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A New Analytical Method for Solving Linear and Nonlinear Fractional Partial Differential Equations

Shehu Maitama and Ibrahim Abdullahi*

Department of Mathematics, Faculty of Science, Northwest University, Kano, Nigeria

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Abstract: In this paper, a new analytical method called the Natural Homotopy Perturbation Method (NHPM) for solving linear and the nonlinear fractional partial differential equation is introduced. The proposed analytical method is an elegant combination of a well-known method, Homotopy Perturbation Method (HPM) and the Natural Transform Method (NTM). In this new analytical method, the fractional derivative is computed in Caputo sense and the nonlinear terms are calculated using He's polynomials. Exact solution of linear and nonlinear fractional partial differential equations are successfully obtained using the new analytical method, and the result is compared with the result of the existing methods.

Keywords: Natural homotopy perturbation method, He's polynomials, Mittag-Leffler function, linear and nonlinear fractional partial differential equations.

1 Introduction

In recent years, there is a rapid development in the concept of fractional calculus and its applications [1,2,3,4]. The fractional calculus which deals with derivatives and integrals of arbitrary orders [5] plays a vital role in many field of applied science and engineering. The linear and nonlinear fractional partial differential equations have wide applications in fluid mechanics, acoustic, electromagnetism, signal processing, analytical chemistry, biology and many other areas of physical science and engineering [6]. In last few decades, many analytical and numerical methods has been developed and successfully applied to solve linear and nonlinear fractional partial differential equations such as, Adomian Decomposition Method [7,8,9,10,11], G'/G-Expansion Method [12], Homotopy Analysis Method [13], Jacobi Spectral Collocation Method [14], Laplace Decomposition Method [15], and New Spectral Algorithm [16]. Moreover, Homotopy Perturbation Method [17,18], Yang– Laplace Transform [19], Local Fractional Variational Iteration Method [20,21], Cylindrical-Coordinate Method [22], Modified Laplace Decomposition Method [15], Spectral Legendre-Gauss-Lobatto Collocation Method [23,24], Homotopy Perturbation Sumudu Transform Method [25,26], and Fractional Complex Transform Method [27] are applied to linear and nonlinear fractional partial differential equations.
However, despite the potential of the numerical methods, they can not be considered as universal methods for solving linear and nonlinear fractional partial differential equations because of many deficiencies and some computational difficulties such as unnecessary linearization, discretization of variables, transformation or taking some restrictive assumptions.
In this paper, a new analytical method called the Natural Homotopy Perturbation Method (NHPM) for solving linear and nonlinear fractional partial differential equations without the above-mentioned deficiencies is introduced. The constructive analytical method is applied directly to linear and nonlinear fractional partial differential equations. The proposed analytical method gives a series solution which converges rapidly to an exact or approximate solution with elegant computational terms. In this new analytical method, the nonlinear terms are computed using He's Polynomials [28,29,30]. Exact solution of linear and nonlinear fractional partial differential equation are successfully obtained using the new analytical method. Thus, the proposed analytical method is a powerful mathematical method for solving linear and nonlinear fractional partial differential equations and is a refinement of the existing methods.

* Corresponding author e-mail: shehu.maitama@yahoo.com

2 Natural transform

In this section, the basic definition and properties of the Natural transform are presented.
Definition: The Natural Transform of the function $v(t)$ for $t \in (0,\infty)$ is defined over the set of functions,
$A = \left\{ v(t) : \exists\, M,\, \tau_1,\, \tau_2 > 0,\; |v(t)| < Me^{\frac{|t|}{\tau_j}},\; if\ t \in (-1)^j \times [0,\infty) \right\}$,
by the following integral:

$$\mathbb{N}^+[v(t)] = V(s,u) = \frac{1}{u}\int_0^\infty e^{\frac{-st}{u}} v(t)\,dt;\; s>0,\; u>0. \tag{1}$$

Here s and u are the Natural transform variables [31,32].
The basic properties of the Natural transform method are given below.

Property 1: *If $V(s,u)$ is the Natural transform and $F(s)$ is the Laplace transform of the function $f(t) \in A$, then* $\mathbb{N}^+[f(t)] = V(s,u) = \frac{1}{u}\int_0^\infty e^{-\frac{st}{u}} f(t)\,dt = \frac{1}{u}F\left(\frac{s}{u}\right)$.

Property 2: *If $V(s,u)$ is the Natural transform and $G(u)$ is the Sumudu transform of the function $v(t) \in A$, then* $\mathbb{N}^+[v(t)] = V(s,u) = \frac{1}{s}\int_0^\infty e^{-t} v\left(\frac{ut}{s}\right)dt = \frac{1}{s}G\left(\frac{u}{s}\right)$.

Property 3: *If* $\mathbb{N}^+[v(t)] = V(s,u)$, *then* $\mathbb{N}^+[v(\alpha t)] = \frac{1}{\alpha}V(s,u)$.

Property 4: *If* $\mathbb{N}^+[v(t)] = V(s,u)$, *then* $\mathbb{N}^+[v'(t)] = \frac{s}{u}V(s,u) - \frac{v(0)}{u}$.

Property 5: *If* $\mathbb{N}^+[v(t)] = V(s,u)$, *then* $\mathbb{N}^+[v''(t)] = \frac{s^2}{u^2}V(s,u) - \frac{s}{u^2}v(0) - \frac{v'(0)}{u}$.

Remark: The Natural transform is a linear operator. That is, if α and β are non–zero constants, then
$\mathbb{N}^+[\alpha f(t) \pm \beta g(t)] = \alpha\mathbb{N}^+[f(t)] \pm \beta\mathbb{N}^+[g(t)] = \alpha F^+(s,u) \pm \beta G^+(s,u)$.
Moreover, $F^+(s,u)$ and $G^+(s,u)$ are the Natural transforms of $f(t)$ and $g(t)$, respectively [31].

Table 1. List of some special Natural transforms

Functional Form	Natural transform Form
1	$\frac{1}{s}$
t	$\frac{u}{s^2}$
e^{at}	$\frac{1}{s-au}$
$\frac{t^{n-1}}{(n-1)!}, n=1,2,\ldots$	$\frac{u^{n-1}}{s^n}$
$\sin(t)$	$\frac{u}{s^2+u^2}$

3 Basic Definitions of Fractional Calculus

In this section, the basic definitions of fractional calculus are presented.
Definition 1: A function $f(x),\ x>0$ is said to be in the space $C_\alpha^m,\ m \in N \bigcup\{0\}$, if $f^{(m)} \in C_\alpha$.
Definition 2: A real function $f(x),\ x>0$ is said to be in the apace $C_\alpha\ \alpha \in \mathbb{R}$ if there exist a real number $p\ (>\alpha)$ such that $f(x) = x^p f_1(x)$ where $f_1(x) \in C[0,\infty)$. Clearly $C_\alpha \subset C_\beta$ if $\beta \le \alpha$.
Definition 3: The left sided Riemann-Liouville fractional integral operator of order $\mu > 0$, of a function $f(t) \in C_\alpha$, and $\alpha \ge -1$ is define as [10,33].

$$D_t^{-\mu} f(t) = \frac{1}{\Gamma(\mu)}\int_0^t \frac{f(\tau)}{(t-\tau)^{1-\mu}}d\tau,\ \mu>0,\ t>0, \tag{2}$$

$$D^0 f(t) = f(t). \tag{3}$$

Definition 4: The (left sided) Caputo fractional derivative $f,\quad f\varepsilon\, C_{-1}^m$, $m\varepsilon\ \mathbb{N}\bigcup\{0\}$, is defined as [4,5].

$$D_t^\mu f(t) = \begin{cases} D_t^{\mu-m}\left[\frac{\partial^m f(t)}{\partial t^m}\right], & m-1<\mu<m,\ m \in\ \mathbb{N}, \\ \frac{\partial^m f(t)}{\partial t^m}, & \mu = m. \end{cases} \tag{4}$$

Note that [6,10,4,5]:

$$D^{-\mu} t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\mu+1)} t^{\gamma+\mu},\ \mu>0,\ \gamma>-1,\ t>0. \tag{5}$$

$$D^{-\mu}D_t^{\mu}f(t)=f(t)-\sum_{k=0}^{m-1}v^{(k)}(0+)\frac{t^k}{k!},\ m-1<\mu\leq m,\ m\in\mathbb{N}. \tag{6}$$

Definition 5: The Natural transform of the Caputo fractional derivative is defined as:

$$\mathbb{N}^+\left[D_t^{\alpha}v(t)\right]=\frac{s^{\alpha}}{u^{\alpha}}V(s,u)-\sum_{k=0}^{m-1}\frac{s^{\alpha-(k+1)}}{u^{\alpha-k}}v^{(k)}(0+), \tag{7}$$

$(m-1<\alpha\leq m)$.

Definition 6: Mittag-Leffler function E_{α} with $\alpha>0$ is defined by the following series representation, valid in the whole complex plane [5]:

$$E_{\alpha}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)},\ \alpha>0,\ z\in\mathbb{C}. \tag{8}$$

4 Analysis of the Method

In this section, the basic idea of the Natural Homotopy Perturbation Method is clearly illustrated by the standard nonlinear fractional partial differential equation of the form:

$$D_t^{\alpha}v(x,t)+M(v(x,t))+F(v(x,t))=g(x,t), \tag{9}$$

subject to the initial condition

$$v(x,0)=f(x). \tag{10}$$

where $F(v(x,t))$ represent the nonlinear terms, $D_t^{\alpha}=\frac{\partial^{\alpha}}{\partial t^{\alpha}}$ is the Caputo fractional derivative of the function $v(t)$, $M(v(x,t))$ is the linear differential operator, and $g(x,t)$ is a source term.

Applying the Natural transform to Eq. (9) subject to the given initial condition we get:

$$V(x,s,u)=\frac{1}{s}f(x)+\frac{u^{\alpha}}{s^{\alpha}}\mathbb{N}^+\left[g(x,t)\right]-\frac{u^{\alpha}}{s^{\alpha}}\mathbb{N}^+\left[M(v(x,t))+F(v(x,t))\right]. \tag{11}$$

Taking the inverse Natural transform of Eq. (11), we have:

$$v(x,t)=G(x,t)-\mathbb{N}^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\mathbb{N}^+\left[M(v(x,t))+F(v(x,t))\right]\right], \tag{12}$$

where $G(x,t)$ is a term arising from the source term and the prescribed initial condition.

Now we apply the Homotopy Perturbation Method.

$$v(x,t)=\sum_{n=0}^{\infty}p^n v_n(x,t). \tag{13}$$

The nonlinear terms $F(v(x,t))$ is decomposed as:

$$F(v(x,t))=\sum_{n=0}^{\infty}p^n H_n(v), \tag{14}$$

where $H_n(v)$ is the He's polynomial and be computed using the following formula:

$$H_n(v_1,v_2,\cdots,v_n)=\frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[F\left(\sum_{j=0}^{n}p^j v_j\right)\right]_{p=0},n=0,1,2,\cdots \tag{15}$$

Substituting Eq. (13) and Eq. (14) into Eq. (12), we get:

$$\sum_{n=0}^{\infty} p^n v_n(x,t) = G(x,t) - p\left(\mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbb{N}^+\left[\sum_{n=0}^{\infty} p^n M(v(x,t)) + \sum_{n=0}^{\infty} p^n H_n(v)\right]\right]\right). \tag{16}$$

Using the coefficient of the likes powers of p in Eq. (16), the following approximations are obtained:

$$\begin{aligned}
p^0 &: v_0(x,t) = G(x,t),\\
p^1 &: v_1(x,t) = -\mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbb{N}^+\left[M(v(x,t)) + H_0(v)\right]\right],\\
p^2 &: v_2(x,t) = -\mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbb{N}^+\left[M(v(x,t)) + H_1(v)\right]\right],\\
p^3 &: v_3(x,t) = -\mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbb{N}^+\left[M(v(x,t)) + H_2(v)\right]\right],\\
&\vdots,
\end{aligned}$$

and so on.
Hence, the series solution of Eq. (9) is given by:

$$v(x,t) = \lim_{N\to\infty}\sum_{n=0}^{N} v_n(x,t). \tag{17}$$

5 Applications

In this section, the application of the Natural Homotopy Perturbation Method to linear and nonlinear fractional partial differential equations are clearly demonstrated to show its simplicity and high accuracy.

Example 1*Consider the following linear fractional partial differential equation of the form:*

$$D_t^\alpha v - 2v_{xx} - 2v_{yy} = 0, \quad -\infty < x,y < \infty, \; t > 0, \tag{18}$$

subject to the initial condition

$$v(x,y,0) = sin(x)sin(y), \; v_t(x,y,0) = 0, \quad \alpha \; \varepsilon \; (1,2). \tag{19}$$

Applying the Natural transform to Eq. (18) subject to the given initial condition, we get:

$$V(x,y,s,u) = \frac{sin(x)sin(y)}{s} + \frac{u^\alpha}{s^\alpha}\mathbb{N}^+\left[2v_{xx} + 2v_{yy}\right]. \tag{20}$$

Taking the inverse Natural transform of Eq. (20), we get:

$$v(x,y,t) = sin(x)sin(y) + \mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbb{N}^+\left[2v_{xx} + 2v_{yy}\right]\right]. \tag{21}$$

Now we apply the Homotopy Perturbation Method.

$$v(x,y,t) = \sum_{n=0}^{\infty} p^n v_n(x,y,t). \tag{22}$$

Then Eq. (21) will become:

$$\sum_{n=0}^{\infty} p^n v_n(x,y,t) = sin(x)sin(y) + p\left(\mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbb{N}^+\left[2\sum_{n=0}^{\infty} p^n v_{nxx} + 2\sum_{n=0}^{\infty} p^n v_{nyy}\right]\right]\right). \tag{23}$$

Using the coefficients of the like powers of p in Eq. (23), the following approximations are obtained:

$$
\begin{aligned}
p^0: v_0(x,y,t) &= sin(x)sin(y),\\
p^1: v_1(x,y,t) &= \mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbb{N}^+\left[2v_{0xx}+2v_{0yy}\right]\right]\\
&= -sin(x)sin(y)\frac{(4t^\alpha)}{\Gamma(\alpha+1)},
\end{aligned}
$$

$$
\begin{aligned}
p^2: v_2(x,y,t) &= \mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbb{N}^+\left[2v_{1xx}+2v_{1yy}\right]\right]\\
&= sin(x)sin(y)\frac{(4t^\alpha)^2}{\Gamma(2\alpha+1)},
\end{aligned}
$$

$$
\begin{aligned}
p^3: v_3(x,y,t) &= \mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbb{N}^+\left[2v_{2xx}+2v_{2yy}\right]\right]\\
&= -sin(x)sin(y)\frac{(4t^\alpha)^3}{\Gamma(3\alpha+1)},
\end{aligned}
$$

$$
\begin{aligned}
p^4: v_4(x,y,t) &= \mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbb{N}^+\left[2v_{3xx}+2v_{3yy}\right]\right]\\
&= sin(x)sin(y)\frac{(4t^\alpha)^4}{\Gamma(4\alpha+1)},\\
&\vdots
\end{aligned}
$$

and so on.
Then, the series solution of Eq. (18) is given by:

$$
\begin{aligned}
v(x,y,t) &= \lim_{N\to\infty}\sum_{n=0}^{N} v_n(x,y,t) \\
&= v_0(x,y,t)+v_1(x,y,t)+v_2(x,y,t)+v_3(x,y,t)+\cdots\\
&= sin(x)sin(y)\left(1-\frac{(4t^\alpha)}{\Gamma(\alpha+1)}+\frac{(4t^\alpha)^2}{\Gamma(2\alpha+1)}-\frac{(4t^\alpha)^3}{\Gamma(3\alpha+1)}+\frac{(4t^\alpha)^4}{\Gamma(4\alpha+1)}+\cdots\right)\\
&= sin(x)sin(y)\sum_{m=0}^{\infty}\frac{(-4t^\alpha)^m}{\Gamma(m\alpha+1)}\\
&= sin(x)sin(y)E_\alpha(-4t^\alpha).
\end{aligned}
\tag{24}
$$

When $\alpha = 1$, we obtained the following result:

$$
\begin{aligned}
v(x,y,t) &= \lim_{N\to\infty}\sum_{n=0}^{N} v_n(x,y,t) \\
&= v_0(x,y,t)+v_1(x,y,t)+v_2(x,y,t)+v_3(x,y,t)+\cdots\\
&= sin(x)sin(y)\left(1-\frac{(4t)}{1!}+\frac{(4t)^2}{2!}-\frac{(4t)^3}{3!}+\frac{(4t)^4}{4!}+\cdots\right)\\
&= sin(x)sin(y)e^{-4t},
\end{aligned}
\tag{25}
$$

which is the exact solution of Eq. (18).
The exact solution is in close agreement with the result obtained by (HPSTM) [26].

Example 2*Consider the following nonlinear fractional partial differential equation of the form:*

$$D_t^{\alpha} v - 6vv_x + v_{xxx} = 0, \tag{26}$$

subject to the initial condition

$$v(x,0) = \frac{1}{6}(x-1). \tag{27}$$

Applying the Natural transform on both sides of Eq. (26), we get:

$$V(x,s,u) = \frac{1}{6s}(x-1) + \frac{u^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}[6vv_x - v_{xxx}] \tag{28}$$

Taking the inverse Natural transform of Eq. (28), we get:

$$v(x,t) = \frac{1}{6}(x-1) + \mathbb{N}^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}[6vv_x - v_{xxx}]\right]. \tag{29}$$

Now we apply the Homotopy Perturbation Method.

$$v(x,t) = \sum_{n=0}^{\infty} p^n v_n(x,t). \tag{30}$$

Then, Eq. (29), will become:

$$\sum_{n=0}^{\infty} p^n v_n(x,t) = \frac{1}{6}(x-1) + p\left(\mathbb{N}^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[6\sum_{n=0}^{\infty} p^n H_n(v) - \sum_{n=0}^{\infty} p^n v_{nxxx}(x,t)\right]\right]\right), \tag{31}$$

where $H_n(v)$ is a He's Polynomial which represent the nonlinear term, vv_x.

Some few components of the He's Polynomials are given below:

$$\begin{aligned}
H_0(v) &= v_0 v_{0x},\\
H_1(v) &= v_1 v_{0x} + v_0 v_{1x},\\
H_2(v) &= v_{0x} v_2 + v_{1x} v_1 + v_{2x} v_0,\\
H_3(v) &= v_{0x} v_3 + v_{1x} v_2 + v_{2x} v_1 + v_{3x} v_0,\\
&\vdots,
\end{aligned}$$

and so on.
Using the coefficient of the like powers of p, in Eq.(31), we obtained the following approximations:

$$\begin{aligned}
p^0:\ v_0(x,t) &= \frac{1}{6}(x-1),\\
p^1:\ v_1(x,t) &= \mathbb{N}^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}[6H_0(v) - v_{0xxx}]\right]\\
&= \mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}[6v_0 v_{0x} - v_{0xxx}]\right]\\
&= \frac{(x-1)}{6}\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right),
\end{aligned}$$

$$\begin{aligned}
p^2:\ v_2(x,t) &= \mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}[6H_1(v) - v_{1xxx}]\right]\\
&= \mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}[6(v_1 v_{0x} + v_0 v_{1x}) - v_{1xxx}]\right]\\
&= \frac{(x-1)}{6}\left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\right),
\end{aligned}$$

$$
\begin{aligned}
p^3: v_3(x,t) &= \mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^+\left[6H_2(v) - v_{2xxx}\right]\right] \\
&= \mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^+\left[6(v_{0x}v_2 + v_{1x}v_1 + v_{2x}v_0) - v_{2xxx}\right]\right] \\
&= \frac{(x-1)}{6}\left(\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\right), \\
&\vdots,
\end{aligned}
$$

and so on.
Therefore, the series solution of Eq. (26) is given by:

$$
\begin{aligned}
v(x,t) &= \lim_{N\to\infty}\sum_{n=0}^{N} v_n(x,t) \\
&= v_0(x,t) + v_1(x,t) + v_2(x,t) + v_3(x,t) + \cdots \\
&= \frac{(x-1)}{6}\left(1 + \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots\right) \\
&= \frac{(x-1)}{6}\sum_{m=0}^{\infty}\frac{(t^{\alpha})^m}{\Gamma(m\alpha+1)} \\
&= \frac{(x-1)}{6}E_{\alpha}(t^{\alpha}).
\end{aligned}
$$

When $\alpha = 1$, we obtained the exact solution of Eq.(26) as:

$$
v(x,t) = \frac{1}{6}\left(\frac{x-1}{1-t}\right), \quad |t| < 1. \tag{32}
$$

The exact solution is in close agreement with the result obtained by (ADM) [11] and (NDM) [34].

Example 3*Consider the following nonlinear time-fractional Harry Dym equation of the form:*

$$
D_t^{\alpha}v(x,t) - v^3(x,t)D_x v(x,t) = 0, \;\; 0 < \alpha \leq 1, \tag{33}
$$

subject to the initial condition

$$
v(x,0) = \left(a - \frac{3\sqrt{b}}{2}x\right)^{\frac{2}{3}}. \tag{34}
$$

Applying the Natural transform to Eq.(33) subject to the given initial condition, we get:

$$
V(x,s,u) = \frac{1}{s}\left(a - \frac{3\sqrt{b}}{2}x\right)^{\frac{2}{3}} + \frac{u^{\alpha}}{s^{\alpha}}\mathbb{N}^+\left[v^3(x,t)D_x v(x,t)\right]. \tag{35}
$$

Taking the inverse Natural transform of Eq.(35), we get:

$$
v(x,t) = \left(a - \frac{3\sqrt{b}}{2}x\right)^{\frac{2}{3}} + \mathbb{N}^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\mathbb{N}^+\left[v^3(x,t)D_x v(x,t)\right]\right]. \tag{36}
$$

Now we apply the Homotopy Perturbation Method.

$$
v(x,t) = \sum_{n=0}^{\infty} p^n v_n(x,t). \tag{37}
$$

Then Eq.(36) will become:

$$
\sum_{n=0}^{\infty} p^n v_n(x,t) = \left(a - \frac{3\sqrt{b}}{2}x\right)^{\frac{2}{3}} + p\left(\mathbb{N}^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\mathbb{N}^+\left[\sum_{n=0}^{\infty} p^n H_n(v)\right]\right]\right), \tag{38}
$$

where $H_n(v)$ is the He's polynomials which represent the nonlinear term $v^3(x,t)D_x v(x,t)$.
Some few components of the He's polynomials are given below:

$$\begin{aligned}
H_0(v) &= v_0^3 D_x v_0,\\
H_1(v) &= v_0^3 D_x v_1 + 3v_1 v_0^2 D_x v_0,\\
H_2(v) &= v_0^3 D_x v_2 + 3v_1 v_0^2 D_x v_1 + \left(3v_0 v_1^2 + 3v_0^2 v_2\right) D_x^3 v_0,\\
&\vdots,
\end{aligned}$$

and so on.
Using the coefficients of the like powers of p in Eq.(38), we obtained the following approximations:

$$\begin{aligned}
p^0:\ v_0(x,t) &= \left(a - \frac{3\sqrt{b}}{2}x\right)^{\frac{2}{3}},\\
p^1:\ v_1(x,t) &= \mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbb{N}^+\left[H_0(v)\right]\right]\\
&= \mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbb{N}^+\left[v_0^3 D_x v_0\right]\right]\\
&= -b^{\frac{2}{3}}\left(a - \frac{3\sqrt{b}}{2}x\right)^{-\frac{1}{3}}\frac{t^\alpha}{\Gamma(\alpha+1)},
\end{aligned}$$

$$\begin{aligned}
p^2:\ v_2(x,t) &= \mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbb{N}^+\left[H_1(v)\right]\right]\\
&= \mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbb{N}^+\left[v_0^3 D_x v_1 + 3v_1 v_0^2 D_x v_0\right]\right]\\
&= -\frac{b^3}{2}\left(a - \frac{3\sqrt{b}}{2}x\right)^{-\frac{4}{3}}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},
\end{aligned}$$

$$\begin{aligned}
p^3:\ v_3(x,t) &= \mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbb{N}^+\left[H_2(v)\right]\right]\\
&= \mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbb{N}^+\left[v_0^3 D_x v_2 + 3v_1 v_0^2 D_x v_1 + \left(3v_0 v_1^2 + 3v_0^2 v_2\right) D_x^3 v_0\right]\right]\\
&= b^{\frac{9}{2}}\left(a - \frac{3\sqrt{b}}{2}x\right)^{-\frac{7}{3}}\left(\frac{15}{2}\frac{\Gamma(2\alpha+1)}{2(\Gamma(\alpha+1))^2} - 16\right)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)},\\
&\vdots,
\end{aligned}$$

and so on.
Then, the series solution of Eq.(33) is given by:

$$\begin{aligned}
v(x,t) &= \sum_{n=0}^{\infty} v_n(x,t) \qquad (39)\\
&= v_0(x,t) + v_1(x,t) + v_2(x,t) + v_3(x,t) + \cdots\\
&= \left(a - \frac{3\sqrt{b}}{2}x\right)^{\frac{2}{3}} - b^{\frac{2}{3}}\left(a - \frac{3\sqrt{b}}{2}x\right)^{-\frac{1}{3}}\frac{t^\alpha}{\Gamma(\alpha+1)}\\
&- \frac{b^3}{2}\left(a - \frac{3\sqrt{b}}{2}x\right)^{-\frac{4}{3}}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + b^{\frac{9}{2}}\left(a - \frac{3\sqrt{b}}{2}x\right)^{-\frac{7}{3}}\left(\frac{15}{2}\frac{\Gamma(2\alpha+1)}{2(\Gamma(\alpha+1))^2} - 16\right)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\cdots
\end{aligned}$$

When $\alpha = 1$, we obtained the exact solution of Eq.(33) as:

$$v(x,t) = \left(a - \frac{3\sqrt{b}}{2}(x+bt)\right)^{\frac{2}{3}}. \tag{40}$$

The exact solution is in close agreement with the result obtained by (ADM) [25].

6 Conclusion

In this paper, Natural transform method (NTM) and Homotopy Perturbation Method (HPM) are successfully combined to form a powerful analytical method called the Natural Homotopy Perturbation Method (NHPM) for solving linear and nonlinear fractional partial differential equations. The new analytical method gives a series solution which converges rapidly to an exact or approximate solution with elegant computational terms. In this new analytical method, the fractional derivative are handle in Caputo sense and the nonlinear term are computed using He's Polynomials. The new analytical method is applied successfully and obtained an exact solution of linear and nonlinear fractional partial differential equations. The simplicity and high accuracy of the new analytical method is clearly illustrated. Thus, the Natural Homotopy Perturbation Method is a powerful analytical method for solving linear and nonlinear fractional partial differential equations.

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