Quantum Physics Letters
An International Journal

http://dx.doi.org/10.18576/qpl/060106

Determining Masses of Light Mesons by Using Numerov's Discretization

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Received: 11 May 2016, Revised: 20 Oct. 2016, Accepted: 28 Oct. 2016
Published online: 1 Apr. 2017

Abstract: Some light meson bound states' eigenvalues and eigenfunctions are calculated by applying the new Numerov's discretization method (NDM) of the Hamiltonian. The light quark- antiquark problem is solved in the framework of non-relativistic quark model. Two types of quark- antiquark potentials were tested and compared. The results show a good fit with other groups and with recent experimental data. The NDM could be used to give accurate results when it is applied to light meson candidates.

Keywords: Light mesons, Eigenvalues, Eigenfunctions, NDM, Non-Relativistic quark model.

1 Introduction

An important part of experimental high energy physics is to the determination of the intrinsic properties of the basic constituents of matter going under the name elementary particles. The main problem of theoretical high energy physics is to understand why the elementary particles exist with certain masses, spins and parities along with their characteristic internal quantum numbers like charge, hypercharge and isospin. The study of the fundamental constituents of matter and their interactions is called Particle Physics. The aim of particle physics is to find the basic building blocks of matter and to understand how they are bound together by the forces of nature. This would help us to understand how the universe was created. Mesons that are built out of the light flavours (i.e., u, d and s) are called light mesons. The constituent masses of these quarks, especially those of the u and d quarks are so similar that they cannot be expected to be distinguished according to their quark content but must be expected to encounter mixed states of all three light flavours. The masses and quantum numbers of the various mesons may also be used to make sense of how these particles decay. Because of the difficult of using perturbative and non-perturbative QCD (quantum chromodynamics) directly to compute hadronic properties, so calculating the properties of hadrons will be used models inspired by QCD rather than the full theory itself. In this paper, phenomenological potential models have provided extremely satisfactory results in describing ordinary hadrons, more specially quark-antiquark bound states (mesons). The quark model is one such attempt, and a very successful one. In a quark model of a meson, the wave function is obtained by solving the Schrodinger equation with a Hamiltonian inspired by QCD and it describes the relative motion of the quark and antiquark. As an example of such a model, Godfrey and Isgur [1] have been considered. Their effective potential contains the effects of a Lorentz-vector one gluon exchange interaction at short distances and a Lorentz scalar linear interaction that models confinement. This work is organized as follows. After the introduction, the brief review of the numerical method which used to solve Schrodinger equation has been introduced. In Sec.3, we introduced the potentials models. In Sec. 4 the results and discussion are introduced. Finally in Sec. 5, we give the conclusion.

2 Numerov's Discretization Method (NDM)

The numerical solution of Schrodinger equation for one or more particles is an important problem in the field of Quantum Mechanics and in most cases is the only method

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that could be used to obtain a usable solution. One of these methods is; Matrix Numerov method. Many authors have praised the virtues of the Numerov's method [2,3,4,5,6]. Numerov's method is a numerical method for approximating the solution of the second order differential equation $\psi''(x) = f(x,y)$ with initial conditions $\psi(x_0) = \psi_0, \psi'(x_0) = \psi'_0$. In this section a new approach to deal with Numerov's Discretization method (NDM) as a matrix has been introduced. In numerical physics the method is used to find solutions of the radial Schrodinger Equation for arbitrary potentials [7]. Hence, NDM can be used to solve Schrodinger equation of the form:

$$E\psi(r,\theta,\varphi) = [\frac{-\hbar^2}{2\mu}\nabla^2 + V_{q\bar{q}}(r)]\psi(r,\theta,\varphi). \quad (1)$$

Where $\frac{-\hbar^2}{2\mu}\nabla^2$ is a non-relativistic kinetic energy which depends only on the square of the relative momentum p between the particles, and $V_{q\bar{q}}(r)$ is the potential energy between the two particles. Now, we will make Transforming Numerov's Method into a Matrix Form: For the time-independent $3-D$ Schrodinger equation, we have:

$$f(r) = \frac{-2\mu(E - V(r))}{\hbar^2}. \quad (2)$$

By using a lattice of points x_i evenly spaced by a distance d, the integration formula is

$$\psi_{i+1} = \frac{(\psi_{i-1}(12 - d^2 f_{i-1}) - 2\psi_i(5d^2 f_i + 12))}{(d^2 f_{i+1} - 12)}. \quad (3)$$

From the above equation

$$\psi_{i+1} = \frac{(12\psi_{i-1} - d^2 f_{i-1}\psi_{i-1} - 10d^2 f_i\psi_i - 24\psi_i}{(d^2 f_{i+1}) - 12)}. \quad (4)$$

By using equation (2), we have:

$$\frac{-2\mu d^2}{\hbar^2}\left[(E\psi_{i-1} - V_{i-1}\psi_{i-1}) + (10E\psi_i - 10V_i\psi_i) + (E\psi_{i+1} - V_{i+1}\psi_{i+1})\right] = 12\psi_{i-1} - 2\psi_i + \psi_{i+1} \quad (5)$$

Where $\psi_i = \psi(x_i)$. By rearranging the above equation, then:

$$\frac{(-\hbar^2}{(2\mu)}\frac{(\psi_{i-1} - 2\psi_i + \psi_{i+1})}{d^2} + \frac{(V_{i-1}\psi_{i-1} + 10V_i\psi_i + V_{i+1}\psi_{i+1})}{12} = \frac{E((\psi_{i+1} + 10\psi_i + \psi_{i-1}))}{12}. \quad (6)$$

Now, the well-known Numerov's method will be transformed into a representation of matrix form on a discrete lattice depending only on the grid number d and the matrix size N. To do that, ψ will be represented by a column vector $(\ldots\psi_{i-1}, \psi_i, \psi_{i+1}\ldots)$ and define matrices

$$A_{N,N} = \frac{(I_{-1} - 2I_0 + I_1)}{d^2}$$
$$B_{N,N} = \frac{(I_{-1} + 10I_0 + I_1)}{12},$$
$$V_N = diag(\ldots V_{i-1}, V_i, V_{i+1})).$$

Where I_{-1}, I_0 and I_1 represent sub$-$, main$-$, and up$-$diagonal unit matrices respectively. Equation (6) could be transformed into a matrix form as follow:

$$\frac{-\hbar^2}{2\mu}A_{N,N}\psi_i + B_{N,N}V_N\psi_i = E_i B_{N,N}\psi_i. \quad (7)$$

Multiplying by $B^{-1}_{N,N}$ we get,

$$\frac{-\hbar^2}{2\mu}A_{N,N}B^{-1}_{N,N}\psi_i + V_N\psi_i = E_i\psi_i. \quad (8)$$

The first term is the Numerov's representation of the kinetic energy operator and the second is the Numerov's representation of the potential energy operator. Equation (8) represents our new approach to describe Numerov's Discretization Method (NDM).

3 The Potentials Used

In the non-relativistic approximation, the mesonic wavefunction is the eigen-function of the Schrodinger equation:

$$E\psi(r) = [\frac{-\hbar^2}{2\mu}\nabla^2 + V(r)]\psi(r). \quad (9)$$

The Hamiltonian of the system is:

$$H = \frac{-\hbar^2}{2\mu}\nabla^2 + V_{q\bar{q}}(r). \quad (10)$$

Many authors have used several different potentials in the past. Some of these potentials have been found with a very crucial test is a unifed description of particle spectra.These potentials have been applied on the meson and baryon sectors, and also to tetra-quark states [8]. The results in all of these cases have been encouraging. In this paper, two potentials [9] that yield good overall results for the meson spectrum have been used for this study. The using of different potentials will allow us to check the sensitivity of our results to the inter-quark interaction. The general form for each potential has been more or less imposed by some basic QCD constraints, but the parameters have been determined by a fit to a well-chosen sample of light meson states. Both potentials rely on a non-relativistic expression of the kinetic energy operator, and they need to solve the Schrodinger equation using a Numerov algorithm. As well as they take the general form

$$V_{q\bar{q}}(r) = -\frac{k(1 - \exp?\frac{-r}{r_c})}{r} + \lambda r^p - \Lambda + (\frac{2\pi\acute{k}}{(3m_q m_{\bar{q}})}(1 - \exp\frac{-r}{r_c})\frac{\exp\frac{-r^2}{r_0^2})}{(\pi^{\frac{3}{2}} r_0^3)}\mathbf{sigma_q sigma_{\bar{q}}}. \quad (11)$$

Where $\mathbf{sigma_q}$,$\mathbf{sigma_{\bar{q}}}$ are the Pauli matrices. One peculiarity of these potentials is that the range r_0 of the hyper?ne term is mass dependent through the relation

$$r_0\left(m_q m_{\acute{q}}\right) = a\left(\frac{2m_q m_{\acute{q}}}{m_q + m_{\acute{q}}}\right)^{-b}. \tag{12}$$

The ?rst choice of potential, denoted $AL1$, has the usual Coulomb + linear form for the central part, and its parameters are $k = 0.4968$; $p = 1$; $'k = 1.847$;$\lambda = 0.168GeV^{\frac{5}{3}}$;$\Lambda = 0.818GeV$; $a = 1.682GeV^{b-1}$; $b = 0.223$; $m_u = m_d = 0.321GeV$; $m_s = 0.588GeV$; $r_c = 0$. The second choice of potential, denoted $AP1$, has a con?ning term suggested by the Regge trajectory behavior of orbital states in a non-relativistic treatment. The parameters are $k = 0.422$; $p = \frac{2}{3}$; $'k = 1.797$;$\lambda = 0.401GeV^{\frac{5}{3}}$; $\Lambda = 1.099GeV$; $a = 1.559GeV^{b-1}$; $b = 0.332$; $m_u = m_d = 0.275GeV$; $m_s = 0.565GeV$,;$r_c = 0$. These names are the same than those used in [9]. The letter A means "for All mesons". The letter L or P denotes respectively the Linear or the $\frac{2}{3}$-Power confinement; the number 1 indicate the parameter r_c is equal to zero. Both potentials reproduce spectra of comparable quality and are simple enough to be handled without any difficulty. In particular, it is quite easy to solve the differential equation resulting from the Schrodinger equation. However, the radial part $R_{nls}(r)$ of the meson wavefunction is calculated numerically. Such a form is not easily used for studying more complicated problems. To solve this problem, the regularized part of the exact radial wavefunction must be approximated by a linear combination of Gaussian functions:

$$R_{nls}\left(r\right) = \sum_{i=1}^{N} c_i exp\left(-\alpha_i r^2\right) \tag{13}$$

For a given number N of Gaussian functions, the parameters c_i and α_i are determined by a variational procedure on the energy of the considered state; $N = 1$ is a rather rough approximation, but $N = 2$ or $N = 3$ greatly improves the results [10]. The approximation with 3 Gaussian terms gives essentially the exact wavefunction. Table 1 shows a few of the masses obtained in this way, for the potentials $AL1$ and $AP1$. Experimental values are shown in the fifth column. The masses obtained using the expansion of the wavefunction in Gaussian functions are shown for $N = 3$.

4 Results and Discussion

The theoretical spectra of some light mesons are calculated in two types of potentials as it is previously explained. This new obtained theoretical spectra is compared to new published experimental data [11]. These theoretical spectra that obtained in different potentials types are fitted by using the experimental spectra to give the most suitable spectra with experiments. The χ^2 relation is used to easily compare among the results obtained by using potentials $AL1$,$AP1$ and this relation can be defined as

$$x^2 = \frac{1}{n}\sum_{k=1}^{n}\left(Mass_k^{theo.} - Mass_k^{Exp.}\right)^2 \tag{14}$$

In this formula, the summation runs over a selected sample of m mesons; n is the number of experimental data in the group. $^{Mass_k}(Exp.)$ is the experimental mass of meson labeled k in the sample, while$^{Mass_k}(theo.)$ is the corresponding theoretical mass depending upon the free parameters. The calculated χ^2 equals to (0.0063) and (0.0059) for the first and second type of potentials, respectively. Then according to the values of χ^2 the second type of potentials $(AP1)$ is found to be the smallest, then it is said to be the best because it gives convergence between theory and experiments [11]. This mean that using $P = \frac{2}{3}$ is better than $P = 1$.

Table 1: Theoretical versus experimental values of the masses (GeV) when both Potentials of (AL1) and (AP1) are used.

State	Name	Theoretical Masses of NRAL model		Expt. Mass [11]
		AL1	AP1	
$1\,^3S_1$	*P(770)*	0.7975	0.831	0.775± 0.0025
$1\,^1S_0$	*π(138)*	0.2055	0.2211	0.138 ± 0.001
$1\,^1P_1$	*b1*	1.1283	1.1481	1.229± 0.003
$1\,^1D_2$	π_2	1.6224	1.6103	1.672± 0.03
$1\,^3D_3$	P_3	1.6405	1.626	1.688± 0.021
$1\,^3F_4$	*F4*	2.007	1.949	2.018± 0.011
$1\,^3P_1$	*F1*	1.3688	1.4003	1.426±0.009
$1\,^3P_1$	*F1*	1.2392	1.2581	1.282±0.005
$1\,^3P_2$	*F2*	1.2392	1.2581	1.275±0.012
$1\,^3F_2$	*F2*	2.0468	2.0117	$2.011\,^{+0.06}_{-0.08}$
$2\,^1S_0$	*π (1300)*	1.3454	1.3391	1.302±0.001
$3\,^1S_0$	*π (1800)*	2.0584	1.9599	1.812 ±0.012
$1\,^3S_1$	φ	1.0585	1.0928	1.019± 0.019
$2\,^3S_1$	φ	1.7344	1.733	1.68± 0.02
$1\,^3D_3$	φ_3	1.8111	1.8204	1.854 ± 0.07
$1\,^3P_2$	*f2'*	1.4691	1.4993	1.525±0.05
$1\,^3P_2$	*a2*	1.2392	1.2581	$1.318\,^{+0.005}_{-0.006}$
$1\,^3S_1$	*k**	0.9362	0.975	0.891±0.0026
$1\,^3P_0$	*k0**	1.3688	1.4003	1.425±0.005
$1\,^1D_2$	*k2*	1.7307	1.7382	1.77± 0.008
$1\,^3D_2$	*k2*	1.7444	1.7491	1.816±0.013
$1\,^3F_4$	*k4**	2.0835	2.0498	2.045±0.009
	$\chi 2$	0.0063	0.0059	

. The wave function of is plotted in Figure 1 and 2 as the radial wavefunction $U(x)$ is calculated within two choices of potential $AL1$ and $AP1$ respectively, according to its parameters. Where the x here is calculated in fm units.

5 Conclusion

In this paper, some light meson bound states' eigenvalues and eigenfunctions are calculated by applying the new Numerov's discretization method (NDM) of the

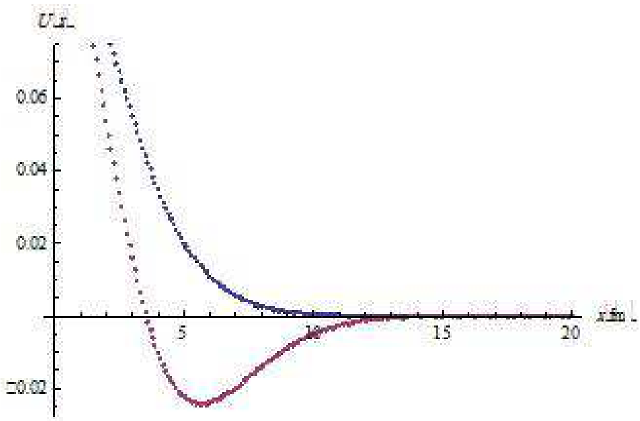

Fig. 1: 2S- state and reduced redial wave function calculated within two choices of potential AL1 and AP1 respectively, with the approximation with 3 Gaussian terms according to its parameters.

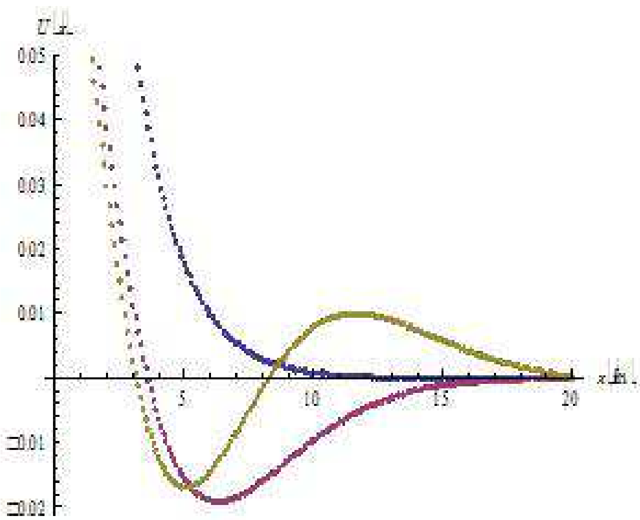

Fig. 2: 3S- state reduced redial wave function calculated within two choices of potential AL1 and AP1 respectively, with the approximation with 3 Gaussian terms according to its parameters.

Hamiltonian. Mesons that are built out of the light flavours (i.e., u, d and s) are called light mesons. The constituent masses of these quarks, especially those of the u and d quarks are so similar that they cannot be expected to be distinguished according to their quark content but must be expected to encounter mixed states of all three light flavours. The masses and quantum numbers of the various mesons may also be used to make sense of how these particles decay. In this work the two potentials are taken into account to compare the resulting theoretical spectra with the experimental data. Both potentials reproduce spectra of comparable quality and are simple enough to be handled without any difficulty. In addition to, it is found that the second choice of potential $(AP1)$ is better than the first potential $(AL1)$. It is suggested that the NDM is a reasonable method for solving Schrodinger equation so we reintroduced it by transforming it into a matrix form to solve radial Schrodinger's equation; it was found that the new method (NDM) is simple to calculate and plot accurate eigenvalues and eigenfunctions. The general agreement between the prediction of the model and the data is very good. It is recommend using the NDM for solving radial Schrodinger equation because it is easy to use, saves the time and is very accurate. Finally, we advise using NDM to obtain other mesons spectra and their properties

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