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Backward Bifurcation in a Fractional Order Epidemiological Model

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Abstract: An epidemiological fractional order model which displays backward bifurcation for some parameters values, is studied in this paper. Because integer order of such model does not convey any information about the effect of the memory or learning mechanism of human population which influences disease transmission, we use the fractional order model in which the memory effect is considered well. As the fractional derivative is considered as the memory index, so the goal of this paper is to study the impact of fractional order derivative on the backward bifurcation phenomenon and on the basic reproduction number R_0.

Keywords: Fractional calculus, computer virus, numerical solution, predictor-corrector method.

1 Introduction

Recent outbreaks of infectious diseases threaten human health and economic activities. So, it is an essential to understand the dynamics of the infectious diseases and predict their future behaviors [1]. Mathematical modeling allows us to predict the progress of an outbreak and to quantify the uncertainty in these predictions [2,3,4,5,6,7,8,9]. Classical models often show only forward bifurcation. A key measure in such models to monitor the spread of diseases in a population is the basic reproduction number R_0 [10,11,12]. For such classical epidemic models, the infection persists if the basic reproduction number $R_0 > 1$, and dies out if $R_0 < 1$. Endemicity of disease may occur even in case of $R_0 < 1$ [10]. So, different control measures must be considered [13,14,15,12,16]. R_0 must be reduced to be below sub-threshold R_0^* such that $R_0^* < R_0 < 1$ to avoid the endemic region $[R_0^*, 1]$. The phenomenon of backward bifurcation can be distinguished in many infectious diseases models [13]. In such case, the reproduction number is not enough to predict the behavior of the epidemic [17, 18]. Backward bifurcation appears in some models for some factors, like the incomplete degree protection for vaccination models [16]. This means that the occurrence of a backward bifurcation may have important public health implications. The following model which is presented in [10] shows backward bifurcations for some certain values of parameters:

$$\begin{aligned} \frac{dS_0}{dt} &= \mu - (\mu + \beta I) S_0, \\ \frac{dI}{dt} &= \beta (S_0 + rS_1) I - (\alpha + \mu) I, \\ \frac{dS_1}{dt} &= \alpha I - (\mu + r\beta I) S_1, \end{aligned} \tag{1}$$

where

$$S_0 + I + S_1 = 1,$$

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S_0 is number of individuals who have never been infected,
I is the infectious individuals,
S_1 is the susceptible individuals with minimum one prior infection.
The authors in [10] assumed that β is the contact rate between S_0 and I, the death rate is μ and the recovery rate is α. When $r=0$, the model is considered as SIR model while $r=1$, the model can be considered as SIS model if $r=1$.
In the integer order model (1), the memory effect is ignored, so to consider the memory effect we should use the same model but with fractional order as in section two in which epidemiological model with memory is presented and also, we explain the reasons of using fractional order models. Basic reproduction number, forward bifurcation and backward bifurcation are illustrated in section three while section four is dedicated for numerical solutions and discussion.

2 Epidemiological Model with Memory

Nevertheless, fractional calculus is an old science [19,20,21,2,3,4,5,6], it has been applied in many of sophisticated applications as it has the property of memory effect which is not found in classical calculus [22,23,24]. Memory effect is plentiful in biological and epidemiological systems [23,15,25,26,27,28], so it can be considered as an excellent tool for describing memory phenomena [29]. In other words, fractional order models are called models with memory [25,26,27,28,30,31,32].
If the order approaches zero, then the system rely powerfully on its previous history but if the order approaches one, the system has a short memory reliance [29,24,1]. In [2,22], it is shown that the fractional order model results are closer to the measured collected data. Hence, we present the following model that is based on integer order model presented in [10] as follows

$$\begin{aligned} D^q(S_0) &= \mu-(\mu+\beta I)S_0,\\ D^q(I) &= \beta\,(S_0+rS_1)I-(\alpha+\mu)I,\\ D^q(S_1) &= \alpha I-(\mu+r\beta I)S_1, \end{aligned} \tag{2}$$

where $0<q\leq 1$.
Caputo fractional order derivative is used in model (2).
$D^q f(x)=J^{m-q}(D^m f(x))$, where $q\in\,]m-1,m]$, m is a natural number,
where the fractional integral operator $J^q f(x)$ is given by the singular integral operator of convolution type as:

$$J^q f(x)=\frac{1}{\Gamma(q)}\int_0^x (x-t)^{q-1} f(t)dt,\quad q>0,\ x>0,$$
$$J^0 f(x)=f(x).$$

The operator $J^q f(x)$ is called integral with memory.
Model (2) has some defect as the left side of (2) has dimension $(t)^{-q}$ where the dimension of the other side is $(t)^{-1}$. So, it is recommended to use the procedure presented in [22,33]. A true form of model (2) can be written is as follows:

$$\begin{aligned} D^q(S_0) &= \mu^q-(\mu^q+\beta^q I)S_0,\\ D^q(I) &= \beta^q(S_0+rS_1)I-(\alpha^q+\mu^q)I,\\ D^q(S_1) &= \alpha^q I-(\mu^q+r\beta^q I)S_1, \end{aligned} \tag{3}$$

The systems (2) and (3) are called systems with memory [16].

3 Basic Reproduction Number R_0 and Backward Bifurcation

R_0 is the mean number of new infections from an infected individual in a completely susceptible population. It carries information about the continuity of a disease [33]. Using the method of next generation matrix [10], basic reproduction number for (1) and (2) can be driven to be

$$R_0=\frac{\beta}{\alpha+\mu}\quad\text{and}\quad R_0^*=\frac{\beta^*}{\alpha+\mu}, \tag{4}$$

where β^* is the critical contact rate.
Similarly, for th system (3):

$$\overline{R}_0 = \frac{\beta^q}{\alpha^q + \mu^q} \text{ and } \overline{R}_0^* = \frac{(\beta^*)^q}{\alpha^q + \mu^q}. \tag{5}$$

From expression (4) it is clear that $R_0 \propto \beta$ which means that, R_0 increases linearly as successful contact rate β. Relations (5), show that $\overline{R}_0 \propto \beta^q$. $\overline{R}_0$ increases at a power law rate as β increases, which is slower than the linear rate. Therefore, the memory and learning mechanism in human population, may control the disease spread rate. If $q \to 1$, then the system has a short memory dependence while $q \to 0$ implies that, the system has a perfect memory.

3.1 Endemic equilibria and their asymptotic stability

In (1) and (2), the disease-free equilibrium $E_0 = (1,0,0)$ and endemic equilibria $E = (S_0^*,\ I^*, S_1^*)$ are such that

$$S_0^* = \frac{\mu}{\mu + \beta I^*},$$

$$I^* = \frac{1}{2}\left(\left(1 - \frac{1}{rR_0} - \frac{\mu}{(\alpha+\mu)R_0}\right) + \sqrt{\left(1 - \frac{1}{rR_0} - \frac{\mu}{(\alpha+\mu)R_0}\right)^2 + \frac{4\mu\left(1-\frac{1}{R_0}\right)}{(\alpha+\mu)rR_0}}\right),$$

$$S_1^* = \frac{\mu\, I^*}{\mu + r\beta I^*}.$$

3.2 Forward bifurcation

If $r \in]0, 1+\mu/\alpha, [$then the systems (1) and (2) has the following endemic states [31]:

1. $E_0 = (1,0,0)$ which is globally stable in case that $R_0 < 1$.
2. The endemic equilibrium $E = (S_0^*,\ I^*, S_1^*)$ which is globally asymptotically stable in case that $|\arg(\lambda)| > \frac{q\pi}{2}$.

3.3 Backward bifurcation

For $r \in]1+\mu/\alpha, \infty[$the systems (1) and (2) display a backward bifurcation, there is a particular endemic steady state or two positive steady states which depend on the roots of the (6) [10]:

$$F(\beta, I) = r\beta^2 I^2 + (r\mu - (r\beta - (\alpha+\mu)))\beta I + \mu(\alpha + \mu - \beta) = 0. \tag{6}$$

The feasible (i.e. positive, real) solution of this equation (6) as in [10] is

$$I^{*\pm} = \frac{1}{2}\left(\left(1 - \frac{1}{rR_0} - \frac{\mu}{(\alpha+\mu)R_0}\right) \pm \sqrt{\left(1 - \frac{1}{rR_0} - \frac{\mu}{(\alpha+\mu)R_0}\right)^2 + \frac{4\mu\left(1-\frac{1}{R_0}\right)}{(\alpha+\mu)rR_0}}\right).$$

And similarly, for the system (3) we get:

$$\overline{I}^{*\pm} = \frac{1}{2}\left(\left(1 - \frac{1}{r\overline{R}_0} - \frac{\mu^q}{(\alpha^q+\mu^q)\overline{R}_0}\right) \pm \sqrt{\left(1 - \frac{1}{r\overline{R}_0} - \frac{\mu^q}{(\alpha^q+\mu^q)\overline{R}_0}\right)^2 + \frac{4\mu^q\left(1-\frac{1}{\overline{R}_0}\right)}{(\alpha^q+\mu^q)r\overline{R}_0}}\right).$$

The condition for the local stability of the E_0, E is that the eigenvalues $\lambda_i$of the Jacobian matrices related to E_0, E satisfy the condition $|\arg\lambda_i| > q\frac{\pi}{2}$. This shows fractional order models are more stable or stable as their corresponding integer order.

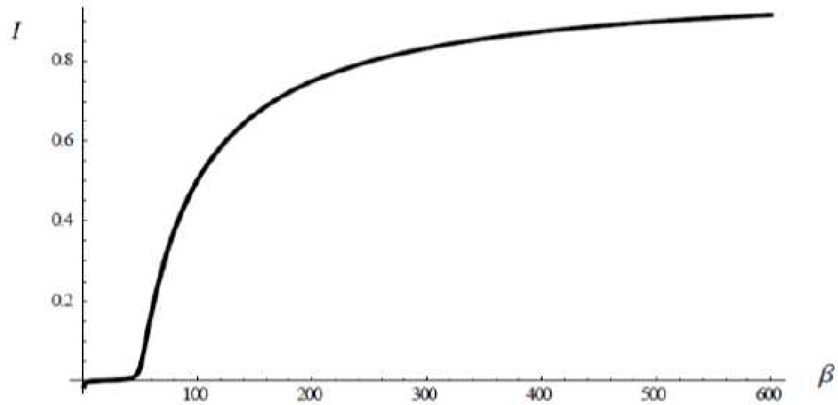

Fig. 1: Bifurcation diagram showing forward bifurcation when $r = 0.2$ and $q = 1$.

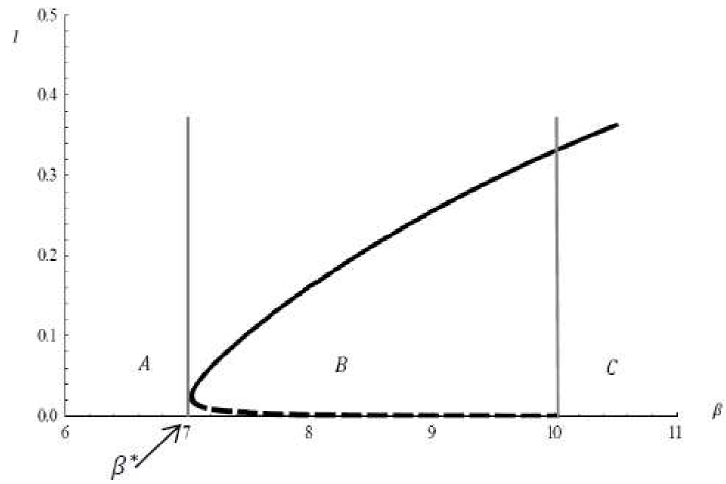

Fig. 2: Bifurcation diagram showing backward bifurcation when $r = 1.5$ and $q = 1$.

Table 1: A summary of the backward bifurcation shown in Figure 2

Region	$\boldsymbol{\beta}$	$\boldsymbol{R_0}$	**Stability of steady state**
A	$\beta < 7$	$R_0 < 0.698$	The disease-free equilibrium points are stable.
B	$7 < \beta < 10.015$	$0.698 < R_0 < 1$	The disease-free and endemic equilibrium points are stable while the other endemic equilibrium point is unstable.
C	$\beta > 10.015$	$R_0 > 1$	The disease-free equilibrium is unstable while the endemic equilibrium is stable.

4 Numerical Simulation and Discussion

We set $\alpha = 10,\ \mu = 0.015,\ \beta = 1.8$ as in [10].

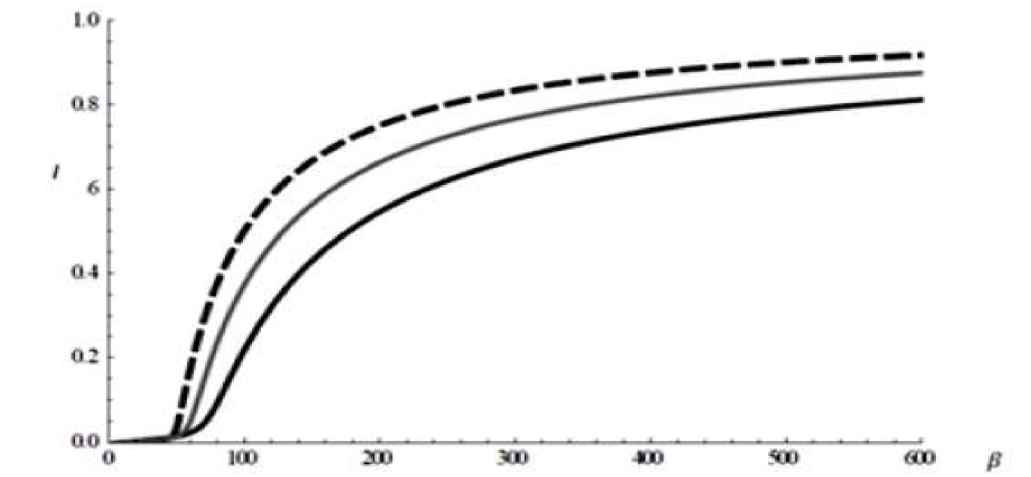

Fig. 3: Bifurcation diagram showing forward bifurcation when $r = 1.5$ for $q = 1$(dashed black line), $q = 0.9$ (gray solid line), $q = 0.8$ (solid black line).

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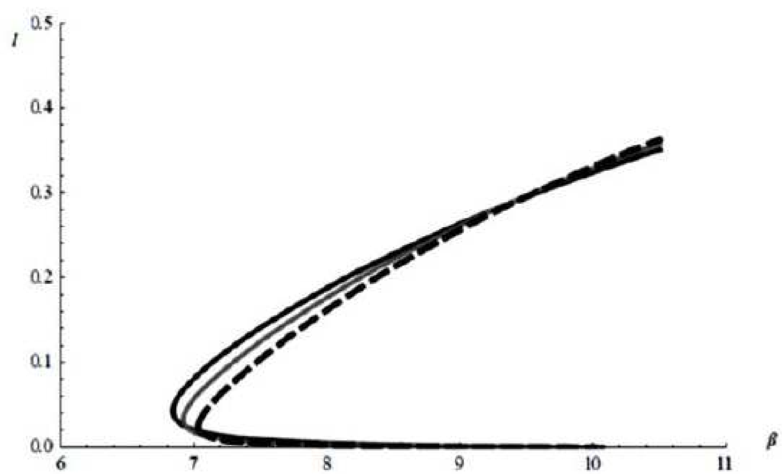

Fig. 4: Bifurcation diagram showing backward bifurcation when $r = 1.5$ for $q = 1$(dashed black line), $q = 0.9$ (gray solid line), $q = 0.8$ (solid black line).

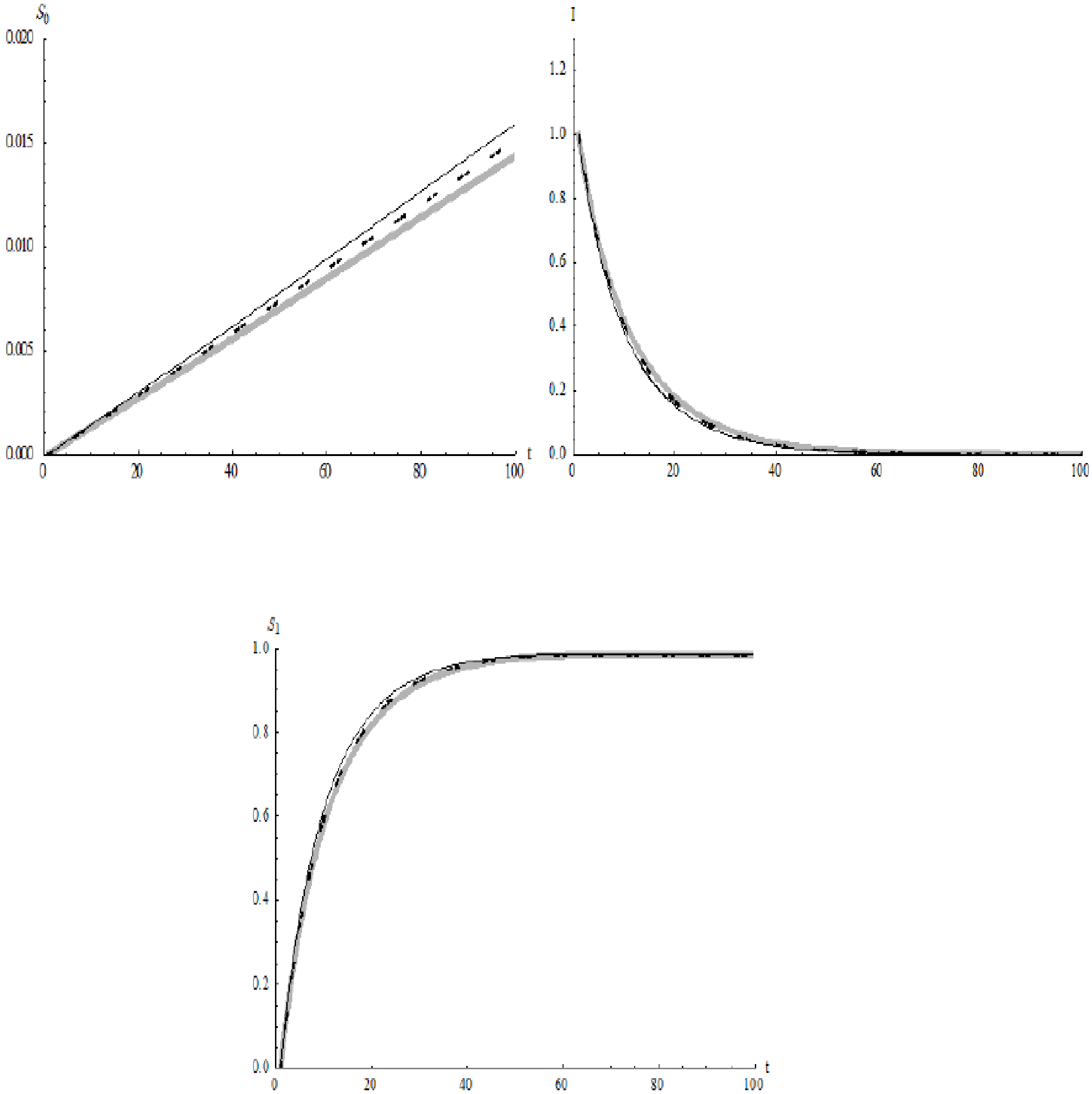

Fig. 5: The solution of system (1) for q = 1 (the gray line), q = 0.99, (the dashed line), q = 0.95 (the black solid line).

5 Conclusion

In this paper, the behaviors of backward bifurcation that arises in epidemiological models with memory has been studied. Fractional order derivative has been proved to be an attractive tool for describing memory phenomena in biology and epidemiology. The impact of backward bifurcation on a fractional order epidemiological model has been studied here. The existence of a backward bifurcation has essential effects that helps to co eradicate infectious diseases. If the disease becomes endemic, so to control the disease, we have to decrease R_0 to enter the region where there is no endemic steady state. The effect of the index of memory which is considered as the physical meaning of the fractional order derivative is discussed here. Memory effect on bifurcation diagrams are studied as shown in Figure 3 and Figure 4. If $q \to 1$, then the system has a short memory dependence while $q \to 0$ implies that, the system depends on the previous memory (see Figure 3 and Figure 4). We argue that fractional order models are suitable to describe behavior of infectious diseases where memory is important to be considered.

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