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Mathematical Model of Vector-Borne Plant Disease with Memory on the Host and the Vector

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Abstract: In this paper, we introduce a fractional order model of vector-borne plant diseases. Memory in both the host, and the vector population provides essential tools to understand the behavior of plant diseases. We use the presented model to study the effects of memory on the host and the vector. The fractional order derivative which is considered as the index of memory is described in the Caputo sense.

Keywords: Fractional calculus, plant diseases models, numerical solutions of fractional order models.

1 Introduction and Preliminaries

Plants are an essential resource for human well-being. They are the major sources of oxygen and products that people use [1]. Plants can get infected by various diseases just like people do caused by pathogens which cannot be seen or recognized without magnification [2]. Plant infectious diseases are very destructive to plants. There are different kinds of plant diseases such as Tobacco Mosaic Virus, Cucumber Mosaic Virus and Barley Yellow Dwarf, that can be detected by different symptoms like wilting or yellowing [3]. Plant diseases pose a serious threat to food security, human health and world economies [1,4]. However, it is impossible to study the large scale dynamics of plant infectious diseases without a formal structure of mathematical model. It is proven that such models help to predict the future behavior of natural disasters [4,5] like plant diseases. Mathematical models are essential and helpful to assist the decision makers to put their strategies and to activate their programs. Nowadays, there is a rapid expansion of studies of mathematical modeling applied to the biological control system [1,4].

We recall that the fractional calculus refers to integration or differentiation of non-integer order [6,7]. Fractional calculus is three centuries old as the conventional calculus, but not very popular among science and engineering community. The beauty of this subject is that fractional derivatives (and integrals) are not a local property or quantity [8]. In the last years has found use in studies of viscoelastic materials, as well as in many fields of science and engineering including fluid flow, rheology, diffusive transport, electrical networks, electromagnetic theory and probability [7]. The fractional calculus and its applications are undergoing rapid developments with more and more persuasion applications in the real world. Below, we will give the definition of fractional-order integration and fractional-order differentiation. The two most commonly used definitions are Riemann - Liouville and Caputo.
Definition 2.1 Riemann –Liouville fractional integration of order α is defined as:

$$J^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt, \;\; 0<\alpha<1, \; x>0,$$

$$J^0 f(x) = f(x).$$

This is called integral with memory

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Definition 2.2 Riemann–Liouville and Caputo fractional derivatives of order αcan be defined respectively as:

$$D^{\alpha} f(x) = D^{m}\left(J^{m-\alpha} f(x)\right),$$

$$D_{*}^{\alpha} f(x) = J^{m-\alpha}\left(D^{m} f(x)\right),$$

where $m-1 < \alpha \leq m,\ m \in N$.
Properties of the operator J^{α}can be found in [9,10], we mention only the following:
(1) $J^{\alpha} J^{\beta} f(x) = J^{\alpha+\beta} f(x)$,
(2) $J^{\alpha} J^{\beta} f(x) = J^{\beta} J^{\alpha} f(x)$,
(3) $J^{\alpha} t^{\gamma} = \frac{\Gamma(+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma},\ \alpha > 0,\ \gamma > -1,\ t > 0.$

Few of the biological models are of fractional order. The concept of fractional calculus has tremendous potential to change the way we see the model.

The major reason of using fractional order models is that they are naturally related to systems with memory which exist in most biological systems [2].

Hence, we introduce fractional-order into the integer order model of vector-borne plant disease model described in [11]:

$$\begin{aligned} D^{\alpha}(s) &= \mu(K-s) - \left(\frac{\beta_p y}{1+\gamma_p y} + \frac{\beta_s x}{1+\gamma_s x}\right) s + dx, \\ D^{\alpha}(x) &= \left(\frac{\beta_p y}{1+\gamma_p y} + \frac{\beta_s x}{1+\gamma_s x}\right) s - wx, \\ D^{\alpha}(y) &= \frac{\beta_1 x}{1+\gamma_1 x}\left(\frac{\Lambda}{m} - y\right) - my, \end{aligned} \tag{1}$$

where $0 < \alpha \leq 1$, is considered as the index of memory [12].

The parameters of the model are defined below:

Table 1

parameter	description
s	number of the susceptible plant hosts
x	number of the infected plant hosts
y	density of the infected insect vectors
β_P	biting rate of an infected vector on the susceptible host plants
β_s	infection incidence between infected and susceptible hosts
β_1	infection ratio between infected hosts and susceptible vectors
γ_p	determines the level at which the force of infection saturates
γ_1	determines the level at which the force of infection saturates
γ_s	determines the level at which the force of infection saturates
γ	the conversion rate of infected hosts to recovered hosts
μ	natural death rate of plant hosts
m	natural death rate of insect vectors
Λ	birth or immigration of insect vectors
d	disease-induced mortality of infected hosts
w	w=$d+\mu+\theta$, θ is the conversion rate of infected hosts to recovered hosts

Model (1) has some flaws, since the left-hand side and the right-hand side of the system have different dimensions. So, the system (1) can be mathematically corrected using the procedure described by Diethelm [13] as follows:

$$\begin{aligned} D^{\alpha}(s) &= \mu^{\alpha}(K-s) - \left(\frac{\beta_p^{\alpha} y}{1+\gamma_p y} + \frac{\beta_s^{\alpha} x}{1+\gamma_s x}\right) s + d^{\alpha} x, \\ D^{\alpha}(x) &= \left(\frac{\beta_p^{\alpha} y}{1+\gamma_p y} + \frac{\beta_s^{\alpha} x}{1+\gamma_s x}\right) s - wx, \\ D^{\alpha}(y) &= \frac{\beta_1 x}{1+\gamma_1 x}\left(\frac{\Lambda}{m^{\alpha}} - y\right) - m^{\alpha} y. \end{aligned} \tag{2}$$

In the remaining sections we discuss the properties of the proposed model. We discuss the basic reproduction number in Section 2, the non-negative solutions in Section 3, the equilibrium points and stability in Section 4 and existence of uniformly stable solution in Section 5, respectively. In Section 6 we present the numerical results. The conclusions are presented in Section 7.

2 The Basic Reproduction Number

The basic reproduction number is defined as the expected number of cases produced by an infection in a completely uninfected population. When $R_0 < 1$, the infection will die out in the long run. But if $R_0 > 1$, the disease is able to invade the susceptible population. We are able to calculate the basic reproduction number of system (1) by the next generation method [14,15,11]. The rate at which new infections are created is determined by the matrix F, and the rates of transfer into and out of the class of infected states are represented by the matrix V; these are given by

$$F = \begin{pmatrix} \beta_s K & \beta_p K \\ 0 & 0 \end{pmatrix}$$

and

$$V = \begin{pmatrix} \omega & 0 \\ -\frac{\beta_1 \Lambda}{m} & m \end{pmatrix}.$$

Therefore, the next generation matrix is

$$FV^{-1} = \begin{pmatrix} \frac{\beta_s K}{\omega} + \frac{\beta_1 \beta_p \Lambda K}{m^2 \omega} & \frac{\beta_p K}{m} \\ 0 & 0 \end{pmatrix}$$

from which we get the basic reproduction number as

$$R_0 = \frac{\beta_s K}{\omega} + \frac{\beta_1 \beta_p \Lambda K}{m^2 \omega}. \tag{3}$$

For the system (2), the basic reproduction number can be also determined to be

$$R_0^* = \frac{\beta_s^\alpha K}{\omega} + \frac{\beta_1 \beta_p^\alpha \Lambda K}{(m^2)^\alpha \omega}. \tag{4}$$

The effect of the fractional order α (which is defined as the memory of the system [12]) is depicted in (4). So, the system (2) is more realistic as it reflects the effects of the memory of the plant host and insect vector.

3 Non-Negative Solutions

Denote $R_+^3 = \{\chi \in R^3 \mid \chi \geq 0\}$ and let $\chi(t) = (s(t), x(t), y(t))^T\}$. To prove the Theorem 1, we need the following generalized mean value theorem [16] and corollary.

Lemma 1. (Generalized Mean Value Theorem [16]). Suppose that $f(\chi) \in C[a,b]$ and $D_a^\alpha f(\chi) \in C[a,b]$, $for\ 0 < \alpha \leq 1$, then we have

$$f(\chi) = f(a) + \frac{1}{\Gamma(\alpha)} (D_a^\alpha f)(\xi)(\chi - a)^\alpha$$

with $a \leq \xi \leq \chi$, $\forall \chi \in (a,b]$.

Corollary 1. Suppose that $f(\chi) \in C[a,b]$ and $D_a^\alpha f(\chi) \in C(a,b]$, $for\ 0 < \alpha \leq 1$. If $D_a^\alpha f(\chi) \geq 0$, $\forall \chi \in (a,b)$, then $f(\chi)$ is non-decreasing for each $\chi \in [a,b]$.

Proof. This is clear from **Lemma 1.**

Theorem 1. There is a unique solution $\chi(t) = (s,x,y)^T$ to system (2) on $t \geq 0$ and the solution will remain in R_+^3.

Proof. From Theorem 3.1 and Remark 3.2 in [17], we know the solution on $(0,+\infty)$ of (2) is existent and unique. Next, we will show the nonnegative orthant R_+^3 is a positively invariant region. What is needed for this is to show that on each hyper plane bounding the nonnegative orthant, the vector field points into R_+^3 . From Eq. (2), we find

$$D^\alpha s|_{s=0} = \mu^\alpha K + d^\alpha x \geq 0,$$
$$D^\alpha x|_{x=0} = \frac{\beta_p^\alpha y}{1+\gamma_p y} s \geq 0,$$
$$D^\alpha y|_{y=0} = \frac{\beta_1 x}{1+\gamma_1 x}\left(\frac{\Lambda}{m^\alpha}\right) \geq 0.$$

4 Equilibrium Points and Stability

To evaluate the equilibrium points of (2), let

$$D^{\alpha}s=0,$$

$$D^{\alpha}x=0,$$

$$D^{\alpha}y=0.$$

Then, $E_0=(K,0,0)$ and $\mathrm{E}^*=(s^*,x^*,y^*)$ are the equilibrium points, where

$$s^*=K-\left(1+\frac{\gamma}{\mu^{\alpha}}\right)x^*,$$

$$x^*=\frac{-B+\sqrt{B^2-4AC}}{2A},$$

$$y^*=\frac{\beta_1\Lambda x^*}{m^{\alpha}\beta_1x^*+(m^2)^{\alpha}(1+\gamma_1x^*)},$$

where

$$A=(\mu^{\alpha}+\gamma)\left(\beta_p^{\alpha}\beta_1\gamma_s\Lambda+m\beta_1\beta_s^{\alpha}+(m^2)^{\alpha}\beta_s^{\alpha}\gamma_1+\beta_s^{\alpha}\beta_p^{\alpha}\beta_1\Lambda\right)>0,$$

$$B=(\mu^{\alpha}+\gamma)\left(\beta_p^{\alpha}\beta_1\Lambda+\beta_s^{\alpha}(m^2)^{\alpha}\right)-\mu^{\alpha}K\,(m\beta_s^{\alpha})\,\beta_1+\gamma_1\beta_s^{\alpha}(m^2)^{\alpha}+$$

$$\beta_s^{\alpha}\beta_p^{\alpha}\beta_1\Lambda+\beta_p^{\alpha}\beta_1\gamma_s\Lambda)+\mu^{\alpha}\omega\left(m^{\alpha}\beta_1+\gamma_1(m^2)^{\alpha}+\gamma_p\beta_1\Lambda+(m^2)^{\alpha}\,\gamma_s\right),$$

C=$(\mu m^2)^{\alpha}\omega\,(1-R_0^*)$.
The Jacobian matrix $J(E_0)$ for system (2) is given by

$$J(E_0)=\begin{pmatrix}-\mu^{\alpha} & d^{\alpha}-\beta_s^{\alpha}K & -\beta_p^{\alpha}K,\\ 0 & \beta_s^{\alpha}K-\omega & \beta_p^{\alpha}K,\\ 0 & \frac{\Lambda\beta_1}{m^{\alpha}} & -m^{\alpha}\end{pmatrix}.$$

The uninfected steady state is asymptotically stable if all of the eigenvalues λ of the Jacobian matrix $J(E_0)$ satisfy the following condition [6,18]:

$$|\arg(\lambda)|>\frac{\alpha\pi}{2}.$$

5 Existence of Uniformly Stable Solution

We will prove the existence and uniqueness of solution for the reduced system (2). Consider the following Lemma:
Lemma 5.1. (Theorem 8.11, [19]) Let $0<\alpha_j<1$, for $j=1,2,\ldots,k$ and consider the initial value problem given by the multi-order fractional differential system (in the Caputo sense)

$$D_*^{\alpha_j}y_j=f_j(x,y_1(x),\ldots,y_k(x))),\ j=1,2,\ldots,k \tag{5}$$

with initial conditions

$$y_j(0)=c_j,\qquad j=1,2,\ldots,k.$$

Assume that the functions $f_j = [0,x] \times R^k \to R,\ j = 1,2,\ldots,k$ are continuous and satisfy Lipschitz conditions with respect to all their arguments except for the first. Then, the initial value problem (5) has a uniquely determined continuous solution. Since each $f_i = [0,T_1] \times R^3 \to R_+\ ; i = 1,2,3$ is continuous, to prove that systems (2) have a unique continuous solution, we need to show that each f_i satisfies the Lipschitz condition with respect to each of its argument except for the first.
Let

$$x_1(t) = s(t),\ \ x_2(t) = x(t), x_3(t) = y(t),$$

$$D^\alpha x_1(t) = f_1(x_1(t), x_2(t), x_3(t)),\quad t > 0 \quad \text{and} \quad x_1(0) = x_{\circ 1}, \tag{6}$$

$$D^\alpha x_2(t) = f_2(x_1(t), x_2(t), x_3(t)),\quad t > 0 \quad \text{and} \quad x_2(0) = x_{\circ 2}, \tag{7}$$

$$D^\alpha x_3(t) = f_3(x_1(t), x_2(t), x_3(t)),\quad t > 0 \quad \text{and} \quad x_3(0) = x_{\circ 3}, \tag{8}$$

Let $D = \{x_1, x_2, x_3 \in R:\ |x_i(t) \le a, t \in [0,T], i = 1,2,3|\}$.
The functions f_i satisfy the Lipschitz condition on R_+^3 if

$$|f_i(s_1,x_1,y_1) - f_i(s_2,x_2,y_2)| \le K_2 \|X_1(t) - X_2(t)\|_2, \tag{9}$$

where $\|X_1(t) - X_2(t)\|_2 = |s_1 - s_2| + |x_1 - x_2| + |y_1 - y_2|$ and K_2 is *the Lipschitz constant*.
Condition (9) is equivalent to show that: each $\frac{\partial f_i}{\partial x_j}(x_1(t), x_2(t), x_3(t))$ exists and satisfies the following relation:
$\frac{\partial f_i}{\partial x_j}(x_1(t), x_2(t), x_3(t)) \le K_n,\quad \forall\, i, j = 1,2,3\ \ and\ n = 1,2,3,\ldots,9$ [16], where

$$x_1(t) = s(t),\ x_2(t) = x(t),\ \ x_3(t) = y(t).$$

This implies that each of the three functions f_1, f_2, f_3 satisfies the Lipschitz condition with respect to the three arguments $x_{1,}x_2\ and\ x_3$, and then each of the three functions f_1, f_2, f_3 is absolutely continuous with respect to the three arguments $x_{1,}x_2\ and\ x_3$.
Consider the following initial value problem which represents the fractional order model which describes a vector-borne plant disease model (6) and (7) and (8).
Definition 5.1. By a solution of the fractional order model which describes a vector-borne plant disease model (6) and (7) and (8), we mean a column vector$(x_1(t)\ \ x_2(t)\ \ x_3(t))^\tau$, $x_1, x_2 and\ x_3 \in \mathbb{C}[0,T]$ is the class of continuous functions defined on the interval $[0,T]$ and τ denote the transpose of the matrix [16].

Theorem 5.1. The fractional order model which describes a vector-borne plant disease model (2) has a unique uniformly Lyapunov stable solution [12].
Proof. Write the model (6) and (7) and (8) in the matrix form

$$D^\alpha X(t) = F(X(t)), t > 0 \text{ and } X(0) = x_0,$$

where
$X(t) = (x_1(t)\ \ x_2(t)\ \ x_3(t))^\tau$,
$F((X(t)) = (f_1(x_1(t),\ x_2(t), x_3(t))\quad f_2(x_1(t),\ x_2(t), x_3(t))\quad f_3(x_1(t),\ x_2(t), x_3(t)))^\tau$.
Applying the Theorem 2.1 [20], we deduce that the fractional order model which describes a vector-borne plant disease model (6) and (7) and (8) has an unique solution. Also, by Theorem 3.2 [20] this solution is uniformly Lyapunov stable.

6 Numerical Results

In this section, GEM [16, 21] is applied to get approximate solutions of the system (1). Consider that:
$\gamma_1 = 0.1,\ \gamma_p = 0.2,\ \gamma_s = 0.2,\ d = 0.1$.The initial conditions are (x_0, y_0, z_0) =(700, 200, 10).

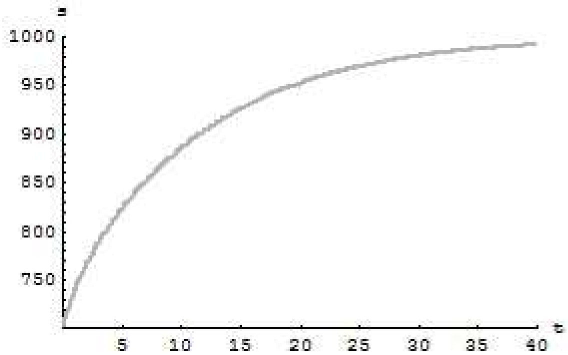


Fig (1.a)

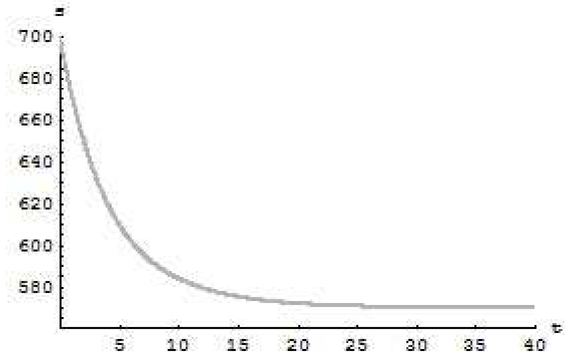
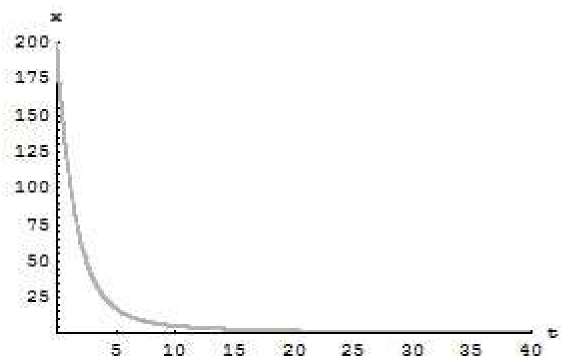
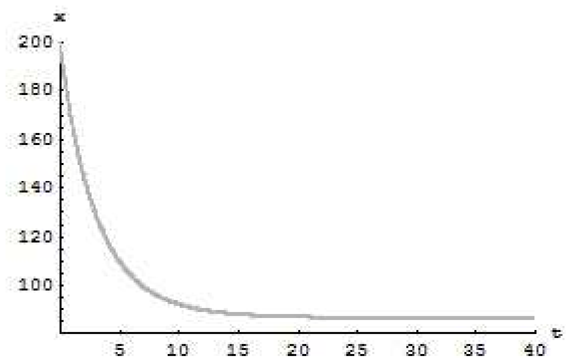


Fig (1.b)

Fig.1. The number of the susceptible plant hosts; s(t), when $\alpha = 1, \beta_1 = 0.0025,\ \beta_p = 0.0025,\ \beta_s = 0.0001\,(1.a),\ \beta_1 = 0.01,\ \beta_p = 0.02,\ \beta_s = 0.01,\ (1.b)$.



Fig(2.a)



Fig(2.b)

Fig.2. The number of the infected plant hosts x(t) when $\alpha = 1,\ \beta_1 = 0.0025,\ \beta_p = 0.0025,\ \beta_s = 0.0001\,(2.a),\ \beta_1 = 0.01,\ \beta_p = 0.02,\ \beta_s = 0.01,\ (2.b)$.

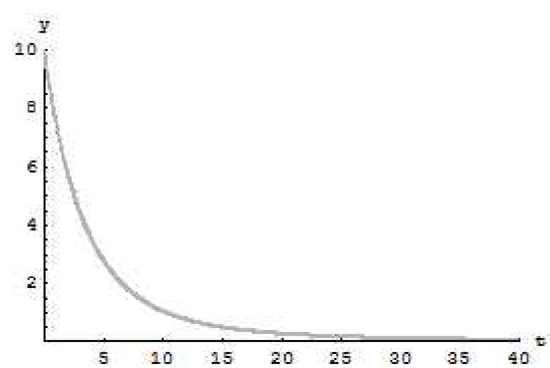


Fig (3.a)

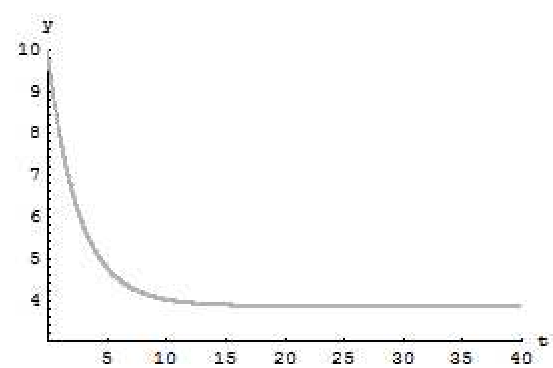


Fig (3.b)

Fig.3. The density of the infected insect vectors y (t) when $\alpha = 1, \beta_1 = 0.0025, \beta_p = 0.0025,\ \beta_s = 0.0001\,(3.a),\ \beta_1 = 0.01,\ \beta_p = 0.02,\ \beta_s = 0.01\,(3.b)$.

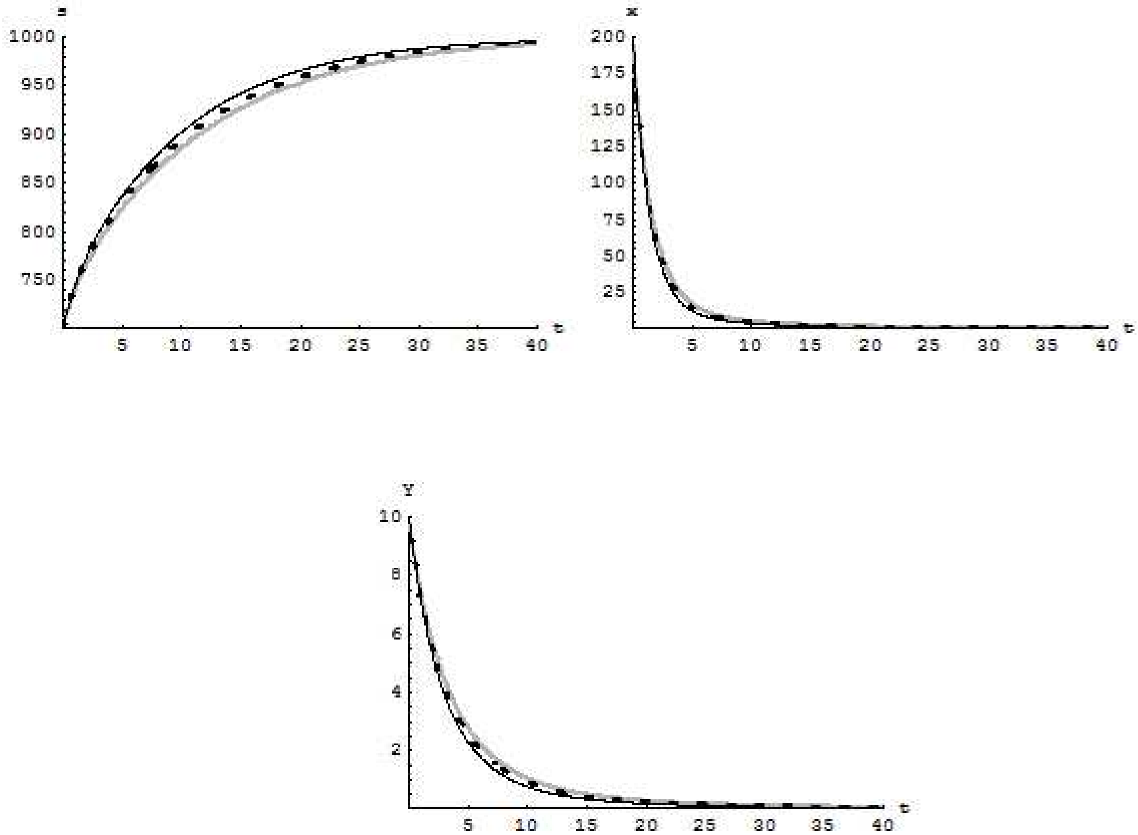


Fig.4. The number of the susceptible plant hosts; s (t), the number of the infected plant hosts x(t), the density of the infected insect vectors y (t), in the 1st case when $\beta_1 = 0.0025$, $\beta_p = 0.0025$, $\beta_s = 0.0001$ for $\alpha = 1$ (the gray line) $\alpha = 0.98$, (the dashed line), $\alpha = 0.95$ (the black solid line).

Now, GEM [21] is applied to get approximate solutions of the system (2). Consider that $\gamma_1 = 0.1$, $\gamma_p = 0.2$, $\gamma_s = 0.2$, $d = 0.1$. The initial conditions are (x_0, y_0, z_0) =(700, 200, 10).

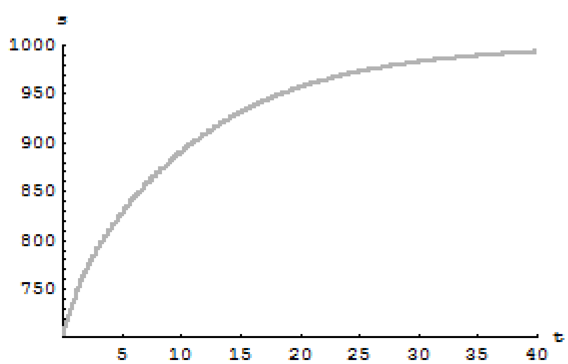


Fig 5.a

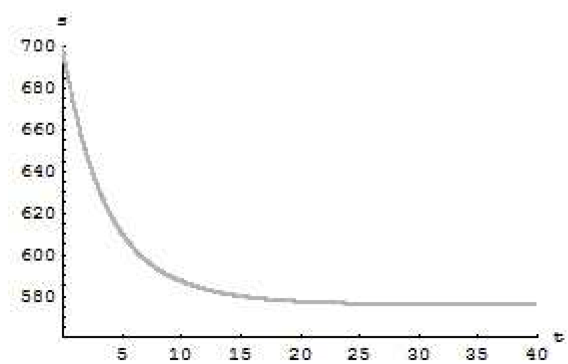


Fig 5.b

Fig.5. The number of the susceptible plant hosts; s(t), when $\alpha_2 = 1, \beta_1 = 0.0025$, $\beta_p = 0.0025$, $\beta_s = 0.0001$ $(5.a)$, $\beta_1 = 0.01$, $\beta_p = 0.02$, $\beta_s = 0.01$, $(5.b)$.

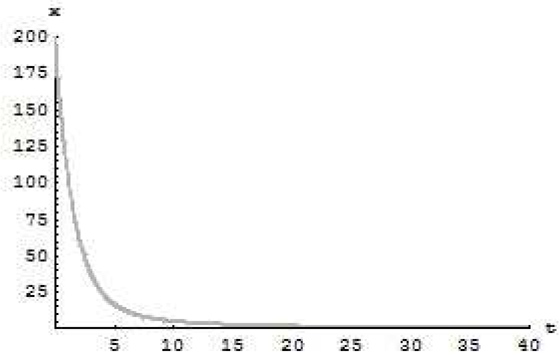


Fig 6.a

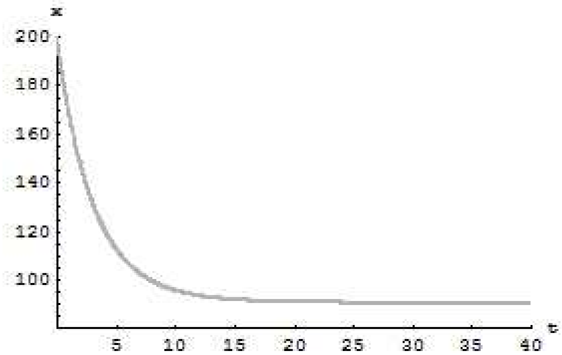


Fig 6.b

Fig.6. The number of the infected plant hosts x(t) when $\alpha_2 = 1,\ \beta_1 = 0.0025,\ \beta_p = 0.0025,\ \beta_s = 0.0001\,(6.a),\ \beta_1 = 0.01,\ \beta_p = 0.02,\ \beta_s = 0.01,\ (6.b)$.

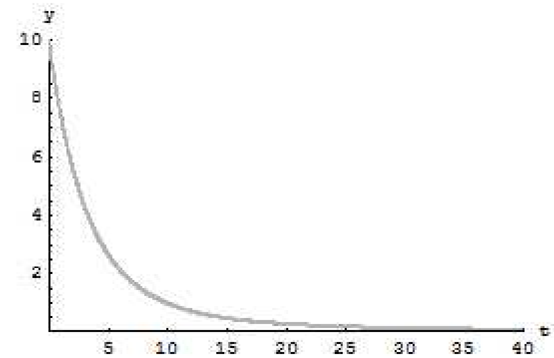


Fig 7.a

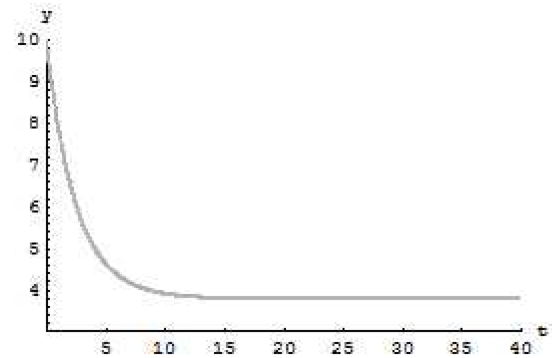


Fig 7.b

Fig.7. The density of the infected insect vectors y (t) when $\alpha_2 = 1,\ \beta_1 = 0.0025,\ \ \beta_p = 0.0025,\ \beta_s = 0.0001\,(7.a),\ \beta_1 = 0.01,\ \beta_p = 0.02,\ \beta_s = 0.01,\ (7.b)$.

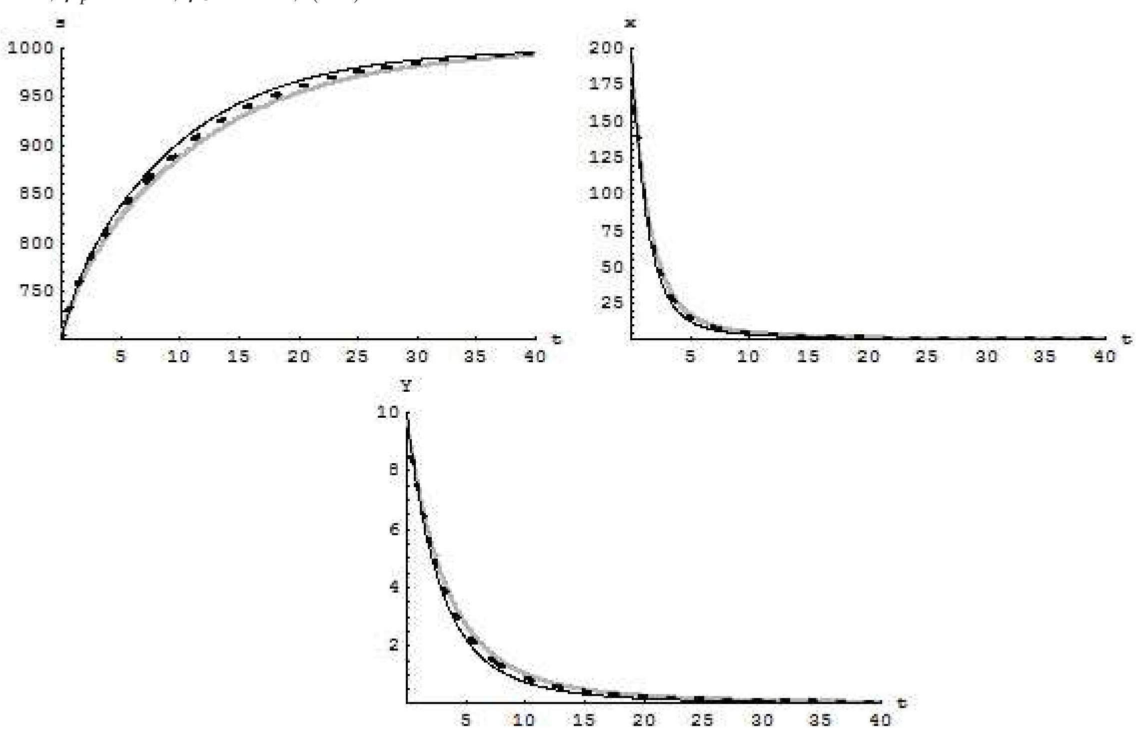


Fig.8. The number of the susceptible plant hosts; s (t), the number of the infected plant hosts x(t), the density of the infected insect vectors y (t), in the 1st case when $\beta_1 = 0.0025$, $\beta_p = 0.0025$, $\beta_s = 0.0001$ for $\alpha = 1$ (the gray line) $\alpha = 0.98$, (the dashed line), $\alpha = 0.95$ (the black solid line).

7 Conclusion

In this paper, we studied the behavior of vector-borne plant disease fractional order model. The influence of the fractional order derivative, which is considered as the index of memory on the vector and host is studied here (see the Figures 1-8). We argued that the fractional order models are more suitable than integer order ones in plant diseases models where memory effects are essential.

References

[1] K. Wilkinson, W. P. Grant, L. E. Green, S. Hunter, M. J. Jeger, P. Lowe, G. F. Medley, P. Mills, J. Phillipson, G. M. Poppy and J. Waage, Infectious diseases of animals and plants: an interdisciplinary approach, *Phil. Trans. R. Soc. B* **366**, 1933–1942 (2011).

[2] J. W. Pscheidt, Plant diseases Kentucky master gardener manual chapter 6, Development cooperative extension service university of Kentucky college of agriculture, Lexington, 40546.

[3] Encyclopedia.com: http://www.encyclopedia.com/topic/Plant_diseases.aspx.

[4] Y. Xiao and F. Van den Bosch, The dynamics of an eco-epidemic model with biological control, *Ecol. Model.* **168**, 203–214 (2003).

[5] L. Xue, H. M. Scott, L. W. Cohnstaedt and C. Scoglio, A network-based meta-population approach to model Rift valley fever epidemics, *J. Theor. Biol.* **306**(5), 129–144 (2012).

[6] E. Ahmed, A. M. A. El-Sayed and H. A. A. El-Saka, Equilibrium points, stability and numerical solutions of fractional-order predator prey and rabies models, *J. Math. Anal. Appl.* **325**, 542–553 (2007).

[7] P. L. Butzer and U. Westphal, An introduction to fractional calculus, Applications of fractional calculus in physics, World Scientific, Singapore, 1-85 2000.

[8] K. S. Miller and B. Ross, An introduction to the fractional calculus and fractional differential equations, John Wiley & Sons, INC 1993.

[9] V. D.Djordjević, J.Jarić, B. Fabry, J. J. Fredberg and D. Stamenović, Fractional derivatives embody essential features of cell rheological behavior, *Ann. Biomed. Eng.* **31**, 692–699 (2003).

[10] A. M. A. El-Sayed, S. Z. Rida and A. A. M. Arafa, On the solutions of time-fractional bacterial chemotaxis in a diffusion gradient chamber, *Int. J. Nonlinear Sci.* **7**, 485–492 (2009).

[11] R. Shi, H. Zhao and S. Tang, Global dynamic analysis of a vector-borne plant disease model, *Adv. Differ. Equ.* **59** (2014).

[12] T. Sardar, S. Rana, S. Bhattacharya, K. Al-Khaled and J. Chattopadhyay, A generic model for a single strain mosquito-transmitted disease with memory on the host and the vector, *Math. Biosci.* **263**, 18–36 (2015).

[13] K. Diethelm and N. J. Ford, Analysis of fractional differential equations, *J. Math. Anal. Appl.* **265**, 229–248 (2002).

[14] J. S. Muldowney, Compound matrices and ordinary differential equations, *Rocky Mount. J. Math.* **20**, 857–872 (1990).

[15] M. Safan and F. A. Rihan, Mathematical analysis of an SIS model with imperfect vaccination and backward bifurcation, *Math. Comput. Simul.* **96**, 195–206 (2014).

[16] Z. Odibat and N. T. Shawagfeh, Generalized Taylor's formula, *Appl. Math. Comput.* **186**, 286–293 (2007).

[17] W. Lin, Global existence theory and chaos control of fractional differential equations, *Aust. J. Math. Anal. Appl.* **332**, 709–726 (2007).

[18] D. Matignon, Stability results for fractional differential equations with applications to control processing, in: Computational Eng. Sys. Appl. vol. 2, Lille, France, p. 963 (1996).

[19] A. M. A. El-Sayed and M. E. Nasr, Existence of uniformly stable solutions of non-autonomous discontinuous dynamical systems, *J. Egyptian Math. Soc.* **19**, 91–94 (2011).

[20] A. M. A. El-Sayed, On the existence and stability of positive solution for a nonlinear fractional-order differential equation and some applications, *Alex. J. Math.* **1**, 1–10 (2010).

[21] Z. Odibat and S. Momani, An algorithm for the numerical solution of differential equations of fractional order, *J. Appl. Math. Inf.* **26**, 15–27 (2008).