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The Martingale Method for Probability of Ultimate Ruin Under Quota -(α,β) Reinsurance Model

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Abstract: In this paper, we consider a quota-(α,β) reinsurance model in discrete times with assumption that claim sequence $X=\{X_n\}_{n\geq 1}$, premium sequence $Y=\{Y_n\}_{n\geq 1}$ are sequence of independent and identically distributed random variables. Furthermore, the sequences $X=\{X_n\}_{n\geq 1}, Y=\{Y_n\}_{n\geq 1}$ are assumed to be independent. By martingale method we obtain some inequalities for ruin probability of the insurance company, ruin probability of the reinsurance company and joint ruin probability. Finally some numerical illustrations are given.

Keywords: Quota reinsurance, ruin probability, joint ruin probability, division ratio, Lundberg's inequality, supermartingale.

1 Introduction

Ruin probability is a main area in risk theory. Appeared in 1903 in the doctoral dissertation of Lundberg ,F [12].This topic, also of interest to many authors, appeared in the research of many mathematicians, see [1,3,11,13,14,16] where upper bounds were estimated for ruin probability having exponential form.

Martingale approach is used to estimate the ruin probability for such models as: i) the classical risk models, ii) the models of claim sizes and premiums with sequences of m-dependent random variables, iii) the modes of interest rates which can be found in the paper [4,5,18,19,7,8].

Besides finding methods to show the ruin probability, the authors also build risk models to reflect the insurance business increasingly more realistic. One of the models is Quota reinsurance, presented in the works, see [2,10,9].

In this paper, we investigate a Quota reinsurance model in discrete time that we shall call it Quota - (α,β) reinsurance or Quota -(α,β) share contract. It is a contract where the share proportion of the premium is different from that of the claim, defined as follows:

The quota - (α,β) reinsurance or quota - (α,β) share contract is a contract between the insurance company and the reinsurer to share premium $Y=\{Y_i\}_{i\geq 1}$ with a fixed proportion $\alpha\in[0,1]$. Where Y_i is the premium at the end of the $i-$ period, $i=1,2,\ldots$ (paid by insured to the insurance company) and losses (claim sizes) $X=\{X_i\}_{i\geq 1}$ with a fixed proportion $\beta\in[0,1]$. Where X_i is the loss at the end of the $i-$ period, $i=1,2,\ldots$(claimed by the insured).

Premium Y_i: αY_i kept by the cedant; $(1-\alpha)Y_i$ transfered to the reinsurer.

Claim loss X_i: βX_i paid by the cedant; $(1-\beta)X_i$ paid by the reinsurer.

$\alpha,\beta\in[0,1]$ are called *division ratios*.

When the surplus process of the insurance company is a sequence of random variables $\{U_n\}_{n\geq 1}$ defined by:

$$U_n=u+\alpha\sum_{i=1}^{n}Y_i-\beta\sum_{i=1}^{n}X_i, \tag{1.1}$$

where u denote the insurance company's initial capital.

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$\mathbb{P}\left[\bigcup_{i=1}^{t}\left(U_i \le 0\right)\right]$ is called *probability of ruin within finite time t*, denoted by $\psi^{(1)}(u,t)$:

$$\psi^{(1)}(u,t) = \mathbb{P}\left[\bigcup_{i=1}^{t}\left(U_i \le 0\right)\right], \tag{1.2}$$

$\mathbb{P}\left[\bigcup_{i=1}^{\infty}\left(U_i \le 0\right)\right]$ is *probability of ultimate ruin*, denoted by $\psi^{(1)}(u)$ and defined as:

$$\psi^{(1)}(u) = \mathbb{P}\left[\bigcup_{i=1}^{\infty}\left(U_i \le 0\right)\right]. \tag{1.3}$$

Similarly, we define the surplus process of the reinsurance company, the probability of ruin within finite time t and the probability ultimate ruin, respectively, by $\{V_n\}_{n\ge 1}, \psi^{(2)}(v,t), \psi^{(2)}(v)$:

$$V_n = v + (1-\alpha)\sum_{i=1}^{n} Y_i - (1-\beta)\sum_{i=1}^{n} X_i, \tag{1.4}$$

$$\psi^{(2)}(v,t) = \mathbb{P}\left[\bigcup_{i=1}^{t}\left(V_i \le 0\right)\right], \tag{1.5}$$

$$\psi^{(2)}(v) = \mathbb{P}\left[\bigcup_{i=1}^{\infty}\left(V_i \le 0\right)\right], \tag{1.6}$$

where v denote the reinsurance company's initial capital.
Evidently: $\lim\limits_{t\to\infty}\psi^{(1)}(u,t) = \psi^{(1)}(u); \lim\limits_{t\to\infty}\psi^{(2)}(v,t) = \psi^{(2)}(v)$.

In the classic model, see [2,10] where α,β are considered equal. In this paper we investigate genralized model with α and β are any constant in $[0,1]$ satisfying condition (1.9).

Recently, some authors have studied the joint ruin probability of insurance and reinsurance companies, see [15,9,20]. We define the finite time t and ultimate joint ruin probabilities, respectively, by $\psi(u,v,t), \psi(u,v)$:

$$\psi(u,v,t) = \mathbb{P}\left\{\bigcup_{i=1}^{t}\left[\left(U_i \le 0\right)\left(V_i \le 0\right)\right]\right\} \tag{1.7}$$

$$\psi(u,v) = \mathbb{P}\left\{\bigcup_{i=1}^{\infty}\left[\left(U_i \le 0\right)\left(V_i \le 0\right)\right]\right\} \tag{1.8}$$

and we have $\lim\limits_{t\to\infty}\psi(u,v,t) = \psi(u,v)$.

In this paper, we consider premium calculation principle based on the expected value principle. Hence, we have:

$$\begin{cases} \mathbb{E}(X_1) < \mathbb{E}(Y_1) \\ \beta\mathbb{E}(X_1) < \alpha\mathbb{E}(Y_1) \\ (1-\beta)\mathbb{E}(X_1) < (1-\alpha)\mathbb{E}(Y_1) \end{cases} \Leftrightarrow \begin{cases} \mathbb{E}(X_1) < \mathbb{E}(Y_1) \\ \beta\frac{\mathbb{E}(X_1)}{\mathbb{E}(Y_1)} < \alpha < \beta\frac{\mathbb{E}(X_1)}{\mathbb{E}(Y_1)} + 1 - \frac{\mathbb{E}(X_1)}{\mathbb{E}(Y_1)} \end{cases} \tag{1.9}$$

This condition allows us to calculate α,β.

In the next, we shall work with the models (1.1) and (1.4) under the following assumptions:

Assumption 1.1. Initial capitals $u > 0, v > 0$.
Assumption 1.2. Let $\alpha,\beta \in [0,1]$ satisfy condition (1.9).
Assumption 1.3. Let $X = \left(X_i\right)_{i\ge 1}$ and $Y = \left(Y_i\right)_{i\ge 1}$ be two sequences of independent and identically distributed (i.i.d), nonnegative random variables defined on the same probability space, $\left(\Omega,\mathscr{F},\mathbb{P}\right)$. Furthermore, $\left(X_i\right)_{i\ge 1}$ and $\left(Y_i\right)_{i\ge 1}$ are assumed to be independents.

The paper is organized as follows: In section 2, first we show Lemma 2.1 about existence and uniqueness of an adjustment coefficient, Lemma 2.2 is used to prove $\left\{Z_n = exp(S_n)\right\}_{n\ge 1}$ to be supermatingale, Lemma 2.3 is maximal inequality for nonnegative supermartingale. Theorem 2.1 will present probability inequality of the insurance company. Similarly to Section 2, Section 3 is dedicated to show probability inequality of the reinsurance company and to derive some probability inequalities for the joint ruin probability. Finally, a numerical example is given to illustrate $\psi^{(1)}(u), \psi^{(2)}(v), \psi(u,v)$ in Section 4.

2 Probability inequalities for ruin probability of the insurance company

Firstly, we present three lemmas, which are necessary for the proof of our main result in Theorem (2.1).

Lemma 2.1. *Suppose that*

$$\alpha\mathbb{E}(Y_1) > \beta\mathbb{E}(X_1) \text{ and } \mathbb{P}\big[(\beta X_1 - \alpha Y_1) > 0\big] > 0, \tag{2.1}$$

then there exists a positive constant $R_0 > 0$ which is the unique root of the equation

$$\mathbb{E}\big\{exp\big[R(\beta X_1 - \alpha\, Y_1)\big]\big\} = 1. \tag{2.2}$$

For all $0 < R_1 \le R_0$, we have

$$\mathbb{E}\big\{exp\big[R_1(\beta X_1 - \alpha\, Y_1)\big]\big\} \le 1. \tag{2.3}$$

Proof. Consider the function $g(R) = \mathbb{E}\big\{exp\big[R(\beta X_1 - \alpha\, Y_1)\big]\big\} - 1$ for $R \in [0,\infty)$.
We have

$$g(0) = 0,$$
$$g'(R) = \mathbb{E}\big\{(\beta X_1 - \alpha Y_1)exp\big[R(\beta X_1 - \alpha\, Y_1)\big]\big\},$$

By assumption of Lemma 2.1 then $g'(0) = \mathbb{E}(\beta X_1 - \alpha Y_1) < 0$. Thus, the function $g(R)$ is decreasing at 0. Further, any turning point of the function is a minimum since

$$g''(R) = \mathbb{E}\left\{(\beta X_1 - \alpha Y_1)^2 exp\big[R(\beta X_1 - \alpha Y_1)\big]\right\} > 0.$$

By $\mathbb{P}(\beta X_1 - \alpha Y_1 > 0) > 0$, we can find some constant $\delta_1 > 0$ such that $\mathbb{P}(\beta X_1 - \alpha Y_1 > \delta_1) > 0$.
We use the property that satisfying if Z is random variables satisfying $Z \ge 0$ then $Z \ge Z.1_A$ with any $A \in \mathscr{F}$

$$\begin{aligned} g(R) &\ge \mathbb{E}\left\{exp\big[R(\beta X_1 - \alpha Y_1)\big].1_{(\beta X_1 - \alpha Y_1 > \delta_1)}\right\} - 1 \\ &\ge \mathbb{E}\big[exp(R.\delta_1).1_{(\beta X_1 - \alpha Y_1 > \delta_1)}\big] - 1 \\ &= exp(R.\delta_1).\mathbb{E}\big[1_{(\beta X_1 - \alpha Y_1 > \delta_1)}\big] - 1 \\ &= exp(R.\delta_1).\mathbb{P}\big(\beta X_1 - \alpha Y_1 > \delta_1\big) - 1 \end{aligned} \tag{2.4}$$

The right- hand of (2.4) $\to \infty$ when $R \to \infty$. Thus $\lim\limits_{R\to\infty} g(R) = \infty$.
Consequently, there exists a unique positive root of the equation (2.2), denoted by R_0.
Since, $g(R)$ is a convex function with $R \in [0,\infty)$ and $g(0) = 0, g(R_0) = 0$. If $0 < R_1 \le R_0$ then $g(R_1) \le 0$ which is equivalent to

$$\mathbb{E}\big\{exp\big[R_1(\beta X_1 - \alpha\, Y_1)\big]\big\} \le 1.$$

The Lemma 2.1 has been proven.

Lemma 2.2. *(see, [6] page 201) In the probabilistic space $(\Omega,\mathscr{F},\mathbb{P})$, let $U = \big(U_1,U_2,...,U_m\big), V = \big(V_1,V_2,...,V_n\big)$ be random vectors, independent of each other, and f be a Borel's function on $\mathbb{R}^m \times \mathbb{R}^n$ where $|\mathbb{E} f(U,V)| \le \infty$. If $u \in \mathbb{R}^m$, the function g is defined by*

$$g(u) = \begin{cases} \mathbb{E} f(u,V) & \text{if } |\mathbb{E} f(u,V)| \le \infty \\ 0 & \text{other} \end{cases} \tag{2.5}$$

then g is Borel's function on $\mathbb{R}^m$ with $g(U) = \mathbb{E}\big[f(U,V)|\sigma(U)\big]$, where $\sigma(U)$ is $\sigma-$ algebra generated by vector U. In other words, under the assumption we can compute $\mathbb{E}\big[f(U,V)|\sigma(U)\big]$ as if U was a constant vector.

Lemma 2.3. *(see, [17] page 493) Let $Y = (Y_n,\mathscr{F}_n)_{n\ge 0}$ be a nonnegative supermartingale. Then for all $\lambda > 0$*

$$\lambda\mathbb{P}\left\{\max_{k\le n} Y_k \ge \lambda\right\} \le \mathbb{E}Y_0, \tag{2.6}$$

$$\lambda\mathbb{P}\left\{\sup_{k\ge n} Y_k \ge \lambda\right\} \le \mathbb{E}Y_n. \tag{2.7}$$

The following theorem will give upper bounds for the ruin probability of the insurance company.
We put:

$$S_n = \beta \sum_{i=1}^{n} X_i - \alpha \sum_{i=1}^{n} Y_i. \tag{2.8}$$

Theorem 2.1. *We consider model (1.1) such that:*
i) The assumptions 1.1-1.3 are satisfied;
ii) The conditions of Lemma (2.1) are also satisfied.
Then:
a) $\{Z_n = exp(R_1 S_n)\}_{n\geq 1}$ *is a supermartingale;*
b) The ruin probability of the insurance company within finite time $\psi^{(1)}(u,t) \leq exp(-R_1 u)$*;*
c) The ultimate ruin probability of the insurance company $\psi^{(1)}(u) \leq exp(-R_1 u)$.

Proof. It follows immediately from result of Lemma 2.1 that there exists a constant R_1 $(0 < R_1 \leq R_0)$, such that:

$$\mathbb{E}\{exp[R_1(\beta X_1 - \alpha Y_1)]\} \leq 1. \tag{2.9}$$

We have

$$\begin{aligned}
\mathbb{E}\big[Z_{n+1}|Z_1,Z_2,...,Z_n\big] &= \mathbb{E}\big\{exp\big[R_1(\beta \sum_{i=1}^{n+1} X_i - \alpha \sum_{i=1}^{n+1} Y_i)\big]|Z_1,Z_2,...,Z_n\big\} \\
&= \mathbb{E}\big\{Z_n exp\big[R_1(\beta X_{n+1} - \alpha Y_{n+1})\big]|Z_1,Z_2,...,Z_n\big\} \\
&= Z_n \mathbb{E}\big\{exp\big[R_1(\beta X_{n+1} - \alpha Y_{n+1})\big]|Z_1,Z_2,...,Z_n\big\}.
\end{aligned} \tag{2.10}$$

We now apply the Lemma 2.2 for the case

$$\begin{aligned}
f(U,V) &= exp\big[R_1(\beta X_{n+1} - \alpha Y_{n+1})\big], \\
U &= (X_1,...,X_n,Y_1,...,Y_n), \\
V &= (X_{n+1},Y_{n+1}),
\end{aligned}$$

where U,V are mutually independent.
Corresponding to the value

$$(X_1 = x_1,...,X_n = x_n, Y_1 = y_1,...,Y_n = y_n),$$

$$u = (x_1,...,x_n,y_1,...,y_n),$$

we consider

$$g(u) = \mathbb{E}\Big\{exp\big[R_1(\beta X_{n+1} - \alpha Y_{n+1})\big]\Big\} = \mathbb{E}\Big\{exp\big[R_1(\beta X_1 - \alpha Y_1)\big]\Big\},$$

by assumption

$$\mathbb{E}\Big\{exp\big[R_1(\beta X_1 - \alpha Y_1)\big]\Big\} \leq 1,$$

consequently

$$g(U) = \mathbb{E}\Big\{exp\big[R_1(\beta X_{n+1} - \alpha Y_{n+1})\big]|X_1,...,X_n,Y_1,...,Y_n\Big\} \leq 1.$$

In the other hand

$$g(U) = \mathbb{E}\Big\{exp\big[R_1(\beta X_{n+1} - \alpha Y_{n+1})\big]|Z_1,Z_2,...,Z_n\Big\} \leq 1. \tag{2.11}$$

Combining (2.10) and (2.11) imply that

$$\mathbb{E}(Z_{n+1}|Z_1,Z_2,...,Z_n) \leq Z_n \text{ for all } n \geq 1. \tag{2.12}$$

So $\{Z_n\}_{n\geq 1} = \{exp(R_1 S_n)\}_{n\geq 1}$ is non- negative supermartingale.
In addition,

$$\psi^{(1)}(u,t) = \mathbb{P}\left[\bigcup_{n=1}^{t}(S_n \geq u)\right] = \mathbb{P}\left[\max_{1\leq k\leq t}(S_k) \geq u\right] = \mathbb{P}\left\{\max_{1\leq k\leq t}\left[exp(R_1 S_k)\right] \geq exp(R_1 u)\right\}. \tag{2.13}$$

Applying Lemma 2.3 for non- negative supermatingale $\{Z_n\}_{n\geq 1} = \{exp(R_1 S_n)\}_{n\geq 1}$, we have:

$$\psi^{(1)}(u,t) \leq \left\{exp(-R_1 u).\mathbb{E}(Z_1)\right\} = \left\{exp(-R_1 u).\mathbb{E}\left(exp\left[R_1(\beta X_1 - \alpha Y_1)\right]\right)\right\} \leq exp(-R_1 u). \tag{2.14}$$

By letting $t \to \infty$ in (2.14). Thus,

$$\lim_{t\to\infty}\psi^{(1)}(u,t) \leq \lim_{t\to\infty}(exp(-R_1 u)) \Leftrightarrow \psi^{(1)}(u) \leq exp(-R_1 u). \tag{2.15}$$

This completes the proof.

Remark 2.1:

–Our result seems to be new even is the case of clasical model when $\alpha = \beta$;
–With $0 < R_1 \leq R_0$ then the right-hand side of (2.14) and (2.15) are the smallest when $R_1 = R_0$.

Similar to the Section 2, we derive inequality for ruin probability of the reinsurance company in the Section 3 following.

3 Probability inequalities for ruin probability of the reinsurance company

We put:

$$S'_n = (1-\beta)\sum_{i=1}^{n} X_i - (1-\alpha)\sum_{i=1}^{n} Y_i. \tag{3.1}$$

We need Lemma 3.1 for the proof of Theorem 3.1 below.

Lemma 3.1. *Under the conditions*

$$(1-\alpha)\mathbb{E}(Y_1) > (1-\beta)\mathbb{E}(X_1) \text{ and } \mathbb{P}\left\{[(1-\beta)X_1 - (1-\alpha)Y_1] > 0\right\} > 0, \tag{3.2}$$

then there exists a positive constant $R'_0 > 0$ which is the unique root of the equation

$$\mathbb{E}\left\{exp\left[R\left((1-\beta)X_1 - (1-\alpha)Y_1\right)\right]\right\} = 1. \tag{3.3}$$

For all $0 < R_2 \leq R'_0$, we have

$$\mathbb{E}\left\{exp\left[R_2\left((1-\beta)X_1 - (1-\alpha)\,Y_1\right)\right]\right\} \leq 1. \tag{3.4}$$

The proof of Lemma 3.1 is similar as Lemma 2.1.
Now we state theorem below for the case of reinsurance company.

Theorem 3.1. *We consider model (1.4) such that:*
i) The assumptions 1.1-1.3 are satisfied;
ii) The conditions of Lemma (3.1) are also satisfied.
Then:
a) $\left\{Z'_n = exp(R_2 S'_n)\right\}_{n\geq 1}$ is a supermartingale;
b) The ruin probability of the reinsurance company within finite time $\psi^{(2)}(v,t) \leq exp(-R_2 v)$;
c) The ultimate ruin probability of the reinsurance company $\psi^{(2)}(v) \leq exp(-R_2 v)$.

The proof of Theorem 3.1 is similar to that of Theorem 2.1.
Remark 3.1
With assumptions of Theorem 2.1, Theorem 3.1 we have the probability inequality for ultimate joint ruin probability. Indeed:

$$\begin{aligned}
\psi(u,v) &= \mathbb{P}\left(\bigcup_{i=1}^{\infty}(U_i \leq 0) \cup \bigcup_{i=1}^{\infty}(V_i \leq 0)\right) \\
&\leq \mathbb{P}\left(\bigcup_{i=1}^{\infty}(U_i \leq 0)\right) + \mathbb{P}\left(\bigcup_{i=1}^{\infty}(V_i \leq 0)\right) \\
&\leq exp(-R_1 u) + exp(-R_2 v).
\end{aligned} \tag{3.5}$$

The inequality (3.5) is corollary of Theorem 2.1 and Theorem 3.1.

4 Numerical illustrations

In this Section, we provide two numerical illustration examples for the probability inequality of $\psi^{(1)}(u), \psi^{(2)}(v)$ and $\psi(u,v)$, with $\alpha = 0.4900, \beta = 0.5000$.

Example 4.1

Let $Y = \{Y_n\}_{n>0}$ be a sequence of independent and identically distributed random variables, with Y_1 having a distribution:

Y_1	1	4
$\mathbb{P}$	0.5500	0.4500

Let $X = \{X_n\}_{n>0}$ be a sequence of independent and identically distributed random variables, with X_1 having distribution:

X_1	2	3
$\mathbb{P}$	0.7000	0.3000

When $\beta X_1 - \alpha Y_1$ having distribution:

$\beta X_1 - \alpha Y_1$	0.5100	1.0100	-0.9600	-0.4600
$\mathbb{P}$	0.3850	0.1650	0.3150	0.1350

To (2.2) we have the equation :

$$exp(r*0.5100)*0.3850+exp(r*1.0100)*0.1650+exp(-r*0.9600)*0.3150 \\ +exp(-r*0.4600)*0.1350=1. \tag{4.1}$$

By Maple the equation (4.1) has a roof $R_0 = 0.0051$.
Similarly, we have distribution of $(1-\beta)X_1 - (1-\alpha)Y_1$

$(1-\beta)X_1 - (1-\alpha)Y_1$	0.4900	0.9900	-1.0400	-0.5400
$\mathbb{P}$	0.3850	0.1650	0.3150	0.1350

To the equation (3.3) gives the equation:

$$exp(r*0.4900)*0.3850+exp(r*0.9900)*0.1650+exp(-r*1.0400)*0.3150 \\ +exp(-r*0.5400)*0.1350=1. \tag{4.2}$$

By Maple we find $R_0' = 0.1548$ is roof of the equation (4.2).
The following table shows upper bounds $\psi^{(1)}(u), \psi^{(2)}(v), \psi(u,v)$ for a range of value of u, v:

Table 1: Upper bounds of $\psi^{(1)}(u), \psi^{(2)}(v)$ and $\psi(u,v)$ with X_1 and Y_1 are discrete random variables.

(u,v)	$\psi^{(1)}(u)$	$\psi^{(2)}(v)$	$\psi(u,v)$
$u=30; v=30$	0.8581	0.0096	0.8677
$u=1550; v=50$	0.0004	0.0004	0.0008
$u=70; v=70$	0.6998	0.0000	0.6998

Example 4.2

Let $Y=\{Y_n\}_{n>0}$ be a sequence of independent and identically distributed random variables, with Y_1 is constant $c=1.1$. $X=\{X_n\}_{n>0}$ be a sequence of independent and identically distributed random variables, with X_1 has the exponential distribution:

$$f(x)=\begin{cases}\lambda exp(-\lambda x) & \text{if } x>0\\ 0 & \text{other } x\leq 0\end{cases} \tag{4.3}$$

where $\lambda=1$. $f(x)$ is density function of X_1

To condition of (2.2), we have equation

$$\int_0^{+\infty} exp\big[r(0.5000x-0.4900*1.1000)-x\big]dx=1$$

$$\Leftrightarrow exp(-r*0.5390)-1+r*0.5000=0. \tag{4.4}$$

Use of Maple software, we find a solution $R_0=0.5927$ of (4.4).

Similarly, to condition of (3.3), we have equqtion

$$\int_0^{+\infty} exp\big[r(0.5000x-0.5100*1.1000)-x\big]dx=1$$

$$\Leftrightarrow exp(-r*0.5610)-1+r*0.5000=0. \tag{4.5}$$

and find $R'_0=0.4185$.

When upper bounds $\psi^{(1)}(u), \psi^{(2)}(v), \psi(u,v)$ for a range of value of u,v.

Table 2: Upper bounds of $\psi^{(1)}(u), \psi^{(2)}(v)$ and $\psi(u,v)$ with $Y_1=c$ and X_1 is continuous random variable.

(u,v)	$\psi^{(1)}(u)$	$\psi^{(2)}(v)$	$\psi(u,v)$
$u=8; v=8$	0.0087	0.0352	0.0439
$u=10; v=14.1326$	0.0027	0.0027	0.0054
$u=15; v=15$	0.0001	0.0019	0.0020

5 Conclusions

This paper constructed upper bounds for $\psi^{(1)}(u,t), \psi^{(1)}(u), \psi^{(2)}(v,t)$, and $\psi^{(2)}(v)$ in model (1.1) and (1.4) by the martingale method. Our main results in this paper not olny prove Theorem 2.1 and Theorem 3.1 but also give numerical examples to illustrate for Theorem 2.1 and Theorem 3.1.

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