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Three-Qubit Quantum Entanglement Robustness Coupling with Kerr-like Medium

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Abstract: This study is oriented to discover new behaviour for quantum system under condition of three-qubit quantum entanglement coupling with Kerr-like medium. Jaynes-Cummings model will be used to represent quantum entanglement coupling with Kerr-like medium. In achieving these objectives, Jaynes-Cummings model will be modified to include three-qubit quantum system coupling with Kerr-like medium for one-qubit transition. The three-qubit state will be interacting with Markovian and non-Markovian environment which represented by Lorenztian spectral density. From the defined three-qubit Jaynes-Cummings model, lower bound concurrence is used to measure quantum entanglement robustness. Based on the analysis quantum entanglement is more robust when Kerr-like medium coupling strength increased for both Markovian environment and non-Markovian environment even though the influence of quantum entanglement robustness is reduce when dipole-dipole interaction is getting stronger.

Keywords: Jaynes-Cummings model, three-qubit quantum state, Lower bound concurrence, Kerr-like medium

1 Introduction

Quantum entanglement play an important role in quantum information processing where robust quantum entanglement will lead to stable quantum information processing. Quantum entanglement will act differently when interact with environment and there is chance where decoherence will occur. Study of quantum entanglement will provide an understanding towards how to achieve a robust quantum entanglement under certain environment condition.

One of the influence environment is Kerr-like medium which also called non-linear cavity field. Kerr-like medium has provided a lot of features of quantum system such as formation of Schrodinger cats which explain the superposition of quantum system and squeezing which able to improve the measurement of quantum behavior [11].

Study of quantum entanglement had conducted in different way such as three-qubit quantum state entanglement behavior [3], decoherence of quantum system , quantum entanglement in quantum spin environment [15] and multi-partite entanglement under stochastic local operations and classical operator [10]. Study of three-qubit quantum state become interesting as three-qubit quantum state is the start towards multi-partite entanglement [4]. Three-qubit quantum entanglement also able to show complex entanglement structure and more robust compare to two-qubit quantum entanglement [1]. Hence, this study focus on three-qubit quantum entanglement coupling with Kerr-like medium which is not being explored yet. This study will observe the quantum entanglement pattern which different Kerr-like medium coupling strength with combination of different environment factor.

Jaynes-Cummings model (JCM) will be used to represent the quantum system in this study due to its simplicity and solvable. JCM will be interacting with leaky multimode cavity field which used to represent Markovian and non-Markovian environment coupling with Kerr-like medium [3]. Single photon transition will only be considered in this study.

This study will start with development of JCM model with include Kerr-like medium and three-qubit quantum state. Next, section is observation of quantum entanglement pattern and then end with conclusion.

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2 Three-qubit Jaynes-Cummings Model

Model will be developed considering the interaction between the atoms for the case of a three-qubit W-state. In this case focus is on the quantum entanglement behavior with coupling of a Kerr-like medium, so the Jaynes-Cummings model will have only a single photon transition number. This section discusses the development of a three-qubit Jaynes-Cummings model coupling with a Kerr-like medium where time dependent coefficient will also be solved. The Hamiltonian rotating wave approximation of the total quantum system is shown in equation (1)-(4).

$$H = H_0 + H_1 + H_2 \tag{1}$$

$$H_0 = \sum_{n=1}^{3} \omega_0 \sigma_n^+ \sigma_n^- + \sum_j \omega_j a_j^\dagger a_j \tag{2}$$

$$H_1 = \sum_{n=1}^{3} \sum_j \alpha_n (g_j \sigma_n^+ a_j + g_j^* \sigma_n^- a_j^\dagger) + \sum_j \chi_j a_j^{\dagger 2} a_j^2 \tag{3}$$

$$H_2 = D_1(\sigma_2^+ \sigma_3^- + \sigma_3^+ \sigma_2^-) + D_2(\sigma_3^+ \sigma_1^- + \sigma_1^+ \sigma_3^-) + D_3(\sigma_1^+ \sigma_2^- + \sigma_2^+ \sigma_1^-) \tag{4}$$

j : Mode of photon.

ω_j : Cavity frequency for cavity mode of j photon.

For the interaction Hamiltonian between the atom and the cavity field, H_1 is the parameter introduced to individualize the atom [3]. This parameter is the dimensionless real constant,α_n . The coupling strength between the qubit and the cavity field is represented by $\alpha_n|g_j|$. D_l with $l \in \{1,2,3\}$ is introduced to represent the dipole-dipole interaction between the qubit with $D_l = [d.d - 3(d.r_{mn})(d.r_{mn})/r_{mn}^2]/r_{mn}^3$. The qubits electric dipole moment is represented by d and r_{mn} represents a two qubit separation [3]. This study includes Kerr-like medium coupling strength represented by $\sum_j \chi_j a_j^{\dagger 2} a_j^2$ term in equation (3).

The initial state of the W state is shown in equation (5) for an empty cavity field, that is, $|\bar{0}\rangle$.$a_l(0)$ with $l = 1,2,3$ being the time-dependent coefficients. A three-qubit state is represented by a W state generalization as a W state is more robust than a GHZ state. The W state is able to maintain a bipartite entanglement after one of the atoms is being traced out.

$$|W(0)\rangle = a_1(0)|100\rangle_{123} + a_2(0)|010\rangle_{123} + a_3(0)|001\rangle_{123} \qquad \sum_{n=1}^{3} |a_n(0)|^2 = 1 \tag{5}$$

The total quantum system when the atom interacts with the cavity field,

$$|\psi(0)\rangle = |W(0)\rangle|\bar{0}\rangle \tag{6}$$

$$|\psi(t)\rangle = e^{-it\omega_0}[a_1(t)|100\rangle_{123} + a_2(t)|010\rangle_{123} + a_3(t)|001\rangle_{123}]|\bar{0}\rangle + \sum_j b_j(t) e^{-it(\omega_j+\chi_j)}|000\rangle_{123}|1_j\rangle_c \tag{7}$$

where $|\bar{0}\rangle$ is the cavity field state with zero photon on mode j and $|1_j\rangle_c$ is the cavity field mode when for one photon with mode j.

Equation (6) is the initial total quantum system and equation (7) is the state of the total system when $t > 0$. In equation (7) a Kerr-like medium is included only in the second term $\sum_j b_j(t) e^{-it(\omega_j+\chi_j)}|000\rangle_{123}|1_j\rangle_c$ which is the development done in this study. Initially when the cavity was zero, there will not be any Kerr-like medium until there is one photon in the cavity. Hence, the first term of equation (7) shows that the W state is acting on the transition frequency. This also means that only a single photon transition will be considered in this study.

Using Schrodinger equation of motion, $b_j(t).\frac{id|\psi(t)\rangle}{dt} = (H_1+H_2)|\psi(t)\rangle$ to derive the time depending coefficient $a_l(t)$ .From equation (7),the Schrodinger equation for the respective time coefficient is equation (8), (9), and (10).

$$i\frac{da_1(t)}{dt} = \alpha_1 \sum_j e^{-i(\omega_j-\omega_0+\chi_j)t} g_j b_j(t) + D_3 a_2(t) + D_2 a_3(t) \tag{8}$$

$$i\frac{da_2(t)}{dt} = \alpha_2 \sum_j e^{-i(\omega_j-\omega_0+\chi_j)t} g_j b_j(t) + D_3 a_1(t) + D_1 a_3(t) \tag{9}$$

$$i\frac{da_3(t)}{dt} = \alpha_3 \sum_j e^{-i(\omega_j-\omega_0+\chi_j)t} g_j b_j(t) + D_2 a_1(t) + D_1 a_2(t) \tag{10}$$

When t becomes large,$e^{-i\omega_j t}$ the term $\sum_j \chi_j b_j(t) e^{-i\omega_j t}$ becomes smaller. The value of this term asymptotically goes to zero. Hence, equation (11) will become zero which will be substituted into equation (11),

$$\sum_j \chi_j b_j(t) e^{-i\omega_j t} \tag{11}$$

$$i\frac{db_j(t)}{dt} = e^{-i(\omega_j-\omega_0+\chi)}\alpha_n g_j^* \sum_{n=1}^{3} a_n(t) + \sum_j \chi_j b_j(t) e^{-i\chi t} \tag{12}$$

Integrating $b_j(t)$ in equation (12) with the initial condition of $b_j(t) = 0$ will produce,

$$b_j(t) = -i\int_0^t e^{-i(\omega_0-\omega_j-\chi)t'} g_j^* \sum_{n=1}^{3} \alpha_n a_n(t) dt \tag{13}$$

Substituting equation (13) into equation (8) - (10) will become equation as showed in (14) - (16) where $J(\omega)$ is the spectral density of a cavity structure.

$$\frac{da_1(t)}{dt} = -\alpha_1 \int_0^t dt' [\sum_{n-1}^{3} \alpha_n a_n(t) \int d\omega J(\omega) e^{-i[(\omega_j-\omega_0)(t-t')+\chi(t-t')]}] - iD_3 a_2(t) - iD_2 a_3(t) \tag{14}$$

$$\frac{da_2(t)}{dt} = -\alpha_2 \int_0^t dt' [\sum_{n-1}^{3} \alpha_n a_n(t) \int d\omega J(\omega) e^{-i[(\omega_j-\omega_0)(t-t')+\chi(t-t')]}] - iD_3 a_1(t) - iD_1 a_3(t) \tag{15}$$

$$\frac{da_3(t)}{dt} = -\alpha_3 \int_0^t dt' [\sum_{n-1}^{3} \alpha_n a_n(t) \int d\omega J(\omega) e^{-i[(\omega_j-\omega_0)(t-t')+\chi(t-t')]}] - iD_2 a_1(t) - iD_1 a_2(t) \tag{16}$$

From the spectral density of cavity field which used to represent the photon leak to the environment shown in equation (17). This equation is called Lorentzian broadening [13].

$$J(\omega) = \frac{R^2}{\pi} \frac{\Gamma}{(\omega-\omega_c)^2+\Gamma^2} \tag{17}$$

A cavity supported mode is represented by ω_0 and Γ is the half width at half height of the field spectrum profile inside the cavity. Besides, R is the atom-cavity coupling strength and T_c is the cavity correlation time with $T_c = \Gamma^{-1}$. Other than cavity correlation time, the qubit relaxation time will be $T_q = (2R\sqrt{\alpha_1^2+\alpha_2^2+\alpha_3^2})^{-1}$. The cavity correlation time and the qubit relaxation time are used to determine whether the environment is a Markovian or non-Markorian, which represents a weak and strong environment, respectively. A Markovian mean is when $T_c < T_q$ while a non-Markovian is when $T_c > T_q$ [5].

Next, based on equations (14)-(16) the Laplace transformation is taken from both sides of the equations to produce a set of equations for $\{\widetilde{a}_n(z); n = 1,2,3\}$. The Laplace transformation will transform the equations into a complex argument of z where z represents a pole in Lorenztian broadening. The notation is $\widetilde{f}(z) = L[f(\tau)] = \int_0^\infty f(\tau)e^{-z\tau}d\tau$. The set of equation is as follows:

$$\widetilde{a}_n(z) = \frac{\sum_{m=1}^{3} A_{nm}(z) a_m(0)}{\sum_{m=0}^{4} B_m z^m} \tag{18}$$

where $l,m,n \in \{1,2,3\}$ with condition $l \neq m \neq n \neq l$

$$A_{nn}(z) = (z^2+d_n^2)[z+1+i(\delta-K)] + \frac{G^2}{4}[(1-r_n^2)z - 2id_n r_l r_m] \tag{19}$$

$$A_{nm}(z) = A_{mn}(z) = -[z+1+i(\delta-K)](d_m d_n + izd_l) + \frac{iG^2}{4}[d_m r_m + d_n r_n - d_l r_l - ir_m r_n z] \tag{20}$$

and

$$B_0 = \frac{G^2}{4}[d_1^2 r_1^2 + d_2^2 r_2^2 + d_3^2 r_3^2 \; 2(d_1 d_2 r_1 r_2 + d_1 d_3 r_1 r_3 + d_2 d_3 r_2 r_3)] - 2id_1 d_2 d_3[1+i(\delta-K)] \tag{21}$$

$$B_1 = -\frac{G^2}{4}(d_1 r_1 r_2 + d_2 r_3 r_1 + d_3 r_2 r_1) + (d_1^2+d_2^2+d_3^2)[1+i(\delta-K)]2id_1 d_2 d_3 \tag{22}$$

$$B_2 = -\frac{G_2}{4} + d_1^2 + d_2^2 + d_3^2, B_3 = 1+i(\delta-K), B_4 = 1 \tag{23}$$

For convenience, dimensionless quantities are introduced based on above equation,

$$d_l = \frac{D_l}{\Gamma}, \delta = \frac{\omega_c-\omega_0}{\Gamma}, \beta = \sqrt{\sum_{n=1}^{3}\alpha_n^2}, r_n = \frac{\alpha_n}{\alpha}, G = \frac{2R\alpha}{\Gamma}, K = \frac{\chi}{\Gamma} \tag{24}$$

by definition $\sum_{n=1}^{3} r_n^2 = 1$. Applying the inverse Laplace transform will get the time-dependent coefficient.

$$a_n(\tau) = \sum_j (z-z_j)\widetilde{a}_n(z)e^{z_j\tau} \tag{25}$$

$\tau = \Gamma t$ and z_j are a pole of $\widetilde{a}_n(z)$. Using the residue theory let $\sum_{m=0}^{4} B_m z^m = 0$, which will produce a value of z_j which is needed for the measurement of the quantum entanglement. A single-photon collective normalized state of the cavity field is represented as

$$|\bar{1}\rangle = \frac{e^{i\omega_c\chi t}}{b(\tau)} \sum_j b_j(\tau) e^{-i\omega_j\chi t}|1_j\rangle_c, \tag{26}$$

for

$$b(\tau) = \sqrt{1-\sum_{n=1}^{3}|a_n(\tau)|^2} \tag{27}$$

The total state of the system developed in this study after obtaining the time-dependent coefficient will be

$$|\psi(\tau)\rangle = e^{-i(\bar{\omega}_0+K)\tau}[a_1(\tau)|100\rangle_{123} + a_2(\tau)|010\rangle_{123} + a_2(\tau)|001\rangle_{123}]|\bar{0}\rangle + e^{-i(\bar{\omega}_c+K)\tau}b(\tau)|000\rangle_{123}|\bar{1}\rangle \tag{28}$$

with$\bar{\omega}_0 = \frac{\omega_0}{\Gamma}$ and$\bar{\omega}_c = \frac{\omega_c}{\Gamma}$.

After the development of a three-qubit quantum system model with coupling to a Kerr-like medium, measurement of quantum entanglement robustness will be conducted. Initially the quantum system was in a separable state between the qubit, $|W(0)\rangle$ and the empty field. The coefficient,$a_n(0)$ of the qubit is a non-zero value, which means that initially the entanglement already existed. Then, over time the qubit and the cavity field will be interacting with each other which will change the entanglement properties. This change is also transforming the quantum state from the pure to the mixed state [3]. In measuring the entanglement and its changes, a lower bound concurrence (LBC) of a three-qubit quantum state is used. In this case the concurrence will be included to the mixed state by using the convex roof construction,

$$C_3 = min_{[p_j,\psi_j]} \sum_j p_j C_3(|\psi_j\rangle) \tag{29}$$

where $|\psi_j\rangle$ is the quantum state of the qubit which includes for the mixed state and p_j is the probability of the quantum state with $p_j > 0$ and $\sum_j p_j = 1$. In this study the concurrence used is the lower bound concurrence which is on linear entropy. A lower bound concurrence (LBC) uses a bipartite entanglement to calculate the overall entanglement. The expression for the LBC which will be used in this study is

$$\underline{C}_3(\rho(\tau)) = \sqrt{\frac{8}{3}\sum_{m,n;m<n}^{3} |a_m(\tau)a_n(\tau)|^2} \tag{30}$$

3 Observation of Quantum Entanglement Robustness

A quantum entanglement robustness is discussed in this section. This section is divided into two areas: the first one will be the study of a tripartite entanglement for negative detuning frequency, and the second one is the study of tripartite entanglement for positive detuning freqency. Different values in the dipole-dipole interaction and Kerr-like medium coupling are considered to observe the quantum entanglement robustness.

3.1 Three-Qubit Entanglement for Negative Detuning Frequency

This section analyzes the entanglement of a three-qubit quantum system in influencing the Kerr-like medium under the environment of non-Markovian, $G = 8.0$ and Markovian,$G = 0.8$. The quantum entanglement is measured via LBC with $\delta = -1$, $r_1 = \frac{1}{\sqrt{2}}$,$r_2 = \frac{1}{\sqrt{2}}$,$r_3 = \frac{1}{\sqrt{6}}$,$a_1(0) = a_2(0) = \frac{1}{2}$ and $a_3(0) = \frac{1}{\sqrt{2}}$. LBC will have a value from zero to one, with one having the most robust quantum entanglement and vice versa. The value of r_n is chosen with a combination of value $a_n(0)$ to achieve a robust entanglement [3], while the value of $a_n(0)$ is used based on the quantum state in quantum teleportation. For a dipole-dipole interaction, d will vary as shown in Figure 1e and Figure 2e and several Kerr-like medium coupling strength is used to analyze the effect of the quantum entanglement of the three-qubit quantum system. The value of pole, z_j will vary according to the different values of the Kerr-like medium and the dipole-dipole interaction which contain three negative real numbers and a complex value except for a zero dipole-dipole interaction where z_j consists of only two poles.

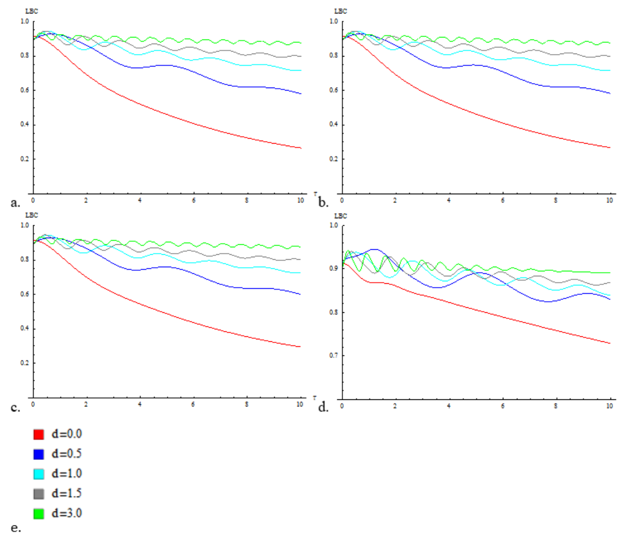

Fig. 1: Lower Bound Concurrence (LBC) $G = 0.8$, various d as shown in e. and a. $K = 0.00$, b. $K = 0.01$, c. $K = 0.10$, d. $K = 0.25$. Time scale, $1 \leq \tau \leq 10$.

Figure 1 shows the entanglement strength for various dipole-dipole interactions and different Kerr-like medium coupling strength. It is showed that in Figure 1a, Figure 1b and Figure 1c is identical which tells weak Kerr-like medium coupling strength has minimum influence towards quantum entanglement. As usual strong dipole-dipole interaction lead to robust quantum entanglement (Nguyen et al., 2011). Starting from $K > 0.10$, quantum entanglement showed sign of changes compare to K as showed in Figure 1d. At the right d the decoherence rate is reduced, which shows a robust quantum entanglement while for $d > 0.5$, quantum entanglement is less robust.

Figure 2 shows the quantum entanglement robustness for various Kerr-like medium coupling and dipole-dipole interactions in a non-Markovian environment. Figure 2a shows that where no Kerr-like medium coupling exists, a stronger dipole-dipole interaction leads to a robust quantum entanglement. As the Kerr-like medium coupling slightly increases to $K = 0.01$ (see Figure 2b), the quantum entanglement behavior still shows an identical pattern as in Figure 2a. When the Kerr-like medium coupling strength further increases to become

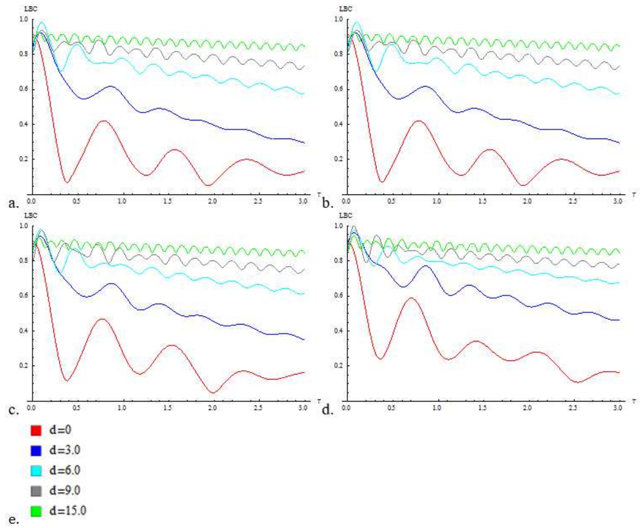

Fig. 2: Lower Bound Concurrence (LBC) for $G = 8.0$, various d as shown in e. and a. $K = 0.00$, b. $K = 0.01$, c. $K = 1.00$, d. $K = 3.00$. Time scale, $1 \le \tau \le 3$.

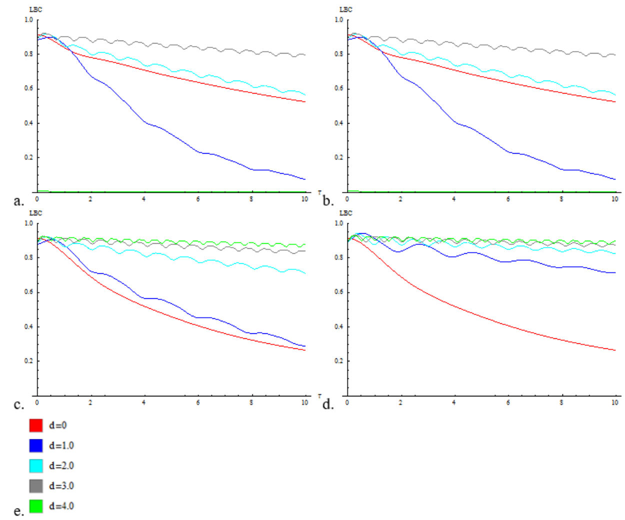

Fig. 3: Lower Bound Concurrence (LBC) for $\delta = 2.0$ and $G = 0.8$, various d as shown in e. and a. $K = 0.00$, b. $K = 0.01$, c. $K = 1.00$, d. $K = 3.00$. Time scale, $1 \le \tau \le 10$.

$K = 0.01$, Figure 2c shows this has caused slight changes for $d = 9.0$ where at $\tau < 0.5$ there is a drop in the LBC before it increases back. For a strong Kerr-like medium coupling, $K = 3.00$, some changes are shown in Figure 2d where the minimum level of the LBC increases in comparison when $K < 3.00$. However, as d becomes stronger, the fluctuation pattern becomes almost the same. In this case, some differences in the fluctuation pattern can be observed. Hence, in the non-Markovian environment, only when $K > 0.01$ the impact on the quantum entanglement is strong. Overall, a strong dipole-dipole interaction reduces the impact of the Kerr-like medium on the quantum entanglement.

3.2 *Three-Qubit Entanglement for Positive Detuning Frequency*

This section discusses the quantum entanglement when detuning frequency is positive. Positive detuning has shown a different behavior on the quantum entanglement behavior from previous section where an in increase in the dipole-dipole interaction no longer produces a robust quantum entanglement.

Figure 3 shows the LBC in the Markovian environment with a different value of d and K. In Figure 3a and 3b, the pattern is no longer observed where a stronger d leads to a robust quantum entanglement, as shown in Section 3.1. Instead, the LBC shows that the decoherence rate is random for the strong and weak dipole-dipole interaction. Both Figure 3a and Figure 3b produce the same pattern of output and quantum entanglement strength. An increase in the Kerr-like medium coupling from $K = 1.00$ to $K = 3.00$ increases the robustness for $d > 0.0$. Initially at $d = 0.0$ the same rate of decoherence was observed (see Figure 3c and Figure 3d). However, when $d > 0.0$ there is an increase in the entanglement robustness with a lesser decoherence rate (compare Figure 3d with Figure 3c). Figure 3d shows a more robust quantum entanglement than $K < 0.01$.

Figure 4 shows the LBC in the function of τ, which in this case will be in the non-Markovian environment. The result showed that an increase from $K = 0.00$ to $K = 0.10$ increases the amplitude. When $d > 6.0$, Figure 4a and Figure 4b show that the quantum entanglement is more robust with a huge increase. When K is further increased the cross, as shown in Figure 4a and 4b, disappears and it is followed by an increase in the quantum entanglement robustness with an increase in d, as shown in Figure 4c and Figure 4d. Then, when $d > 0.0$, the quantum entanglement becomes stronger for $K = 3.00$ than for $K \le 1.00$. When d reaches 9.0 the differences between Figure 4c and 4d become lesser.

4 Conclusion

Different parameters are observed and studied for the robustness quantum entanglement for three-qubit Jaynes-Cummings model coupling with Kerr-like medium. Overall, a weak Kerr-like medium coupling does not influence much the quantum entanglement in all environments and this also applied for strong dipole-dipole interactions. Further enhancement of the Kerr-like medium coupling changes the quantum

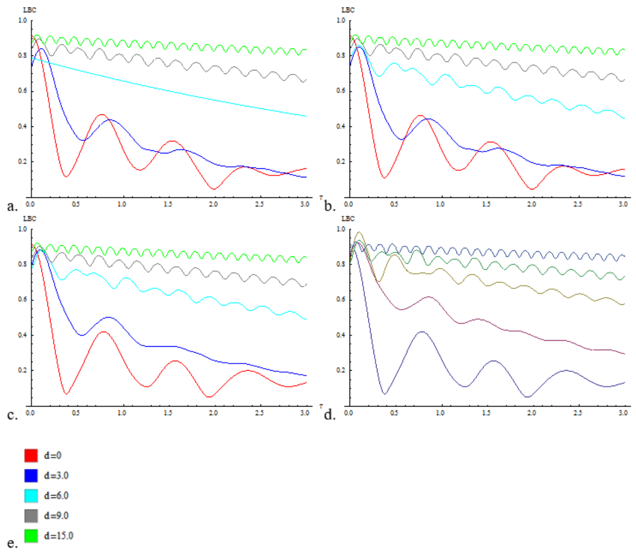

Fig. 4: Lower Bound Concurrence (LBC) for $\delta = 2.0$ and $G = 8.0$, various d as shown in e. and a. $K = 0.00$, b. $K = 0.10$, c. $K = 1.00$, d. $K = 3.00$. Time scale, $1 \leq \tau \leq 3$.

entanglement with different parameters of dipole-dipole interaction, environment or detuning frequency.

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