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# Transmuted Two-Parameter Lindley Distribution

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**Abstract:** In this paper, a transmuted two-parameter Lindley distribution (TTLD) is suggested as a modification of the well known two-parameter Lindley distribution (TLD). The necessary mathematical properties including the moments, the moment generating function, hazard rate function, reliability function, and the order statistics for the TTLD are derived. Also, the maximum likelihood estimators for the TTLD parameters and the Renyi entropy are derived. A real data set is used for illustration. It is found that the TTLD is more better than the TLD and Lindley distribution (LD) to fit this set of real data.

**Keywords:** Two-parameter Lindley distribution, Transmuted two-parameter Lindley distribution, Transmutation map, Order statistics, Reliability, Renyi entropy, Quantile.

## 1 Introduction

Numerous lifetime data used in statistical analysis depends on a particular statistical distribution. Knowledge of suitable distribution of real data will extremely ameliorates the efficiency and the power of the statistical tests involved with it. Therefore, several distributions are suggested for modeling lifetime data. However, there are still many life time data that does not follow any distribution and hence there is a need to extend some new distributions. Here we suggested a new distribution for fitting lifetime data using one of the well known distribution function generation methods. The two-parameter Lindley distribution is defined by [1] for modeling waiting and survival times data.

**Definition 1:** A random variable $X$ is said to have a TLD with parameters $\alpha$ and $\theta$ if its probability density function (pdf) is given by

$$f_{TLD}(x) = \frac{\theta^2}{\theta+\alpha}(1+\alpha x)\,e^{-\theta x}, \qquad x>0\,,\theta>0\,,\alpha>-\theta, \tag{1}$$

and the respective cumulative distribution (cdf) function is

$$F_{TLD}(x) = 1-\frac{\theta+\alpha+\alpha\theta x}{\theta+\alpha}e^{-\theta x}\,, \qquad x>0\,,\theta>0\,,\alpha>-\theta. \tag{2}$$

The $r$th moment of the two-parameter Lindley distribution random variable is

$$E\left(X^r_{TLD}\right) = \frac{\theta+\alpha+\alpha r}{\theta^r\left(\theta+\alpha\right)}\Gamma\left(r+1\right),\ r=1,2,\ldots \tag{3}$$

**Definition 2:** A random variable $X$ is said to have a LD with parameter $\theta$ if its pdf is given by

$$f_{LD}(x) = \frac{\theta^2}{\theta+1}(1+x)\,e^{-\theta x}\,, \qquad x>0\,,\theta>0, \tag{4}$$

and the respective cdf is defined as

$$F_{LD}(x) = 1-\frac{\theta+1+\theta x}{\theta+1}\,e^{-\theta x}\,, \qquad x>0\,,\theta>0. \tag{5}$$

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It can be noted that the LD is a special case from the TLD when $\alpha = 1$. For more details about the two-parameter Lindley distribution see [1].

**Definition 3**: A random variable $X$ is said to have transmuted distribution if its cdf is given by

$$F(x) = (1+\lambda)G(x) - \lambda [G(x)]^2, \ \ |\lambda| \leq 1, \tag{6}$$

with corresponding pdf is defined as

$$f(x) = g(x)[1+\lambda - 2\lambda G(x)] \tag{7}$$

where $G(x)$ is the cdf of the base distribution, $f(x)$ and $g(x)$ are the corresponding probability density functions associated with the cdf's $F(x)$ and $G(x)$, respectively.

Observe that when $\lambda = 0$ the distribution of the base random variable can be obtained. An extensive information about the transmutation map method can be found in [2].

Many researchers deal with the generalization of some known distributions based on the transmutation map method. [3] suggested the transmuted Lomax distribution. [4] proposed transmuted Pareto distribution as a generalization of the Pareto distribution. [5] defined the transmuted Inverse Rayleigh distribution as a generalization of the Inverse Rayleigh distribution. [6] presented transmuted Kumaraswamy distribution. [7] proposed transmuted modified Weibull distribution. [8] defined transmuted generalized Lindley distribution. [9] introduced the transmuted extreme value distribution with an application to climate data. [10] proposed the transmuted generalized inverse Weibull distribution. [11] suggested transmuted Lindley distribution. [12] suggested the transmuted Generalized Gamma distribution. They investigated the mathematical properties and the maximum likelihood estimators of the unknown parameters in these distributions. [13] proposed transmuted Quasi Lindley distribution as a generalization of the Quasi Lindley distribution.

In the this paper, we presented the mathematical formulation of the TTLD and provided some possible applications. In Section 2 we demonstrated the transmuted two-parameter Lindley distribution. The statistical properties including the $r$th moment, the moment generating function, variance, skewness and kurtosis are discussed in Section 3. The distributions of order statistics are given in Section 4. The reliability analysis is given in Section 5. The random number generation and the maximum likelihood estimates are investigated in Section 6. The Rnyi entropy for the TTLD is defined in Section 7. The usefulness of the proposed TTLD is demonstrated by using real data set in Section 8. Finally, some conclusions are provided in Section 9.

## 2 The transmuted two-parameter Lindley distribution

**Definition 4:** A random variable $X$ is said to have transmuted two-parameter Lindley distribution (TTLD) if its cumulative distribution function is

$$\begin{aligned} F_{TTLD}(x) &= (1+\lambda)\left[1 - \frac{(\alpha+\theta+\alpha\theta x)\mathrm{e}^{-\theta x}}{\alpha+\theta}\right] - \lambda\left[1 - \frac{(\alpha+\theta+\alpha\theta x)\mathrm{e}^{-\theta x}}{\alpha+\theta}\right]^2 \\ &= \frac{\mathrm{e}^{-2\theta x}\left[\left(\mathrm{e}^{\theta x}-1\right)\theta + \alpha\left(\mathrm{e}^{\theta x}-\theta x-1\right)\right]\left[\mathrm{e}^{\theta x}(\alpha+\theta) + (\alpha+\theta+\alpha\theta x)\lambda\right]}{(\alpha+\theta)^2}. \end{aligned} \tag{8}$$

The pdf corresponding to (8) becomes

$$f_{TTLD}(x) = \frac{\theta^2}{\theta+\alpha}(1+\alpha x)e^{-\theta x}\left[1+\lambda-2\lambda\left(1-\frac{\theta+\alpha+\alpha\theta x}{\theta+\alpha}e^{-\theta x}\right)\right], \ x>0, \theta>0, \alpha>-\theta. \tag{9}$$

It is clear that if $\lambda = 0$ and $\alpha = 1$ in (8) and (9), then respectively we will have the cdf and pdf of the Lindley distributed random variable .

## 3 Some mathematical properties of the TTLD

In this section, we presented the $r$th moment for the transmuted two-parameter Lindley distribution random variable. Also, the moment generating function, mean, variance, the coefficients of skewness, kurtosis, and variation are derived. The shapes of the pdf and cdf of the TTLD functions are discussed and some numerical results are obtained.

**Theorem 1:** The $r$th moment of the TTL distributed random variable can be expressed as

$$E\left(X^r\right)=\frac{1}{(\theta+\alpha)\theta^r}\left\{\theta\left(1-\lambda+\tfrac{\lambda}{2^r}\right)\Gamma(r+1)+\alpha\left(1-\lambda+\tfrac{\lambda(2\theta+\alpha)}{2^{r+1}(\theta+\alpha)}\right)\Gamma(r+2)+\tfrac{\lambda\alpha^2}{2^{r+2}(\theta+\alpha)}\Gamma(r+3)\right\}. \tag{10}$$

**Proof:**

$$\begin{aligned}
E\left(X^r\right)&=\int_0^\infty x^r f(x)dx\\
&=\int_0^\infty x^r\frac{\theta^2}{\theta+\alpha}(1+\alpha x)e^{-\theta x}\left[1+\lambda-2\lambda\left(1-\frac{\theta+\alpha+\alpha\theta x}{\theta+\alpha}e^{-\theta x}\right)\right]dx\\
&=\frac{\theta^2}{\theta+\alpha}\int_0^\infty x^r\left[(1-\lambda)e^{-\theta x}+xe^{-\theta x}+2\lambda e^{-2\theta x}+2\alpha\lambda\left(\frac{2\theta+\alpha}{\theta+\alpha}\right)xe^{-2\theta x}+\frac{2\lambda\theta\alpha^2}{\theta+\alpha}x^2e^{-2\theta x}\right]dx\\
&=\frac{\theta^2}{\theta+\alpha}\left[\begin{array}{l}\int_0^\infty(1-\lambda)x^re^{-\theta x}dx+\int_0^\infty(\alpha-\lambda\alpha)x^{r+1}e^{-\theta x}dx+2\lambda\int_0^\infty x^re^{-2\theta x}dx\\ +2\alpha\lambda\int_0^\infty\frac{2\theta+\alpha}{\theta+\alpha}x^{r+1}e^{-2\theta x}dx+\int_0^\infty\frac{2\lambda\theta\alpha^2}{\theta+\alpha}x^{r+2}e^{-2\theta x}dx\end{array}\right]\\
&=\theta^2\left[\begin{array}{l}\left(\frac{1-\lambda}{(\theta+\alpha)\theta^{r+1}}+\frac{2\lambda\theta+2\lambda\alpha}{(\theta+\alpha)^2(2\theta)^{r+1}}\right)\Gamma(r+1)+\left(\frac{\alpha-\lambda\alpha}{(\theta+\alpha)\theta^{r+2}}+\frac{4\lambda\theta\alpha+2\lambda\alpha^2}{(\theta+\alpha)^2(2\theta)^{r+2}}\right)\Gamma(r+2)\\ +\frac{2\lambda\theta\alpha^2}{(\theta+\alpha)^2(2\theta)^{r+3}}\Gamma(r+3)\end{array}\right].
\end{aligned}$$

So,

$$E\left(X^r\right)=\frac{1}{(\theta+\alpha)\theta^r}\left\{\theta\left(1-\lambda+\tfrac{\lambda}{2^r}\right)\Gamma(r+1)+\alpha\left(1-\lambda+\tfrac{\lambda(2\theta+\alpha)}{2^{r+1}(\theta+\alpha)}\right)\Gamma(r+2)+\tfrac{\lambda\alpha^2}{2^{r+2}(\theta+\alpha)}\Gamma(r+3)\right\}.$$

**Theorem 2:** The moment generating function of the TTLD is defined as

$$M_X(t)=\frac{1}{(2\theta-t)(\theta+\alpha)}\left[\theta^2(1-\lambda)(1+\alpha)+\frac{2\lambda}{\theta+\alpha}\left(\theta+\alpha+2\theta\alpha+\alpha^2+\frac{2\theta\alpha^2}{2\theta-t}\right)\right]. \tag{11}$$

**Proof:**

$$\begin{aligned}
M_X(t)&=\int_0^\infty e^{tx}f(x)dx\\
&=\int_0^\infty\frac{\theta^2e^{tx}}{\theta+\alpha}\left[(1-\lambda)e^{-\theta x}+\alpha(1-\lambda)xe^{-\theta x}+2\lambda e^{-2\theta x}+2\lambda\alpha\left(\frac{2\theta+\alpha}{\theta+\alpha}\right)xe^{-2\theta x}+\frac{2\lambda\theta\alpha^2}{\theta+\alpha}x^2e^{-2\theta x}\right]dx\\
&=\frac{\theta^2}{\theta+\alpha}\left[\begin{array}{l}(1-\lambda)\int_0^\infty e^{-(\theta-t)x}dx+\alpha\frac{1-\lambda}{\theta+\alpha}\int_0^\infty xe^{-(\theta-t)x}dx+2\lambda\int_0^\infty e^{-(2\theta-t)x}dx\\ +2\alpha\lambda\frac{2\theta+\alpha}{\theta+\alpha}\int_0^\infty xe^{-(2\theta-t)x}dx+\frac{2\lambda\theta\alpha^2}{\theta+\alpha}\int_0^\infty x^2e^{-(2\theta-t)x}dx\end{array}\right]\\
&=\frac{\theta^2}{\theta+\alpha}\left[\frac{1-\lambda}{\theta-t}+\frac{\alpha-\lambda\alpha}{\theta-t}\Gamma(2)+\frac{2\lambda\theta+2\lambda\alpha}{(\theta+\alpha)(2\theta-t)}+\frac{4\lambda\theta\alpha+2\lambda\alpha^2}{(\theta+\alpha)(2\theta-t)}\Gamma(2)+\frac{2\lambda\theta\alpha^2}{(\theta+\alpha)(2\theta-t)^2}\Gamma(3)\right]\\
&=\frac{1}{2\theta-t}\left[\frac{\theta^2(1-\lambda)(1+\alpha)}{\theta+\alpha}+\frac{2\lambda}{(\theta+\alpha)^2}\left(\theta+\alpha+2\theta\alpha+\alpha^2+\frac{2\theta\alpha^2}{2\theta-t}\right)\right]\\
&=\frac{1}{(2\theta-t)(\theta+\alpha)}\left[\theta^2(1-\lambda)(1+\alpha)+\frac{2\lambda}{\theta+\alpha}\left(\theta+\alpha+2\theta\alpha+\alpha^2+\frac{2\theta\alpha^2}{2\theta-t}\right)\right].
\end{aligned}$$

Therefore, from (10) we can find the following moments of the TTLD, where the first moment(mean) is

$$E(X)=\frac{\alpha^2(8-3\lambda)-6\alpha\theta(\lambda-2)-2\theta^2(\lambda-2)}{4\theta(\alpha+\theta)^2} \tag{12}$$

and the second moment is given by

$$E\left(X^2\right)=\frac{8(\alpha+\theta)(3\alpha+\theta)-3\lambda\left(5\alpha^2+8\alpha\theta+2\theta^2\right)}{4\theta^2(\alpha+\theta)^2}. \tag{13}$$

Thus, using these relations the variance of transmuted two-parameter Lindley distribution can be written as

$$\begin{aligned} Var(X) &= E\left(X^2\right)-[E(X)]^2 \\ &= \frac{4(\alpha+\theta)^2\begin{bmatrix}8(\alpha+\theta)(3\alpha+\theta)\\ -3\left(5\alpha^2+8\alpha\theta+2\theta^2\right)\lambda\end{bmatrix}-\begin{bmatrix}\alpha^2(8-3\lambda)-6\alpha\theta(\lambda-2)\\ -2\theta^2(\lambda-2)\end{bmatrix}^2}{16\theta^2(\alpha+\theta)^4}. \end{aligned} \tag{14}$$

The third and fourth moments of the TTLD random variable are

$$E\left(X^3\right)=\frac{24(\alpha+\theta)(4\alpha+\theta)-3\left(25\alpha^2+35\alpha\theta+7\theta^2\right)\lambda}{4\theta^3(\alpha+\theta)^2}, \tag{15}$$

and

$$E\left(X^4\right)=\frac{48(\alpha+\theta)(5\alpha+\theta)-15\left(14\alpha^2+18\alpha\theta+3\theta^2\right)\lambda}{2\theta^4(\alpha+\theta)^2}. \tag{16}$$

Moreover, the coefficients of skewness, kurtosis, variation of a random variable $X$ are defined as $\delta_1=\frac{E\left(X^3\right)-3\mu\sigma^2-\mu^3}{\sigma^3}$, $\delta_2=\frac{E\left(X^4\right)-4\mu E\left(X^3\right)+6E\left(X^2\right)\sigma^2+3E\left(X^4\right)}{\sigma^8}$, and $\delta_3=\frac{\sigma}{\mu}$, respectively where $\sigma$ is the standard deviation of $X$.

For the TTLD random variable, respectively, they are defined as

$$\delta_1=-2\frac{\left\{\begin{array}{l}-64(\alpha+\theta)^3\left(2\alpha^3+6\alpha^2\theta+6\alpha\theta^2+\theta^3\right)+24(\alpha+\theta)^2\left(\alpha^4+4\alpha^3\theta+7\alpha^2\theta^2+7\alpha\theta^3+\theta^4\right)\lambda\\ +6(\alpha+\theta)\left(3\alpha^2+6\alpha\theta+2\theta^2\right)\left(3\alpha^3+9\alpha^2\theta+10\alpha\theta^2+2\theta^3\right)\lambda^2+\left(3\alpha^2+6\alpha\theta+2\theta^2\right)^3\lambda^3\end{array}\right\}}{\left\{\begin{array}{l}-\left[\alpha^2(8-3\lambda)-6\alpha\theta(-2+\lambda)-2\theta^2(-2+\lambda)\right]^2\\ -4(\alpha+\theta)^2\left[8(\alpha+\theta)(3\alpha+\theta)-3\left(5\alpha^2+8\alpha\theta+2\theta^2\right)\lambda\right]\end{array}\right\}^{3/2}}, \tag{17}$$

$$\delta_2=-48\,\theta^4(\alpha+\theta)^8\frac{R}{\left\{\begin{array}{l}\theta^4[-4+\lambda(2+\lambda)]+\alpha^4[-8+\lambda(3+2.25\lambda)]+\alpha\theta^3[-24+\lambda(12+6\lambda)]\\ +\alpha^3\theta[-32+\lambda(12+9\lambda)]+\alpha^2\theta^2[-44+\lambda(19+12.\lambda)]\end{array}\right\}^4}, \tag{18}$$

where

$$R=\theta^2\left\{\begin{array}{l}\theta^6[-48+\lambda(24+\lambda(16+\lambda(4+\lambda)))]\\ +\alpha^8[-128+\lambda(48.+\lambda(48+\lambda(13.5+5.0625\lambda)))]\\ +\alpha\theta^5[-576+\lambda(288+\lambda(192+\lambda(48+12\lambda)))]\\ +\alpha^7\theta[-1024+\lambda(384+\lambda(384+\lambda(108+40.5\lambda)))]\\ +\alpha^2\theta^4[-2528+\lambda(1184+\lambda(876+\lambda(230+60\lambda)))]\\ +\alpha^6[-3520+\lambda(1344+\lambda(1310+\lambda(373.5+135\lambda)))]\\ +\alpha^3\theta^3[-5824+\lambda(2544+\lambda(2076+\lambda(576+162\lambda)))]\\ +\alpha^5\theta^1[-6784+\lambda(2672+\lambda(2496+\lambda(720+243\lambda)))]\\ +\alpha^4\theta^2[-7984+\lambda(3288+\lambda(2898+\lambda(831+256.5\lambda)))]\end{array}\right\},$$

and

$$\delta_3=\sqrt{\frac{16(\alpha+\theta)^2\left(2\alpha^2+4\alpha\theta+\theta^2\right)-4(\alpha+\theta)\left(3\alpha^3+9\alpha^2\theta+10\alpha\theta^2+2\theta^3\right)\lambda-\left(3\alpha^2+6\alpha\theta+2\theta^2\right)^2\lambda^2}{\left[6\alpha\theta(\lambda-2)+2\theta^2(\lambda-2)+\alpha^2(3\lambda-8)\right]^2}}. \tag{19}$$

The following figures illustrate the possible shapes of the pdf and cdf of the TTLD by keeping $\alpha = 2, \theta = 1$ for various values of the parameter $\lambda$ .

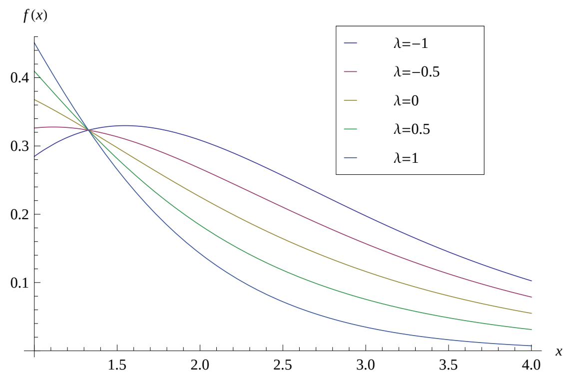


**Fig. 1:** The pdf of the TTLD with $\alpha = 2, \theta = 1$ and $\lambda = -1, -0.5, 0, 0.5, 1$.

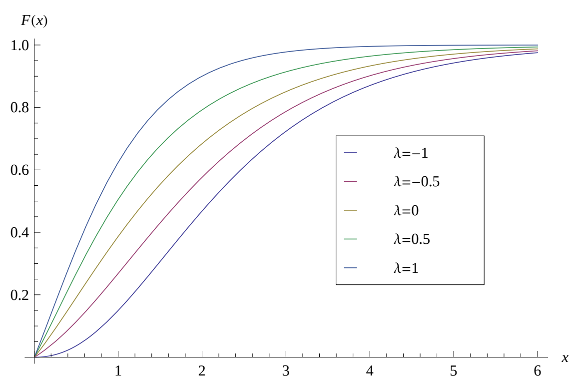


**Fig. 2:** The cdf of the TTLD with $\alpha = 2, \theta = 1$ and $\lambda = -1, -0.5, 0, 0.5, 1$.

The following table includes some values of the mean, variance, the coefficient of variation, coefficients of skewness, and kurtosis of the TTLD for different values of the distribution parameter $\lambda$ when $\theta = 2, \alpha = 3$.

**Table 1:** Some values of the mean, variance, CV, $\delta_1$, $\delta_2$, and $\delta_3$, of the TTLD with $\theta = 2, \alpha = 3$ for $-1 \leq \lambda \leq 1$.

| $\lambda$ | **Mean** | **Var** | $\delta_1$ | $\delta_2$ | $\delta_3$ |
|---|---|---|---|---|---|
| 1 | 0.4450 | 0.144480 | 1.50700 | 296.8320 | 0.854150 |
| 0.9 | 0.4805 | 0.187370 | 1.92766 | 272.1170 | 0.900580 |
| 0.8 | 0.5180 | 0.227744 | 2.03399 | 192.9220 | 0.942855 |
| 0.7 | 0.5515 | 0.265598 | 2.02700 | 137.0500 | 0.934473 |
| 0.6 | 0.5870 | 0.300931 | 1.97507 | 100.7270 | 0.934535 |
| 0.5 | 0.6225 | 0.333744 | 1.90547 | 76.8004 | 0.928041 |
| 0.4 | 0.6580 | 0.346036 | 1.83026 | 60.5726 | 0.916951 |
| 0.3 | 0.6935 | 0.391808 | 1.75509 | 49.2353 | 0.902589 |
| 0.2 | 0.7290 | 0.417059 | 1.68273 | 41.1054 | 0.885873 |
| 0.1 | 0.7645 | 0.439790 | 1.61458 | 35.1493 | 0.867451 |
| 0 | 0.8000 | 0.460000 | 1.55135 | 30.7139 | 0.847791 |
| - 0.1 | 0.8355 | 0.477690 | 1.49344 | 27.3749 | 0.827230 |
| - 0.2 | 0.8710 | 0.492859 | 1.44111 | 24.8505 | 0.806015 |
| - 0.3 | 0.9065 | 0.505508 | 1.39461 | 22.9499 | 0.784325 |
| - 0.4 | 0.9420 | 0.515636 | 1.35420 | 21.5428 | 0.762291 |
| - 0.5 | 0.9775 | 0.523244 | 1.32026 | 20.5400 | 0.740006 |
| - 0.6 | 1.0130 | 0.528331 | 1.29326 | 19.8823 | 0.717536 |
| - 0.7 | 1.0485 | 0.530898 | 1.27387 | 19.5322 | 0.694924 |
| - 0.8 | 1.0840 | 0.530944 | 1.26296 | 19.4706 | 0.672195 |
| - 0.9 | 1.1195 | 0.528470 | 1.26169 | 19.6939 | 0.649361 |
| - 1 | 1.1550 | 0.523475 | 1.27160 | 20.2139 | 0.626421 |

Based on Table 1, it can be noted that the mean and variance values are decreasing as $\lambda$ values are decreasing, while they are increasing as the magnitude of $\lambda$ is increasing. However, the values of $\delta_1$, $\delta_2$, and $\delta_3$ are increasing as the magnitude of $\lambda$ gets large. Also, the skewness values in Table (1) indicate that the two-parameter Lindley distribution is asymmetric distribution as shown in Figure 1.

## 4 Order statistics

The order statistics are crucial in numerous areas of training and statistical theory and they have many applications in life testing and reliability. Let $X_{(1)}, X_{(2)}, ..., X_{(n)}$ be the order statistics of the random sample $X_1, X_2, ..., X_n$ selected from a pdf and cdf $f(x)$ and $F(x)$ , respectively. The pdf of the $j$th order statistics $X_{(j)}$ is defined as

$$f_{(j)}(x) = \frac{n!}{(j-1)!(n-j)!}[F(x)]^{j-1}[1-F(x)]^{n-j} f(x),$$

for $j = 1, 2, ..., n$. See [14]. From (8), (9), and $f_{(j)}(x)$, we have the pdf of the ith TTLD random variable $X_{(j)}$ as

$$\begin{aligned} f_{TTLD(j)}(x) = {} & \frac{n! \mathrm{e}^{-2\theta x}(1+\alpha x)\theta^2}{(j-1)!(n-j)!(\alpha+\theta)^2}\left[2(\alpha+\theta+\alpha\theta x)\lambda - \mathrm{e}^{\theta x}(\alpha+\theta)(\lambda-1)\right] \\ & \times\left\{1+2\lambda+\frac{\mathrm{e}^{-2\theta x}(\alpha+\theta+\alpha\theta x)\left[(\alpha+\theta+\alpha\theta x)\lambda - \mathrm{e}^{\theta x}(\alpha+\theta)(1+3\lambda)\right]}{(\alpha+\theta)^2}\right\}^{n-i} \\ & \times\left\{\frac{\mathrm{e}^{-2\theta x}(\alpha+\theta+\alpha\theta x)\left[-(\alpha+\theta+\alpha\theta x)\lambda + \mathrm{e}^{\theta x}(\alpha+\theta)(1+3\lambda)\right]}{(\alpha+\theta)^2} - 2\lambda\right\}^{i-1}, \end{aligned} \tag{20}$$

for $x > 0$, $\theta > 0$, $\alpha > -\theta$. Therefore, from $f_{(j)}(x)$, the pdf of the smallest order statistics $X_{(1)} = \min\{X_1, X_2, ..., X_n\}$ is

$$f_{TTLD(1)}(x) = \frac{n}{(\alpha+\theta)^2} e^{-2\theta x}(1+\alpha x)\theta^2 \left[2(\alpha+\theta+\alpha\theta x)\lambda - e^{\theta x}(\alpha+\theta)(\lambda-1)\right] \times \left\{1+2\lambda+\frac{e^{-2\theta x}(\alpha+\theta+\alpha\theta x)\left[(\alpha+\theta+\alpha\theta x)\lambda - e^{\theta x}(\alpha+\theta)(1+3\lambda)\right]}{(\alpha+\theta)^2}\right\}^{n-1}, \tag{21}$$

and the pdf of the largest order statistics $X_{(n)} = \max\{X_1, X_2, ..., X_n\}$ has the form

$$f_{TTLD(n)}(x) = \frac{n}{(\alpha+\theta)^2} e^{-2\theta x}(1+\alpha x)\theta^2 \left[2(\alpha+\theta+\alpha\theta x)\lambda - e^{\theta x}(\alpha+\theta)(\lambda-1)\right] \times \left\{\frac{e^{-2\theta x}(\alpha+\theta+\alpha\theta x)\left[-(\alpha+\theta+\alpha\theta x)\lambda + e^{\theta x}(\alpha+\theta)(1+3\lambda)\right]}{(\alpha+\theta)^2} - 2\lambda\right\}^{n-1}. \tag{22}$$

## 5 Reliability analysis

The reliability function $R(t)$ is the probability of an item not failing prior to a time $t$. The reliability function of the transmuted two-parameter Lindley distribution is given by

$$\begin{aligned} R_{TTLD}(t) &= 1 - F_{TTLD}(t) \\ &= \frac{e^{-2t\theta}(\alpha+\theta+t\alpha\theta)\left[e^{t\theta}(\alpha+\theta)(1+3\lambda) - (\alpha+\theta+t\alpha\theta)\lambda\right]}{(\alpha+\theta)^2} - 2\lambda. \end{aligned} \tag{23}$$

The hazard rate function of the transmuted two-parameter Lindley distribution is defined as

$$H_{TTLD}(t) = \frac{f_{TTLD}(t)}{1-F_{TTLD}(t)} \tag{24}$$

$$= \frac{(1+t\alpha)\theta^2\left[2(\alpha+\theta+t\alpha\theta)\lambda - e^{t\theta}(\alpha+\theta)(\lambda-1)\right]}{(\alpha+\theta+t\alpha\theta)\left[(\alpha+\theta+t\alpha\theta)\lambda - e^{t\theta}(\alpha+\theta)(\lambda-1)\right]}, \tag{25}$$

where the hazard rate function is known as an instantaneous failure rate which is used in characterizing life phenomenon. Note that if $t = \lambda = 1$, then $H_{TTLD}(t) = \frac{2(1+\alpha)\theta^2}{\alpha+\theta+\alpha\theta}$. Figure (3) illustrates the shape of the hazard rate function of the TTLD when $\alpha = 1.5, \theta = 0.5$ for different values of $\lambda$.

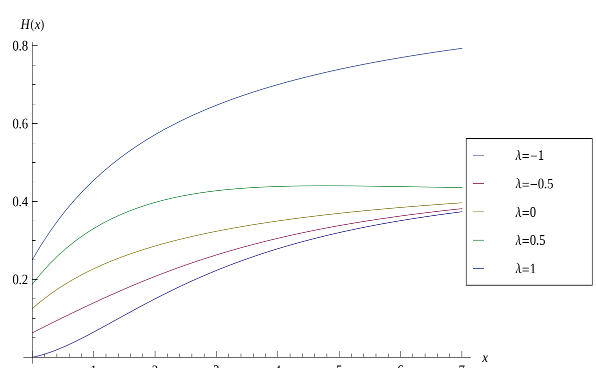


**Fig. 3:** The hazard function of the TTLD with $\alpha = 1.5, \theta = 0.5$ and $\lambda = -1, -0.5, 0, 0.5, 1$.

## 6 Random number generation and maximum likelihood estimation

Using the inversion method, we can generate random numbers from the transmuted two-parameter Lindley distribution by letting

$$\frac{\mathrm{e}^{-2\theta x}\left[\theta\left(\mathrm{e}^{\theta x}-1\right)+\alpha\left(\mathrm{e}^{\theta x}-\theta x-1\right)\right]\left[\mathrm{e}^{\theta x}(\alpha+\theta)+\lambda(\alpha+\theta+\alpha\theta x)\right]}{(\alpha+\theta)^2}=u, \tag{26}$$

where $u$ is a uniformly distributed random variable, $U(0,1)$. Solving (26) for $x$ one can generate random numbers when $\alpha,\theta$, and $\lambda$ are known.

Now, let $X_1,X_2,\ldots,X_n$ be a random sample of size $n$ from the TTLD with parameters $\alpha$ , $\theta$ and $\lambda$ , then the likelihood function is given by

$$\begin{aligned} L_{TTLD} &= \prod_{i=1}^{n}\left[\frac{\theta^2}{\theta+\alpha}(1+\alpha x_i)e^{-\theta x_i}\right]\left[1+\lambda-2\lambda\left(1-\frac{\theta+\alpha+\alpha\theta x_i}{\theta+\alpha}e^{-\theta x_i}\right)\right] \\ &= \left(\frac{\theta^2}{\theta+\alpha}\right)^n\prod_{i=1}^{n}(1+\alpha x_i)\,e^{-\theta n\sum\limits_{i=1}^{n}x_i}\prod_{i=1}^{n}\left[1+\lambda-2\lambda\left(1-\frac{\theta+\alpha+\alpha\theta x_i}{\theta+\alpha}e^{-\theta x_i}\right)\right]. \end{aligned} \tag{27}$$

Hence, the log likelihood function $\Psi=\log(L_{TTLD})$ will be

$$\begin{aligned} \Psi &= \log\left\{\left(\frac{\theta^2}{\theta+\alpha}\right)^n\prod_{i=1}^{n}(1+\alpha x_i)\,e^{-\theta n\sum\limits_{i=1}^{n}x_i}\prod_{i=1}^{n}\left[1+\lambda-2\lambda\left(1-\frac{\theta+\alpha+\alpha\theta x_i}{\theta+\alpha}e^{-\theta x_i}\right)\right]\right\} \\ &= n\log\left(\frac{\theta^2}{\theta+\alpha}\right)+\sum_{i=1}^{n}\log(1+\alpha x_i)-\theta\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}\log\left[1+\lambda-2\lambda\left(1-e^{-\theta_i x}-\frac{\alpha\theta x_i}{\theta+\alpha}e^{-\theta x_i}\right)\right]. \end{aligned} \tag{28}$$

Differentiating Equation (28) with respect to $\theta,\alpha$ and $\lambda$ results in

$$\frac{\partial\Psi}{\partial\theta}=\frac{2n}{\theta}-\frac{n\theta^2}{\theta+\alpha}-\sum_{i=1}^{n}x_i-\sum_{i=1}^{n}\frac{2\lambda\left(1+\frac{\alpha\theta x_i-\alpha}{\theta+\alpha}-\frac{\alpha\theta}{(\theta+\alpha)^2}\right)x_ie^{-\theta x_i}}{1-\lambda+2\lambda e^{-\theta x_i}+2\lambda\frac{\alpha\theta x_i}{\theta+\alpha}e^{-\theta x_i}}, \tag{29}$$

$$\frac{\partial\Psi}{\partial\alpha}=-\frac{n}{\theta+\alpha}+\sum_{i=1}^{n}\frac{x_i}{1+\alpha x_i}+\sum_{i=1}^{n}\frac{2\lambda\left(\frac{\theta x_i}{\theta+\alpha}e^{-\theta x_i}-\frac{\alpha\theta x_i}{(\theta+\alpha)^2}e^{-\theta x_i}\right)}{1+\lambda-2\lambda\left(1-e^{-\theta x_i}-\frac{\alpha\theta x_i}{\theta+\alpha}e^{-\theta x_i}\right)}, \tag{30}$$

$$\frac{\partial\Psi}{\partial\lambda}=\sum_{i=1}^{n}\frac{2e^{-\theta x_i}+2\frac{\alpha\theta x_i}{\theta+\alpha}e^{-\theta x_i}-1}{1+\lambda-2\lambda\left(1-e^{-\theta x_i}-\frac{\alpha\theta x_i}{\theta+\alpha}e^{-\theta x_i}\right)}. \tag{31}$$

The maximum likelihood estimator $\hat{\alpha},\hat{\theta},\hat{\lambda}$ of $\alpha,\theta,\lambda$ can be obtained by equating the above nonlinear system to zero and $\frac{\partial\Psi}{\partial\theta}=0,$ , $\frac{\partial\Psi}{\partial\alpha}=0$, $\frac{\partial\Psi}{\partial\lambda}=0$ and solving these equations simultaneously.

## 7 Renyi entropy

The entropy of a random variable $X$ is a measure of variation of the uncertainty. A large entropy value indicates greater uncertainty in the data. The Renyi entropy is defined as

$$I_R(p)=\frac{1}{1-p}\log\left(\int_0^\infty f(x)^p dx\right), \tag{32}$$

where $p>0$ and $p\neq 0$. The Renyi entropy of the TTLD random variable $X$ is given in the following theorem.

**Theorem 4:** The Renyi entropy of the TTLD random variable $X$ is defined as

$$I_{TTLDR}(p)=\frac{1}{1-p}\log\left[\sum_{j=0}^{p}\sum_{i=0}^{p}\sum_{k=0}^{i}\binom{p}{j}\binom{p}{i}\binom{i}{k}\frac{(2\lambda)^{i}\alpha^{j+k}\theta^{2p-j-1}}{(1-\lambda)^{i-p}(\theta+\alpha)^{p+k}(p+i)^{j+k+1}}\Gamma(j+k+1)\right]. \tag{33}$$

**Proof:**

$$I_{TTLDR}(p)=\frac{1}{1-p}\log\int_{0}^{\infty}\left\{\left(\frac{\theta^2}{\theta+\alpha}(1+\alpha x)e^{-\theta x}\right)\left[(1+\lambda)-2\lambda\left(1-\frac{\theta+\alpha+\alpha\theta x}{\theta+\alpha}e^{-\theta x}\right)\right]\right\}^{p}dx$$

$$=\frac{1}{1-p}\log\left\{\left(\frac{\theta^2}{\theta+\alpha}\right)^{p}\int_{0}^{\infty}(1+\alpha x)^{p}e^{-p\theta x}(1-\lambda)^{p}\left[1+\frac{2\lambda}{(1-\lambda)}\left(1+\frac{\alpha\theta x}{\theta+\alpha}\right)e^{-\theta x}\right]^{p}dx\right\}.$$

By using the Binomial theorem $(a+b)^n=\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^{k}$, we have $(1+\alpha x)^p=\sum_{j=0}^{p}\binom{p}{j}(\alpha x)^j$, and

$$\left[1+\frac{2\lambda}{1-\lambda}\left(1+\frac{\alpha\theta x}{\theta+\alpha}\right)e^{-\theta x}\right]^{p}=\sum_{i=0}^{p}\binom{p}{i}\left(\frac{2\lambda}{1-\lambda}\right)^{i}\left(1+\frac{\alpha\theta x}{\theta+\alpha}\right)^{i}e^{-\theta i x}.$$

Therefore,

$$I_R(p)=\frac{1}{1-p}\log\left\{\left(\frac{\theta^2}{\theta+\alpha}\right)^{p}(1-\lambda)^{p}\int_{0}^{\infty}\left[\sum_{j=0}^{p}\binom{p}{j}(\alpha x)^{j}e^{-p\theta x}\right]\sum_{i=0}^{p}\binom{p}{i}\left(\frac{2\lambda}{1-\lambda}\right)^{i}\left(1+\frac{\alpha\theta x}{\theta+\alpha}\right)^{i}\left(e^{-\theta x}\right)^{i}dx\right\}$$

$$=\frac{1}{1-p}\log\left\{\left(\frac{\theta^2}{\theta+\alpha}\right)^{p}\sum_{j=0}^{p}\sum_{i=0}^{p}\left(\frac{2\lambda}{1-\lambda}\right)^{i}\binom{p}{j}\binom{p}{i}(1-\lambda)^{p}\alpha^{j}\int_{0}^{\infty}x^{j}e^{-p\theta x}\left(1+\frac{\alpha\theta x}{\theta+\alpha}\right)^{i}\left(e^{-\theta x}\right)^{i}dx\right\},$$

and

$$\left(1+\frac{\alpha\theta x}{\theta+\alpha}\right)^{i}=\sum_{k=0}^{i}\binom{i}{k}\left(\frac{\alpha\theta x}{\theta+\alpha}\right)^{k}.$$

Then,

$$I_R(p)=\frac{1}{1-p}\log\left\{\left(\frac{\theta^2}{\theta+\alpha}\right)^{p}\sum_{j=0}^{p}\sum_{i=0}^{p}\left(\frac{2\lambda}{1-\lambda}\right)^{i}\binom{p}{j}\binom{p}{i}(1-\lambda)^{p}\alpha^{j}\int_{0}^{\infty}x^{j}e^{-p\theta x}\sum_{k=0}^{i}\binom{i}{k}\left(\frac{\alpha\theta x}{\theta+\alpha}\right)^{k}\left(e^{-\theta x}\right)^{i}dx\right\}$$

$$=\frac{1}{1-p}\log\left\{\left(\frac{\theta^2}{\theta+\alpha}\right)^{p}\sum_{j=0}^{p}\sum_{i=0}^{p}\sum_{k=0}^{i}\frac{(2\lambda)^{i}}{(1-\lambda)^{i-p}}\binom{p}{j}\binom{p}{i}\binom{i}{k}\alpha^{j}\left(\frac{\alpha\theta}{\theta+\alpha}\right)^{k}\int_{0}^{\infty}x^{j+k}e^{-\theta x(p+i)}\,dx\right\}. \tag{34}$$

Now, let $u=\theta(p+i)x$ and $x=\frac{u}{\theta(p+i)}$, then $du=\theta(p+i)dx$ and $dx=\frac{du}{\theta(p+i)}$, and when $x=0,\infty$, then $u=0,\infty$, respectively. Hence,

$$\int_{0}^{\infty}x^{j+k}e^{-\theta x(p+i)}\,dx=\frac{1}{[\theta(p+i)]^{j+k+1}}\int_{0}^{\infty}u^{j+k}e^{-u}du=\frac{1}{[\theta(p+i)]^{j+k+1}}\Gamma(j+k+1).$$

Thus,

$$I_{TTLDR}(p)=\frac{1}{1-p}\log\left\{\sum_{j=0}^{p}\sum_{i=0}^{p}\sum_{k=0}^{i}\binom{p}{j}\binom{p}{i}\binom{i}{k}\frac{(2\lambda)^{i}\alpha^{j+k}\theta^{2p-j-1}}{(1-\lambda)^{i-p}(\theta+\alpha)^{p+k}(p+i)^{j+k+1}}\Gamma(j+k+1)\right\}.$$

## 8 An application

In this section, the usefulness of the TTL distribution is illustrated using the 72 exceedances for the years 1958-1984, rounded to one decimal place of flood peaks (in m3/s) of the Wheaton River (WR) near Carcross in Yukon Territory, Canada. The data is provided in Table 2 and its descriptive statistics are given in Table 3.

**Table 2:** 72 exceedances of Wheaton River flood data.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.4 | 2.2 | 14.4 | 20.6 | 0.7 | 12 | 1.9 | 1.7 | 13 | 1.1 | 9.3 | 5.3 |
| 11.6 | 18.7 | 8.5 | 14.1 | 1.1 | 1.7 | 2.5 | 1.4 | 14.4 | 25.5 | 37.6 | 22.1 |
| 15 | 2.2 | 39 | 11 | 22.9 | 1.1 | 1.7 | 0.6 | 0.1 | 0.3 | 0.6 | 7.3 |
| 0.4 | 1.7 | 7 | 2.8 | 9.9 | 30 | 10.4 | 9 | 10.7 | 20.1 | 3.6 | 14.1 |
| 25.5 | 30.8 | 13.3 | 3.4 | 21.5 | 2.7 | 27.6 | 5.6 | 36.4 | 4.2 | 64 | 11.9 |
| 27.1 | 2.5 | 27.4 | 20.2 | 5.3 | 2.5 | 9.7 | 1.5 | 27.5 | 1 | 27 | 16.8 |

**Table 3:** Descriptive statistics of the WR data.

| **Mean** | **Variance** | **Median** | **Skewsens** | **Kurtisos** |
|---|---|---|---|---|
| 12.204 | 151.222 | 9.5 | 1.473 | 5.89 |

The skewness value of the WR data is 1.473 and hence the distribution skewed to the left. The better distribution corresponds to smaller values of the criterion, Cramr-von Mises criterion $(W^*)$, Anderson-Darling criterion $(A^*)$, Bayesian information criterion (BIC), consistent Akaike information criterion (CAIC), Akaike information criterion (AIC), the maximized log-likelihood (MLL), and Hannan-Quinn information criterion (HQIC), where

$$\text{AIC} = -2MLL + 2w,\ \text{CAIC} = -2MLL + \frac{2wn}{n-w-1},$$

$$\text{BIC} = -2MLL + wLog(n),\ \text{HQIC} = 2Log\left\{Log(n)[w-2MLL]\right\},$$

where $w$ is the number of parameters and $n$ is the sample size. The values of these statistics are obtained for the WR data using the Lindley distribution (LD), two-parameter Lindley distribution (TLD), and the transmuted two-parameter Lindley distribution (TTLD) and the results presented in Table 4.

**Table 4:** The statistics AIC, CAIC, dIC, HQIC, $W^*$, $A^*$, K-S, and -2MLL for the WR data

| **Model** | **AIC** | **CAIC** | **BIC** | **HQIC'** | $W^*$ | $A^*$ | **K-S** | **-2MLL** |
|---|---|---|---|---|---|---|---|---|
| **LD** | 724.9826 | 725.0397 | 727.2593 | 725.889 | 0.7449 | 3.769413 | 0.8894 | 361.4913 |
| **TLD** | 540.8397 | 541.0136 | 545.3931 | 542.6524 | 2.2134 | 10.0858 | 0.9184 | 268.4199 |
| **TTLD** | **508.9973** | **509.3502** | **515.8273** | **511.7163** | **0.1379** | **0.7854** | **0.1052** | **251.4986** |

The MLEs of the models LD, TLD, and TTLD parameters are obtained, and the respective standard deviation and confidence intervals (CI) are obtained using the WR data.

**Table 5:** The MLEs of the parameters, and standard deviation for the WR data.

| **Model** | **Parameter** | **MLE** | **Std. Dev.** | **Inf.** 95% **CI** | **Sup.** 95% **CI** |
|---|---|---|---|---|---|
| **LD** | $\hat{\theta}_{LD}$ | 4.446613 | 0.293494 | 3.871375 | 5.0219 |
| **TLD** | $\hat{\theta}_{TLD}$ | 6.219696 | 0.509369 | 5.221352 | 7.218 |
| | $\hat{\alpha}_{TLD}$ | 93.470986 | 56.003077 | -16.293 | 203.235 |
| **TTLD** | $\hat{\theta}_{TTLD}$ | 0.296136 | 7.17526 | -13.7671 | 14.3594 |
| | $\hat{\alpha}_{TTLD}$ | 3.014284 | 81.047327 | -155.8356 | 161.8641 |
| | $\hat{\lambda}_{TTLD}$ | 0.901162 | 0.085557 | 0.7335 | 1.0689 |

The results in Tables 4 and 5 showed that the TTLD has the smallest values of the AIC, CAIC, BIC, HQIC, $W^*$, $A^*$, and K-S among all fitted distributions. therefore, the TTLD can be described as the best distribution for fitting the WR data.

## 9 Conclusions

Here a new distribution is suggested as a generalization of the two-parameter Lindley distribution, the so-called the transmuted two-parameter Lindley distribution. The reason for introducing the TTLD is to provide more flexibility distribution in modeling real data. Some statistical properties of the new distribution are derived such as the moments, the $r$th moment, the coefficient of skewness, kurtosis, variation and the distribution of order statistics. Also, the hazard rate and reliability functions, generation of random numbers, and the maximum likelihood estimation of the TTLD are defined as well as the Renyi entropy is proved. An application to real data set showed that the proposed distribution can be used effectively to better fit than the LD and TLD distributions.

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