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Iterative methods for solving nonlinear equations by using quadratic spline function

Rostam K. Saeed and Kawa M. Aziz

University of Salahaddin/Erbil - College of Science – Department of Mathematics-Erbil-Iraq.
Email: rostamkarim64@uni-sci.org

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Abstract: In this paper, iterative methods of order three and four constructed based on quadratic spline function for solving nonlinear equations. Several numerical examples are given to illustrate the efficiency and performance of the iterative methods; the methods are also compared with some other known iterative methods.

Keywords: nonlinear equations, Order of Convergence, interpolation polynomial.

1 Introduction

Solving nonlinear equations is one of the most important problems in numerical analysis. To solve nonlinear equations, iterative methods such as Newton's method are usually used. Newton's method for computing a simple zero α of a nonlinear equation $f(x)=0$ has been modified in a different of ways [1-10]. Taghvafard [11], proposed two iterative methods both of order three by sing cubic spline function.

In this paper, by suing quadratic spline function, third and fourth order convergence iteration formulas proposed, which can be used as an alternative to Newton method or in cases where Newton method is not successful. Newton's well-known method to find α iteratively is defined as follows:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{1.1}$$

Equation (1) converges quadratically in some neighborhood of α . [7]

Some modifications of Newton's method have been developed in [8], by considering different quadrature formulas for the computation of the integral arising from Newton's theorem:

$$f(x) = f(x_n) + \int_{x_n}^{x} f'(t)\,dt. \tag{1.2}$$

2 Iterative methods and convergence analysis

In this section, we derive iterative methods as follows:

Let x_n is the initial approximation to the exact root α . We use (1.1) to find another approximation y_n to α :

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{2.1}$$

Now, approximate $f(x)$ by using quadratic spline function on two points x_n and y_n as follows:

$$f(x) \approx \frac{m_{n+1} - m_n}{2(y_n - x_n)}(x - x_n)^2 + (x - x_n)m_n + f(x_n), \tag{2.2}$$

where

$$m_{n+1} = -m_n + 2\frac{f(y_n) - f(x_n)}{y_n - x_n}, \; m_n \text{ is arbitrary and } n \ge 1. \tag{2.3}$$

By using Taylor series expansion for $f(y_n)$ about x_n of order two we write:

$$f(y_n) \approx f(x_n) + (y_n - x_n) f'(x_n) + \frac{(y_n - x_n)^2}{2} f''(x_n). \tag{2.4}$$

Differentiate (2.2) with respect to *x*, replace *x* by *t*, then substitute it in (1.2) we get:

$$f(x) = f(x_n) + \int_{x_n}^{x} (m_n + \frac{(m_n - m_{n+1})(t - x_n)}{x_n - y_n}) dt. \tag{2.5}$$

Suppose x_{n+1} is another approximation to α, we put

$$f(x_{n+1}) \approx 0, \tag{2.6}$$

and since m_n is arbitrary, we let

$$m_n = f'(x_n). \tag{2.7}$$

In equation (2.5) replace x by x_{n+1}, compute the integral and using equations (1.1), (2.3), (2.4), (2.6) and (2.7) yields:

$$\frac{f''(x_n)}{2} x_{n+1}^2 + (f'(x_n) - f''(x_n)\, x_n) x_{n+1} + \frac{f''(x_n) x_n^2}{2} - f'(x_n) x_n + f(x_n) = 0. \tag{2.8}$$

Then,

$$x_{n+1} = x_n - \frac{f'(x_n) - f'(x_n)\sqrt{1 - \frac{2 f''(x_n) f(x_n)}{f'(x_n)^2}}}{f''(x_n)}, \tag{2.9}$$

where

$$\sqrt{1 - \frac{2 f''(x_n) f(x_n)}{f'(x_n)^2}} \approx -\frac{2 f'(x_n)^4 f''(x_n) f(x_n) - 2 f'(x_n)^6 + f'(x_n)^2 f''(x_n)^2 f(x_n)^2 + f''(x_n)^3 f(x_n)^3}{2 f'(x_n)^6}.$$

From equation (2.9) we suggest the following one step iteration method of third order for solving nonlinear equation $f(x) = 0$.

Algorithm 1:

INPUT initial approximation x_0; tolerance ε; maximum number of iterations N.

OUTPUT approximate solution x_{n+1} or message of failure.

Step 1: Set *n*=0 and $i = 1$.

Step 2: While $i \le N$ do steps 3-5.

Step 3: calculate

$$x_{n+1} = x_n - \frac{2 f'(x_n)^4 f(x_n) + f'(x_n)^2 f''(x_n) f(x_n)^2 + f''(x_n)^2 f(x_n)^3}{2 f'(x_n)^5}, \tag{2.10}$$

Step 4: If $|x_{n+1} - x_n| < \varepsilon$; then OUTPUT ($x_{n+1}$); stop.

Step 5: Set $n = n+1$, $i = i+1$ and go to Step 2.

Step 6: OUTPUT ('Method failed after N iterations, $N =$' N); stop.

We derive another new third order iterative method as follows: In equation (1.2) replace x by x_{n+1}, and evaluate the integral using (2.3), (2.4), (2.6) and choose

$$m_n = \frac{f(y_n) - f(x_n)}{y_n - x_n}, \tag{2.11}$$

and

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f(x_n)^2 f''(x_n)}{2 f'(x_n)^3}, \text{ (named Chebyshev method [9])} \tag{2.12}$$

we get the linear equation with respect to x_{n+1}

$$f(x_n) - f'(x_n) x_n + f'(x_n) x_{n+1} + \frac{f''(x_n) f(x_n) x_n}{2 f'(x_n)} - \frac{f''(x_n) f(x_n) x_{n+1}}{2 f'(x_n)} + \frac{f''(x_n)^2 f(x_n)^2 x_n}{4 f'(x_n)^3}$$

$$- \frac{f''(x_n)^2 f(x_n)^2 x_{n+1}}{4 f'(x_n)^3} = 0 \,. \tag{2.13}$$

Now, from equation (2.13) we suggest another one step iteration method of third order for solving nonlinear equation $f(x) = 0.$

Algorithm 2:

INPUT initial approximation x_0; tolerance ε; maximum number of iterations N.

OUTPUT approximate solution x_{n+1} or message of failure.

Step 1: Set n=0 and $i = 1$.

Step 2: While $i \le N$ do steps 3-5.

Step 3: calculate

$$x_{n+1} = x_n + \frac{4 f'(x_n)^3 f(x_n)}{2 f'(x_n)^2 f''(x_n) f(x_n) - 4 f'(x_n)^4 + f''(x_n)^2 f(x_n)^2}, \tag{2.14}$$

Step 4: If $|x_{n+1} - x_n| < \varepsilon$; then OUTPUT (x_{n+1}); stop.

Step 5: Set $n = n+1,\ i = i+1$ and go to Step 2.

Step 6: OUTPUT ('Method failed after N iterations, $N =$' N); stop.

We derive two new fourth-order iterative methods as follows: Approximate $f(y_n)$ by using Taylor series expansion about x_n of order three we get:

$$f(y_n) \approx f(x_n) + (y_n - x_n) f'(x_n) + \frac{(y_n - x_n)^2}{2} f''(x_n) + \frac{(y_n - x_n)^3}{6} f'''(x_n). \tag{2.15}$$

Repeat all iterations for deriving equation (2.9) but only replace equation (2.4) by (2.15) we get:

$$x_{n+1} = x_n + \frac{3 f'(x_n)^2 - \sqrt{9 f'(x_n)^4 - 18 f'(x_n)^2\, f''(x_n)\, f(x_n) + 6 f'''(x_n)\, f'(x_n)\, f(x_n)^2}}{f'''(x_n)\, f(x_n) - 3 f'(x_n) f''(x_n)}. \tag{2.16}$$

Using Taylor series expansion, we get

$$\sqrt{9 f'(x_n)^4 - 18 f'(x_n)^2\, f''(x_n)\, f(x_n) + 6 f'''(x_n)\, f'(x_n)\, f(x_n)^2} \approx 3 f'(x_n)^2 - 3 f''(x_n)\, f(x_n)$$

$$+ \frac{f'''(x_n) f(x_n)^2}{f'(x_n)} - \frac{3 f''(x_n)^2 f(x_n)^2}{2 f'(x_n)^2} + \frac{f'''(x_n) f''(x_n) f(x_n)^3}{f'(x_n)^3} - \frac{3 f''(x_n)^3 f(x_n)^3}{2 f'(x_n)^4}. \tag{2.17}$$

Now, by substitute (2.17) in (2.16) we suggest one step iteration method of fourth-order for solving nonlinear equation $f(x) = 0.$

Algorithm 3:

INPUT initial approximation x_0; tolerance ε; maximum number of iterations N.

OUTPUT approximate solution x_{n+1} or message of failure.

Step 1: Set n=0 and $i=1$.

Step 2: While $i \le N$ do steps 3-5.

Step 3: calculate

$$x_{n+1}=x_n-\frac{3f''(x_n)^3 f(x_n)^3-3(f'(x_n)f''(x_n)f(x_n))^2}{6f'(x_n)^5 f''(x_n)-2f'(x_n)^4 f'''(x_n)f(x_n)}-\frac{2f'(x_n)^3 f(x_n)^2+2f''(x_n)f'(x_n)f(x_n)^3}{2f'(x_n)^4 f(x_n)}. \quad (2.18)$$

Step 4: If $|x_{n+1}-x_n|<\varepsilon$; then OUTPUT (x_{n+1}); stop.

Step 5: Set $n=n+1,\ i=i+1$ and go to Step 2.

Step 6: OUTPUT ('Method failed after N iterations, $N=$' N); stop.

To derive another fourth-order method: Approximate $f(x)$ by using Hermite interpolation polynomial [7] on the two points x_n and y_n, we get:

$$f(x)\approx H(x)=f(x_n)hx_n(x)+f(y_n)hy_n(x)+f'(x_n)Rx_n(x)+f'(y_n)Ry_n(x), \quad (2.19)$$

where

$$w(x)=(x-x_n)(x-y_n),$$

$$hx_n(x)=\left[1-\frac{w''(x_n)}{w'(x_n)}(x-x_n)\right]\left(\frac{x-y_n}{x_n-y_n}\right)^2,$$

$$hy_n(x)=\left[1-\frac{w''(y_n)}{w'(y_n)}(x-y_n)\right]\left(\frac{x-x_n}{y_n-x_n}\right)^2$$

$$Rx_n(x)=(x-x_n)\left(\frac{x-y_n}{x_n-y_n}\right)^2,$$

and

$$Ry_n(x)=(x-y_n)\left(\frac{x-x_n}{y_n-x_n}\right)^2. \quad (2.20)$$

Approximate $f'(y_n)$ by sung Taylor expansion of the first order about x_n we get:

$$f'(y_n)\approx f'(x_n)+(y_n-x_n)f''(x_n). \quad (2.21)$$

Now, substitute (2.4) and (2.21) in equation (2.19), yields:

$$f(x)\approx f(x_n)(1+hx_n(x))+f(y_n)hy_n(x)+f'(x_n)(y_n-x_n+Rx_n(x)+Ry_n(x))+$$
$$(\frac{(y_n-x_n)^2}{2}+(y_n-x_n)Ry_n(x))f''(x_n).$$

In equation (1.2), replace x by x_{n+1}, x_n by y_n, and find the integral. Then, use the above equation to obtain:

$$f(y_n)-\frac{(y_n-x_{n+1})(2f'(x_n)-2f''(x_n)x_n+f''(x_n)y_n+f''(x_n)\,x_{n+1}}{2}=0. \quad (2.22)$$

Now, from equation (2.22) we suggest one step iteration method of fourth-order for solving nonlinear equation $f(x)=0$.

Algorithm 4:

INPUT initial approximation x_0; tolerance ε; maximum number of iterations N.

OUTPUT approximate solution x_{n+1} or message of failure.

Step 1: Set n=0 and $i=1$.

Step 2: While $i \le N$ do steps 3-5.

Step 3: calculate

$$x_{n+1} = x_n + \frac{\sqrt{9\,f'(x_n)^4 - 18\,f'(x_n)^2\,f''(x_n)\,f(x_n) + \dfrac{3f''(x_n)\,f'''(x_n)\,f(x_n)^3}{f'(x_n)}}}{3f'(x_n)\,f''(x_n)} - \frac{f'(x_n)}{f''(x_n)}. \tag{2.23}$$

Step 4: If $|x_{n+1}-x_n|<\varepsilon$; then OUTPUT (x_{n+1}); stop.

Step 5: Set $n=n+1,\ i=i+1$ and go to Step 2.

Step 6: OUTPUT ('Method failed after N iterations, N =' N); stop.

We consider the convergence analysis proposed in this paper by the following theorem.

Theorem 1: Let $\alpha \in I$ be a simple zero of sufficiently differentiable function $f: I \to R$ for an open interval I. If x_0 is sufficiently close to α, then the methods defined by (2.10) and (2.14) are of third-order, and satisfies the error equation

$$e_{n+1} = -c_3\,e_n^3 + O(e_n^4), \tag{2.24}$$

furthermore, the methods defined by (2.18) and (2.23) are of fourth order, and satisfies the error equations

$$e_{n+1} = \left(\frac{2c_2^4 - 7c_2^2c_3 + c_4c_2 + c_3^2}{c_2}\right)e_n^4 + O(e_n^5), \tag{2.25}$$

and

$$e_{n+1} = (c_4 - 3c_2c_3)\,e_n^4 + O(e_n^5), \tag{2.26}$$

where $e_n = x_n - \alpha$ and $c_k = f^{(k)}(\alpha)/k!\,f'(\alpha)$.

Proof: Let α be a simple zero of f. Since f is sufficiently differentiable, by expanding $f(x_n)$ about α, we get:

$$f(x_n) = f(\alpha) + (x_n-\alpha)\,f'(x_n) + \frac{(x_n-\alpha)^2}{2!}\,f''(x_n) + \frac{(x_n-\alpha)^3}{3!}\,f'''(x_n)\cdots. \tag{2.27}$$

Then

$$f(x_n) = f'(\alpha)[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + \cdots]. \tag{2.28}$$

Differentiating (2.28) three times with respect to x_n we get:

$$f'(x_n) = f'(\alpha)[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + 6c_6e_n^5 + \cdots], \tag{2.29}$$

$$f''(x_n) = f'(\alpha)[2c_2 + 6c_3e_n + 12c_4e_n^2 + 20c_5e_n^3 + 30c_6e_n^4 + 42c_7e_n^5 + \cdots], \tag{2.30}$$

$$f'''(x_n) = f'(r)[6c_3 + 24c_4e_n + 60c_5e_n^2 + 120c_6e_n^3 + 210c_7e_n^4 + 336c_8e_n^5 + \cdots]. \tag{2.31}$$

Using (2.28)-(2.31) in (2.10) and (2.14) respectively, we get the following error equations

$$e_{n+1} = -c_3\,e_n^3 + (5c_2^3 - 3c_4)e_n^4$$

and

$$e_{n+1} = -c_3\,e_n^3 + (2c_2^3 - 3c_4)e_n^4.$$

On the other hand, using (2.28)-(2.31) in (2.18) and (2.23) respectively, we get:

$$e_{n+1} = \left(\frac{2c_2^4 - 7c_2^2c_3 + c_4c_2 + c_3^2}{c_2}\right)e_n^4 + \left(-\frac{2(18c_2^6 - 40c_2^4c_3 + 14c_4c_2^3 + 13c_2^2c_3^2 - 2c_5c_2^2 - 4c_4c_2c_3 + c_3^3}{c_2^2}\right)e_n^5,$$

and

$$e_{n+1} = (c_4 - 3c_2c_3)e_n^4 + (9c_2^2c_3 - 12c_4c_2 - 6_3^2 + 4c_5)e_n^5.$$

This means the methods defined by (2.10) and (2.14) are at least cubically convergent, and the methods defined by (2.18) and (2.23) are at least fourth-order convergence.■

3 Numerical examples

We now present some examples to illustrate the efficiency of our developed methods in this paper which are given by the Algorithms 1-4. We compare the Newton's (NM) method, Golbabai and Javidi method (GJM [4]), Abbasbandy methods (AM [1]), Noor and Noor method (NNM[9]) Chun and Ham method (CYM[2]), Javidi method Algorithm2.2 (JM [6]) and Saeed and Aziz method (SAM [10]). All computations are carried out with double arithmetic precision. Displayed in Table 1 are the number of iterations (*N*) required such that $|x_{N+1} - x_N| < \varepsilon$ and $|f(x_{N+1})| < \varepsilon$ where $\varepsilon = 10^{-15}$.

We use the following functions, some of which are the same as in [1, 2, 4, 6, 9, 10]

$f_1(x) = x^3 + 4x^2 + 8x + 8$, $\quad f_2(x) = x^2 - e^x - 3x + 2 = 0$

$f_3(x) = x^2 - (1-x)^5 = 0$, $\quad f_4(x) = \sin^2(x) - x^2 + 1 = 0$

$f_5(x) = e^x - 3x^2 = 0$, $\quad f_6(x) = e^x + 2^{-x} + 2\cos x - 6$

Table 1: Comparisons between the methods depending on the number of iterations (*N*)

$f(x)$	*N* (*Number of iterations*)										
	x_0	**NM**	**GJM**	**AM**	**NNM**	**CYM**	**JM**	**SAM**	**Alg.1**	**Alg.2**	**Alg.3**	**Alg.4**
f_1	-1	3	5	4	6	div.	6	1	5	5	18	3
	1	7	4	5	5	div.	7	4	7	5	5	7
f_2	0	4	3	3	4	div.	4	2	4	3	3	3
	-1	5	3	3	5	div.	5	3	5	3	3	4
f_3	0.2	5	3	3	5	3	5	3	5	4	2	4
	1	6	4	4	6	3	5	3	5	5	3	6
f_4	-1	6	3	4	7	3	11	4	7	5	4	4
	2	5	3	3	5	3	4	3	6	3	3	4
f_5	2	5	4	4	5	3	5	3	6	4	4	4
	0.5	6	3	5	7	3	6	4	7	5	6	4
f_6	1.5	5	3	4	6	div.	7	3	7	4	2	4
	1	8	4	15	18	div.	div.	div.	74	div.	7	div.

4 Conclusion

We can conclude that the new presented algorithms in this paper perform in most cases better than the methods which we have taken for comparison depending on the number of iterations.

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