}{2}}u(t) = \left(\frac{t}{4}\right)^4\left[\sqrt{v(t)}+\sqrt[3]{D^{\frac{1}{2}}v(t)}\right],\ 0<t<1,\\ D^{\frac{3}{2}}v(t) = \left(\frac{t}{4}\right)^4\left[\sqrt[3]{u(t)}+\sqrt[4]{D^{\frac{1}{2}}u(t)}\right],\ 0<t<1,\\ u(0)=0,\ D^{\frac{1}{2}}u(1)=\sum_{i=1}^{5}\frac{1}{2^i}D^{\frac{1}{2}}u\left(\frac{1}{2^i}\right),\\ v(0)=0,\ D^{\frac{1}{2}}v(1)=\sum_{i=1}^{5}\frac{1}{2^i}D^{\frac{1}{2}}\left(\frac{1}{2^i}\right). \end{cases}$$

Now

$$a_0(t)=b_0(t)=0,\ \ a_1(t)=a_2(t)=b_1(t)=b_2(t)=\frac{t^4}{256}$$

now by Theorem 1, the existence of solution for $\theta_1 = \frac{1}{2}, \theta_2 = \frac{1}{3}$ and $\nu_1 = \frac{1}{4}, \nu_2 = \frac{1}{4}$ is obvious.
Also $K = \frac{1}{256}, L = \frac{1}{256}, \alpha = \frac{3}{2}, \beta = \frac{3}{2}, \delta = \gamma = \frac{1}{2}$

$$G^*_\alpha = \sup_{t\in[0,1]} \left| \int_0^1 G\alpha(t,s)ds \right| = 0.28199, \; G^*_\beta = 0.28199$$

$$\rho_1 = \max\{.0022, 0.0156\} = 0.0156 < 1, \; \rho_2 = 0.0156 < 1.$$

Hence by Theorem 3, BVP(1) has a unique solutions.

*Example 2.*Consider the problem

$$\begin{cases} D^{\frac{9}{2}}u(t) = \left(\frac{t+1}{4}\right)^6 [(v(t))^2 + (D^{\frac{1}{2}}v(t))^2], \; 0 < t < 1, \\ D^{\frac{9}{2}}v(t) = \left(\frac{t}{8}\right)^2 [(u(t))^2 + (D^{\frac{1}{2}}u(t))^2], \; 0 < t < 1, \\ u(0) = u'(0) = u''(0) = 0, \; D^{\frac{1}{2}}u(1) = \sum_{i=1}^{5} \frac{1}{2^i} D^{\frac{1}{2}} u\left(\frac{1}{2^i}\right), \\ v(0) = v'(0) = v''(0) = 0, \; D^{\frac{1}{2}}v(1) = \sum_{i=1}^{5} \frac{1}{2^i} D^{\frac{1}{2}} \left(\frac{1}{2^i}\right). \end{cases}$$

Now $\theta_1 = \theta_2 = \nu_1 = \nu_2 = 2 > 1$. By simple calculation, we can obtain that $a_1 < \frac{1}{2A}, b_1 < \frac{1}{2B}$. Thus by the use of Theorem 2, one can easily show that BVP (2), has a solution.

Conclusion

Fixed point theory plays a vital role in investigation of FDEs. With the help of the some fixed point theorems, we successfully developed some conditions which guarantees the existence of solutions and uniqueness of solution of the problem under consideration. The main result is demonstrated by various test problems. Application of these theorems and investigations of high order fractional order differential equations subjected to more complicated type boundary constrains specially nonlinear boundary conditions are still an open problem and lies in the domain of our future work.

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