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A New Class of Meromorphic Multivalent Functions Defined by Linear Operator

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Abstract: In the present paper, we introduce a new class of meromorphic multivalent functions on the punctured unit disk $\mathbb{U}^* := \{z \in \mathbb{C} : 0 < |z| < 1\}$. We obtain some geometric properties like coefficient inequality, linear combination, extreme points, growth and distortion theorems, δ- neighborhoods, partial sum, weighted mean, arithmetic mean and radii of starlikeness and convexity for the function class $\mathscr{B}_{\lambda,p}^{\alpha,\nu}(\gamma,\beta)$.

Keywords: Meromorphic functions, coefficient inequality, linear combination, extreme points, growth and distortion theorems, arithmetic mean, closure theorem

1 Introduction and Definition

Let $\sum_p$ be the class of functions of the form:

$$f(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} a_k z^{k-p} \quad (p \in \mathbb{N} := \{1,2,3,\cdots\}) \tag{1}$$

which are analytic and p-valent in the punctured unit disk

$$\mathbb{U}^* = \{z \in \mathbb{C} : 0 < |z| < 1\} = \mathbb{U} \setminus \{0\},$$

where $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ is the open unit disk.

Consider a subclass $\mathscr{T}_p$ of functions of the form:

$$f(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} a_k z^{k-p} \quad (a_k \geq 0). \tag{2}$$

For functions $f \in \mathscr{T}_p$ given by (2) and $g \in \mathscr{T}_p$ given by

$$g(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} b_k z^{k-p} \quad (z \in \mathbb{U}^*, b_k \geq 0), \tag{3}$$

we define $f * g$ by

$$\begin{aligned}(f * g)(z) =& \frac{z^p f(z) \star z^p g(z)}{z^p} \\ =& \frac{1}{z^p} + \sum_{k=1}^{\infty} a_k b_k z^{k-p} = (g * f)(z) \quad (z \in \mathbb{U}^*)\end{aligned} \tag{4}$$

where $\star$ denote the usual Hadamard product (or convolution) of analytic functions.

A function f of the form (2) is said to be in the class $\sum_p^*(\delta)$ of meromorphic p-valently starlike functions of order δ in $\mathbb{U}^*$ if and only if

$$\Re\left[-\frac{zf'(z)}{f(z)}\right] > \delta \quad (z \in \mathbb{U}^*;\ 0 \leq \delta < p;\ p \in \mathbb{N}),$$

and is in the class of meromorphically convex of order δ denoted by $\sum_p^k(\delta)$ if and only if

$$-\Re\left[1 + \frac{zf''(z)}{f'(z)}\right] > \delta \quad (z \in \mathbb{U}^*;\ 0 \leq \delta < p;\ p \in \mathbb{N}).$$

El-Ashwah [8] defined the linear operator as

$$\mathscr{I}_p^m(\lambda,l)f(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty}\left(\frac{l+\lambda k}{l}\right)^m a_k z^{k-p}$$
$$(\lambda \geq 0, l > 0, m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, z \in \mathbb{U}^*). \tag{5}$$

By specializing the parameters λ, l and p, we obtain the following operators studied earlier by various researchers. For

- $p = \lambda = 1$, the operator $\mathscr{I}_1^m(1,l) = \mathscr{I}(m,l)$ has been studied in [6,7];

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- $\lambda = l = 1$, the operator $\mathscr{I}_p^m(1,1) = \mathscr{D}_p^m$ has been studied in [5,10,12];
- $p = l = \lambda = 1$, the operator $\mathscr{I}_1^m(1,1) = \mathscr{I}^m$ has been studied in [13];
- $p = l = 1$, the operator $\mathscr{I}_1^m(\lambda,1) = \mathscr{D}_\lambda^m$ has been studied in [1].

Set

$$\phi_p^m(\lambda,l;z) = \frac{1}{z^p} + \sum_{k=1}^{\infty}\left(\frac{l+\lambda k}{l}\right)^m z^{k-p}. \qquad (6)$$

Corresponding to the function $\phi_p^m(\lambda,l;z)$, let us define the function $\phi_{p,\alpha}^{m,\dagger}(\lambda,l;z)$, the generalized multiplicative inverse of $\phi_p^m(\lambda,l;z)$ given by the relation

$$\phi_p^m(\lambda,l;z) * \phi_{p,\alpha}^{m,\dagger}(\lambda,l;z) = \frac{1}{z^p(1-z)^{\alpha+p}} \quad (\alpha > -p; z \in \mathbb{U}^*).$$

Note that if $\alpha = -p+1$, then $\phi_{p,\alpha}^{m,\dagger}(\lambda,l;z)$ is the inverse of $\phi_p^m(\lambda,l;z)$ with respect to the Hadamard product $*$. Using this function we define the following family of transforms $\mathscr{I}_{p,\alpha}^m(\lambda,l)$ defined by

$$\mathscr{I}_{p,\alpha}^m(\lambda,l)f(z) = \phi_{p,\alpha}^{m,\dagger}(\lambda,l;z) * f(z)$$
$$= \frac{1}{z^p} + \sum_{k=1}^{\infty}\frac{(\alpha+p)_k}{(1)_k}\left(\frac{l}{l+\lambda k}\right)^m a_k z^{k-p}$$
$$(\alpha > -p, l > 0, \lambda \geq 0, m \in \mathbb{N}_0; z \in \mathbb{U}^*), \qquad (7)$$

where $f \in \mathscr{T}_p$ is in the form (2) and $(\beta)_n$ denotes the Pochhammer symbol given by

$$(\beta)_k = \frac{\Gamma(\beta+k)}{\Gamma(\beta)} = \begin{cases} 1 & (k=0) \\ \beta(\beta+1)...(\beta+k-1) & (k \in \mathbb{N}). \end{cases}$$

Using the operator $\mathscr{I}_{p,\alpha}^m(\lambda,l)$ we define the subclass of $\mathscr{T}_p$ as follows:

Definition 1.1. A function $f \in \mathscr{T}_p$ given by (2) is said to be in the class $\mathscr{B}_{\lambda,p}^{\alpha,\nu}(\gamma,\beta)$ if it satisfies the inequality

$$\left|\frac{z^{p+2}\left(\mathscr{I}_{p,\alpha}^m(\lambda,l)f(z)\right)'' + z^{p+1}\left(\mathscr{I}_{p,\alpha}^m(\lambda,l)f(z)\right)' - p^2}{\nu z^{p+1}(\mathscr{I}_{p,\alpha}^m(\lambda,l)f(z))' + \gamma(1+\nu)p - p}\right| < \beta,$$

$$(0 \leq \gamma < 1, 0 < \beta \leq 1, \alpha > -p, 0 < \nu \leq 1, p \in \mathbb{N}; z \in \mathbb{U}). \qquad (8)$$

The object of the present paper is to obtain some basic geometric properties of the function class $\mathscr{B}_{\lambda,p}^{\alpha,\nu}(\gamma,\beta)$ such as coefficient inequality, linear combination, extreme points, growth and distortion theorems, δ-neighborhoods and partial sums, arithmetic mean, weighted mean, closure and radii of starlikeness and convexity.

2 Coefficient Inequality

In the following theorem, we obtain the necessary and sufficient conditions for a function $f \in \mathscr{T}_p$ to be in the class $\mathscr{B}_{\lambda,p}^{\alpha,\nu}(\gamma,\beta)$.

Theorem 2.1. Let $f \in \mathscr{T}_p$ be given by (2). Then $f \in \mathscr{B}_{\lambda,p}^{\alpha,\nu}(\gamma,\beta)$ if and only if

$$\sum_{k=1}^{\infty}\left(\frac{l}{l+\lambda k}\right)^m \frac{(\alpha+p)_k}{(1)_k}(k-p)\left[(k-p)+\beta\nu\right]a_k \leq \beta p(1-\gamma)(1+\nu)$$
$$(0 \leq \gamma < 1, 0 < \beta \leq 1, \alpha > -p, 0 < \nu \leq 1, p \in \mathbb{N}). \qquad (9)$$

The result is sharp for the function

$$f(z) = \frac{1}{z^p} + \frac{\beta p(1-\gamma)(1+\nu)}{\left(\frac{l}{l+\lambda k}\right)^m \frac{(\alpha+p)_k}{(1)_k}(k-p)[k-p+\beta\nu]} z^{k-p} \quad (k \geq 1). \qquad (10)$$

Proof. Assume that the inequality (9) holds true and let $|z| = 1$. Then from (8) we have

$$\left|z^{p+2}(\mathscr{I}_{p,\alpha}^m(\lambda,l)f(z))'' + z^{p+1}(\mathscr{I}_{p,\alpha}^m(\lambda,l))' - p^2\right|$$
$$-\beta\left|\nu z^{p+1}(\mathscr{I}_{p,\alpha}^m(\lambda,l)f(z))' + \gamma(1+\nu)p - p\right|$$
$$= \left|\sum_{k=1}^{\infty}\left(\frac{l}{l+\lambda k}\right)^m \frac{(\alpha+p)_k}{(1)_k}(k-p)^2 a_k z^k\right|$$
$$-\beta\left|p(1-\gamma)(1+\nu) - \nu\sum_{k=1}^{\infty}\left(\frac{l}{l+\lambda k}\right)^m \frac{(\alpha+p)_k}{(1)_k}(k-p)a_k z^k\right|$$
$$\leq \sum_{k=1}^{\infty}\left(\frac{l}{l+\lambda k}\right)^m \frac{(\alpha+p)_k}{(1)_k}(k-p)[(k-p)+\beta\nu]a_k$$
$$-\beta p(1-\gamma)(1+\nu) \leq 0,$$

by virtue of (9). Hence, by the principle of maximum modulus, $f \in \mathscr{B}_{\lambda,p}^{\alpha,\nu}(\gamma,\beta)$.

Conversely, let $f \in \mathscr{B}_{\lambda,p}^{\alpha,\nu}(\gamma,\beta)$. Then

$$\left|\frac{z^{p+2}(\mathscr{I}_{p,\alpha}^m(\lambda,l)f(z))'' + z^{p+1}(\mathscr{I}_{p,\alpha}^m(\lambda,l)f(z))' - p^2}{\nu z^{p+1}(\mathscr{I}_{p,\alpha}^m(\lambda,l)f(z))' + \gamma(1+\nu)p - p}\right|$$
$$= \left|\frac{\sum_{k=1}^{\infty}\left(\frac{l}{l+\lambda k}\right)^m \frac{(\alpha+p)_k}{(1)_k}(k-p)^2 a_k z^k}{p(1-\gamma)(1+\nu) - \nu\sum_{k=1}^{\infty}\left(\frac{l}{l+\lambda k}\right)^m \frac{(\alpha+p)_k}{(1)_k}(k-p)a_k z^k}\right| < \beta.$$

Since $\Re(z) < |z|$ for all z, we have

$$\Re\left\{\frac{\sum_{k=1}^{\infty}\left(\frac{l}{l+\lambda k}\right)^m \frac{(\alpha+p)_k}{(1)_k}(k-p)^2 a_k z^k}{p(1-\gamma)(1+\nu) - \nu\sum_{k=1}^{\infty}\left(\frac{l}{l+\lambda k}\right)^m \frac{(\alpha+p)_k}{(1)_k}(k-p)a_k z^k}\right\} < \beta. \qquad (11)$$

We can choose the value of z on the real axis so that $z^{p+2}(\mathscr{I}_{p,\alpha}^m(\lambda,l)f(z))''$ and $z^{p+1}(\mathscr{I}_{p,\alpha}^m(\lambda,l)f(z))'$ are real. Let $z \longrightarrow 1^-$ through real values. Therefore, from (11), we obtain

$$\frac{\sum_{k=1}^{\infty}\left(\frac{l}{l+\lambda k}\right)^m \frac{(\alpha+p)_k}{(1)_k}(k-p)^2 a_k}{p(1-\gamma)(1+\nu) - \nu\sum_{k=1}^{\infty}\left(\frac{l}{l+\lambda k}\right)^m \frac{(\alpha+p)_k}{(1)_k}(k-p)a_k} < \beta,$$

which implies that

$$\sum_{k=1}^{\infty}\left(\frac{l}{l+\lambda k}\right)^m \frac{(\alpha+p)_k}{(1)_k}(k-p)[k-p+\beta\nu]a_k \leq \beta p(1-\gamma)(1+\nu),$$

which proves the inequality (9).
Sharpness follows if we take

$$f(z)=\frac{1}{z^p}+\frac{\beta p(1-\gamma)(1+\nu)}{\left(\frac{l}{l+\lambda k}\right)^m \frac{(\alpha+p)_k}{(1)_k}(k-p)[k-p+\beta\nu]}z^{k-p} \quad (k\geq 1).$$

The proof of Theorem 2.1 is thus completed. □

As an application of Theorem 2.1, we obtain the following:

Corollary 2.2. Let $f\in\mathscr{B}_{\lambda,p}^{\alpha,\nu}(\gamma,\beta)$. Then

$$a_k \leq \frac{\beta p(1-\gamma)(1+\nu)}{\left(\frac{l}{l+\lambda k}\right)^m \frac{(\alpha+p)_k}{(1)_k}(k-p)[k-p+\beta\nu]}$$

$$(0\leq\gamma<1,\ 0<\beta\leq 1,\ \alpha>-p,\ 0<\nu\leq 1,\ p\in\mathbb{N}).$$

3 Linear combination

Let the functions $f,g\in\mathscr{T}_p$ be given by (2) and (3) respectively. For $0\leq t\leq 1$, define the function $h(z)$ by

$$\begin{aligned} h(z)=&(1-t)f(z)+tg(z)\\ =&\frac{1}{z^p}+\sum_{k=1}^{\infty}[(1-t)a_k+tb_k]z^{k-p}\\ =&\frac{1}{z^p}+\sum_{k=1}^{\infty}c_k z^{k-p}, \end{aligned} \tag{12}$$

where for simplicity, we write

$$c_k=(1-t)a_k+tb_k. \tag{13}$$

Clearly $c_k\geq 0$.

Theorem 3.1. The class $\mathscr{B}_{\lambda,p}^{\alpha,\nu}(\gamma,\beta)$ is closed under convex linear combination.

Proof. To prove the class $\mathscr{B}_{\lambda,p}^{\alpha,\nu}(\gamma,\beta)$ is convex, i.e. to show for $f,g\in\mathscr{B}_{\lambda,p}^{\alpha,\nu}(\gamma,\beta)\Longrightarrow h\in\mathscr{B}_{\lambda,p}^{\alpha,\nu}(\gamma,\beta)$, where $h(z)$ is defined as (12) whose coefficient is given by (13). Since $f,g\in\mathscr{B}_{\lambda,p}^{\alpha,\nu}(\gamma,\beta)$, hence by application of Theorem 2.1 we have

$$\sum_{k=1}^{\infty}\left(\frac{l}{l+\lambda k}\right)^m \frac{(\alpha+p)_k}{(1)_k}(k-p)[(k-p)+\beta\nu]a_k \leq \beta p(1-\gamma)(1+\nu) \tag{14}$$

and

$$\sum_{k=1}^{\infty}\left(\frac{l}{l+\lambda k}\right)^m \frac{(\alpha+p)_k}{(1)_k}(k-p)[(k-p)+\beta\nu]b_k \leq \beta p(1-\gamma)(1+\nu). \tag{15}$$

To show $h\in\mathscr{B}_{\lambda,p}^{\alpha,\nu}(\gamma,\beta)$, by virtue of Theorem 2.1 it is sufficient to show that

$$\sum_{k=1}^{\infty}\left(\frac{l}{l+\lambda k}\right)^m \frac{(\alpha+p)_k}{(1)_k}(k-p)[(k-p)+\beta\nu]c_k \leq \beta p(1-\gamma)(1+\nu). \tag{16}$$

Now making use of (14) and (15) in (16) give

$$\begin{aligned} &\sum_{k=1}^{\infty}\left(\frac{l}{l+\lambda k}\right)^m \frac{(\alpha+p)_k}{(1)_k}(k-p)[(k-p)+\beta\nu]c_k\\ =(1-t)&\sum_{k=1}^{\infty}\left(\frac{l}{l+\lambda k}\right)^m \frac{(\alpha+p)_k}{(1)_k}(k-p)[(k-p)+\beta\nu]a_k\\ +t&\sum_{k=1}^{\infty}\left(\frac{l}{l+\lambda k}\right)^m \frac{(\alpha+p)_k}{(1)_k}(k-p)[(k-p)+\beta\nu]b_k\\ &\leq(1-t)[\beta p(1-\gamma)(1+\nu)]+t\beta p(1-\gamma)(1+\nu)\\ &=\beta p(1-\gamma)(1+\nu) \end{aligned}$$

Hence the result follows. □

4 Extreme Points

The determination of the extreme points of a family of multivalent function enable us to solve many extremal problems.

Theorem 4.1. Let $f_{-p}(z)=z^{-p}$ and

$$f_{k-p}(z)=\frac{1}{z^p}+\frac{\beta p(1-\gamma)(1+\nu)}{\left(\frac{l}{l+\lambda k}\right)^m \frac{(\alpha+p)_k}{(1)_k}(k-p)[k-p+\beta\nu]}z^{k-p} \quad (k\geq 1).$$

Then $f\in\mathscr{B}_{\lambda,p}^{\alpha,\nu}(\gamma,\beta)$ if and only if it can be expressed in the form

$$f(z)=\sum_{k=0}^{\infty}d_k f_{k-p}(z), \tag{17}$$

where

$$d_k\geq 0 \qquad \text{and} \qquad \sum_{k=0}^{\infty}d_k=1.$$

Proof. Suppose that

$$f(z)=\sum_{k=0}^{\infty}d_k f_{k-p}(z)$$

where

$$d_k\geq 0 \quad \text{and} \quad \sum_{k=1}^{\infty}d_k=1.$$

Then

$$f(z) = d_0 f_{-p}(z) + \sum_{k=1}^{\infty} d_k f_{k-p}(z)$$
$$= d_0 z^{-p} + \sum_{k=1}^{\infty} d_k$$
$$\left[\frac{1}{z^p} + \frac{\beta p(1-\gamma)(1+\nu)}{\left(\frac{l}{l+\lambda k}\right)^m \frac{(\alpha+p)_k}{(1)_k}(k-p)[k-p+\beta\nu]} z^{k-p}\right]$$
$$= \sum_{k=0}^{\infty} \frac{d_k}{z^p} + \sum_{k=1}^{\infty} \frac{\beta p(1-\gamma)(1+\nu)}{\left(\frac{l}{l+\lambda k}\right)^m \frac{(\alpha+p)_k}{(1)_k}(k-p)[k-p+\beta\nu]} d_k z^{k-p}$$
$$= \frac{1}{z^p} + \sum_{k=1}^{\infty} e_k z^{k-p}, \tag{18}$$

where, for convenience we take

$$e_k = \frac{\beta p(1-\gamma)(1+\nu) d_k}{\left(\frac{l}{l+\lambda k}\right)^m \frac{(\alpha+p)_k}{(1)_k}(k-p)[k-p+\beta\nu]}.$$

By Theorem 2.1, $f \in \mathscr{B}_{\lambda,p}^{\alpha,\nu}(\gamma,\beta)$ if and only if

$$\sum_{k=1}^{\infty} \frac{\left(\frac{l}{l+\lambda k}\right)^m \frac{(\alpha+p)_k}{(1)_k}(k-p)[k-p+\beta\nu]}{\beta p(1-\gamma)(1+\nu)} e_k \le 1,$$

for f given by (18).
Now

$$\sum_{k=1}^{\infty} \frac{\left(\frac{l}{l+\lambda k}\right)^m \frac{(\alpha+p)_k}{(1)_k}(k-p)[k-p+\beta\nu]}{\beta p(1-\gamma)(1+\nu)} e_k$$
$$= \sum_{k=1}^{\infty} \frac{\left(\frac{l}{l+\lambda k}\right)^m \frac{(\alpha+p)_k}{(1)_k}(k-p)[k-p+\beta\nu]}{\beta p(1-\gamma)(1+\nu)} \frac{\beta p(1-\gamma)(1+\nu)}{\left(\frac{l}{l+\lambda k}\right)^m \frac{(\alpha+p)_k}{(1)_k}(k-p)[k-p+\beta\nu]} d_k$$
$$= \sum_{k=1}^{\infty} d_k = \sum_{k=0}^{\infty} d_k - d_0$$
$$= 1 - d_0 \le 1.$$

Conversely, assume that $f \in \mathscr{B}_{\lambda,p}^{\alpha,\nu}(\gamma,\beta)$. Then we show that f can be expressed in the form of (17).

Since $f \in \mathscr{B}_{\lambda,p}^{\alpha,\nu}(\gamma,\beta)$, hence by Corollary 2.2 we have

$$a_k \le \frac{\beta p(1-\gamma)(1+\nu)}{\left(\frac{l}{l+\lambda k}\right)^m \frac{(\alpha+p)_k}{(1)_k}(k-p)[k-p+\beta\nu]}.$$

Take

$$d_k = \frac{\left(\frac{l}{l+\lambda k}\right)^m \frac{(\alpha+p)_k}{(1)_k}(k-p)[k-p+\beta\nu]}{\beta p(1-\gamma)(1+\nu)} a_k \quad (k \ge 1)$$

and

$$d_0 = 1 - \sum_{k=1}^{\infty} d_k,$$

then

$$f(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} a_k z^{k-p}$$
$$= \frac{1}{z^p} + \sum_{k=1}^{\infty} \frac{\beta p(1-\gamma)(1+\nu)}{\left(\frac{l}{l+\lambda k}\right)^m \frac{(\alpha+p)_k}{(1)_k}(k-p)[k-p+\beta\nu]} d_k z^{k-p}$$
$$= \frac{1}{z^p} + \sum_{k=1}^{\infty} [f_{k-p}(z) - z^{-p}] d_k$$
$$= \frac{1}{z^p}\left(1 - \sum_{k=1}^{\infty} d_k\right) + \sum_{k=1}^{\infty} d_k f_{k-p}(z)$$
$$= \frac{d_0}{z^p} + \sum_{k=1}^{\infty} d_k f_{k-p}(z)$$
$$= \sum_{k=0}^{\infty} d_k f_{k-p}(z).$$

Thus, the proof of Theorem 4.1 is completed.

5 Growth and distortion theorems

By making use of Theorem 2.1, we first prove the following growth theorem for the function in the class $\mathscr{B}_{\lambda,p}^{\alpha,\nu}(\gamma,\beta)$.

Theorem 5.1. If $f(z) \in \mathscr{B}_{\lambda,p}^{\alpha,\nu}(\gamma,\beta)$, then for $0 < |z| < 1$ we have

$$\frac{1}{|z|^p} - \frac{\beta p(1-\gamma)(1+\nu)}{\left(\frac{l}{l+\lambda}\right)^m (\alpha+p)(1-p)[1-p+\beta\nu]} |z|^{1-p} \le |f(z)|$$
$$\le \frac{1}{|z|^p} + \frac{\beta p(1-\gamma)(1+\nu)}{\left(\frac{l}{l+\lambda}\right)^m (\alpha+p)(1-p)[1-p+\beta\nu]} |z|^{1-p}. \tag{19}$$

The result is sharp for the function

$$f(z) = \frac{1}{z^p} + \frac{\beta p(1-\gamma)(1+\nu)}{\left(\frac{l}{l+\lambda}\right)^m (\alpha+p)(1-p)[(1-p)+\beta\nu]} z^{1-p}. \tag{20}$$

Proof. Since $f(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} a_k z^{k-p}$, we have

$$|f(z)| = \left|\frac{1}{z^p} + \sum_{k=1}^{\infty} a_k z^{k-p}\right|$$
$$\le \frac{1}{|z|^p} + \sum_{k=1}^{\infty} a_k |z|^{k-p} \quad (a_k \ge 0)$$
$$\le \frac{1}{|z|^p} + |z|^{1-p} \sum_{k=1}^{\infty} a_k. \tag{21}$$

By Theorem 2.1, we have

$$\sum_{k=1}^{\infty} a_k \le \frac{\beta p(1-\gamma)(1+\nu)}{\left(\frac{l}{l+\lambda}\right)^m (\alpha+p)(1-p)[1-p+\beta\nu]}.$$

Thus from (21), we obtain

$$|f(z)| \leq \frac{1}{|z|^p} + \frac{\beta p(1-\gamma)(1+\nu)}{\left(\frac{l}{l+\lambda}\right)^m (\alpha+p)(1-p)[1-p+\beta\nu]}|z|^{1-p}. \tag{22}$$

Similarly, we have

$$|f(z)| \geq \frac{1}{|z|^p} - \sum_{k=1}^{\infty} a_k |z|^{k-p}$$
$$\geq \frac{1}{|z|^p} - |z|^{1-p}\sum_{k=1}^{\infty} a_k$$
$$\geq \frac{1}{|z|^p} - \frac{\beta p(1-\gamma)(1+\nu)}{\left(\frac{l}{l+\lambda}\right)^m (\alpha+p)(1-p)[1-p+\beta\nu]}|z|^{1-p}. \tag{23}$$

The result (19) follows from (22) and (23). This complete the proof of Theorem 5.1 .

The distortion estimates for the functions in the class $\mathscr{B}_{\lambda,p}^{\alpha,\nu}(\gamma,\beta)$ is given in the following theorem.

Theorem 5.2. If $f \in \mathscr{B}_{\lambda,p}^{\alpha,\nu}(\gamma,\beta)$, then for $0<|z|<1$, we have

$$\frac{p}{|z|^{p+1}} - \frac{\beta p(1-\gamma)(1+\nu)}{\left(\frac{l}{l+\lambda}\right)^m (\alpha+p)[1-p+\beta\nu]}|z|^{-p} \leq |f'(z)|$$
$$\leq \frac{p}{|z|^{p+1}} + \frac{\beta p(1-\gamma)(1+\nu)}{\left(\frac{l}{l+\lambda}\right)^m (\alpha+p)[1-p+\beta\nu]}|z|^{-p}. \tag{24}$$

The results is sharp for the function $f(z)$ given by (20).

Proof. Since

$$f(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} a_k z^{k-p} \quad (a_k \geq 0),$$

we have

$$f'(z) = \frac{-p}{z^{p+1}} + \sum_{k=1}^{\infty}(k-p)a_k z^{k-p-1}.$$

Hence

$$|f'(z)| \leq \frac{p}{|z|^{p+1}} + \sum_{k=1}^{\infty}(k-p)a_k|z|^{k-p-1}$$
$$\leq \frac{p}{|z|^{p+1}} + |z|^{-p}\sum_{k=1}^{\infty}(1-p)a_k \tag{25}$$

Since $f \in \mathscr{B}_{\lambda,p}^{\alpha,\nu}(\gamma,\beta)$, it follows from Theorem 2.1 and (25) that

$$|f'(z)| \leq \frac{p}{|z|^{p+1}} + \frac{\beta p(1-\gamma)(1+\nu)}{\left(\frac{l}{l+\lambda}\right)^m (\alpha+p)[1-p+\beta\nu]}|z|^{-p}. \tag{26}$$

Similarly,

$$|f'(z)| \geq \frac{p}{|z|^{p+1}} - \sum_{k=1}^{\infty}(k-p)a_k|z|^{k-p-1}$$
$$\geq \frac{p}{|z|^{p+1}} - |z|^{-p}\sum_{k=1}^{\infty}(1-p)a_k$$
$$\geq \frac{p}{|z|^{p+1}} - \frac{\beta p(1-\gamma)(1+\nu)}{\left(\frac{l}{l+\lambda}\right)^m (\alpha+p)[1-p+\beta\nu]}|z|^{-p}. \tag{27}$$

The result (24) follows from (26) and (27). Thus, the proof of Theorem 5.2 is completed.

6 Neighborhoods and Partial Sums

Goodman [9], Ruscheweyh [11] and more recently Altinas and Owa [2](also, see [3,4]) have investigated the familiar concept of neighborhoods of analytic functions. Here we begin by introducing the δ-neighborhood of a function $f \in \mathscr{T}_p$ of the form (2).

Definition 6.1. Let $0<\beta\leq 1,\ 0<\nu\leq 1,\ 0\leq\gamma<1$, $p\in\mathbb{N}$ and $\delta\geq 0$. We define δ-neighborhood of a function $f\in\mathscr{T}_p$ of the form (2) and denoted by $N_\delta(f)$ as

$$N_\delta(f) = g \in \mathscr{T}_p : g(z) = z^{-p} + \sum_{k=1}^{\infty} b_k z^{k-p} \text{ and}$$
$$\sum_{k=1}^{\infty} \frac{\left(\frac{l}{l+\lambda k}\right)^m \frac{(\alpha+p)_k}{(1)_k}(k-p)[k-p+\beta\nu]}{\beta p(1-\gamma)(1+\nu)}|a_k-b_k| \leq \delta \tag{28}$$

Making use of the Definition 6.1 we now prove the following result.

Theorem 6.2. Let the function f given by (2) be in the class $\mathscr{B}_{\lambda,p}^{\alpha,\nu}(\gamma,\beta)$. If f satisfies the following condition:

$$(f(z)+\varepsilon z^{-p})(1+\varepsilon)^{-1} \in \mathscr{B}_{\lambda,p}^{\alpha,\nu}(\gamma,\beta) \quad (\varepsilon\in\mathbb{C}, |\varepsilon|<\delta, \delta>0), \tag{29}$$

then

$$N_\delta(f) \subset \mathscr{B}_{\lambda,p}^{\alpha,\nu}(\gamma,\beta). \tag{30}$$

Proof It is clearly seen from (8) that a function $f\in\mathscr{B}_{\lambda,p}^{\alpha,\nu}(\gamma,\beta)$ if and only if for any complex number σ with $|\sigma|=1$,

$$\frac{z^{p+2}(\mathscr{I}_{p,\alpha}^m(\lambda,l)f(z))'' + z^{p+1}(\mathscr{I}_{p,\alpha}^m(\lambda,l)f(z))' - p^2}{\beta[\nu z^{p+1}(\mathscr{I}_{p,\alpha}^m(\lambda,l)f(z))' + \gamma(1+\nu)p - p]} \neq \sigma \quad (z\in\mathbb{U}), \tag{31}$$

which is equivalent to

$$\frac{(f*h)(z)}{z^{-p}} \neq 0 \quad (z\in\mathbb{U}) \tag{32}$$

where, for convenience

$$h(z) = z^{-p} + \sum_{k=1}^{\infty} \frac{\left(\frac{l}{l+\lambda k}\right)^m \frac{(\alpha+p)_k}{(1)_k}(k-p)[k-p-\sigma\beta\nu]}{\sigma\beta p(1-\gamma)(1+\nu)} z^{k-p}$$
$$= z^{-p} + \sum_{k=1}^{\infty} e_k z^{k-p}. \tag{33}$$

It is easy to find from (33) that

$$|e_k| = \left| \frac{\left(\frac{l}{l+\lambda k}\right)^m \frac{(\alpha+p)_k}{(1)_k}(k-p)[k-p-\sigma\beta\nu]}{\sigma\beta p(1-\gamma)(1+\nu)} \right|$$
$$\leq \frac{\left(\frac{l}{l+\lambda k}\right)^m \frac{(\alpha+p)_k}{(1)_k}(k-p)[k-p+\beta\nu]}{\beta p(1-\gamma)(1+\nu)} \quad (k \geq 1, p \in \mathbb{N}).$$

Since $(f(z)+\varepsilon z^{-p})(1+\varepsilon)^{-1} \in \mathscr{B}^{\alpha,\nu}_{\lambda,p}(\gamma,\beta)$, by virtue of (32), we get

$$\frac{1}{z^{-p}} \left(\frac{f(z)+\varepsilon z^{-p}}{1+\varepsilon} * h(z) \right) \neq 0. \tag{34}$$

Assume that

$$\left| \frac{(f*h)(z)}{z^{-p}} \right| < \delta.$$

Then by (34) we get

$$\left| \frac{1}{1+\varepsilon} \frac{(f*h)(z)}{z^{-p}} + \frac{\varepsilon}{1+\varepsilon} \right| \geq \frac{|\varepsilon|}{|1+\varepsilon|} - \frac{1}{|1+\varepsilon|} \left| \frac{(f*h)(z)}{z^{-p}} \right|$$
$$> \frac{|\varepsilon|-\delta}{|1+\varepsilon|} \geq 0.$$

This is a contradiction as $|\varepsilon| < \delta$. Therefore

$$\left| \frac{(f*h)(z)}{z^{-p}} \right| \geq \delta. \tag{35}$$

Now, let

$$g(z) = z^{-p} + \sum_{k=1}^{\infty} b_k z^{k-p} \in N_\delta(f),$$

so that

$$\left| \frac{[f(z)-g(z)]*h(z)}{z^{-p}} \right| = \left| \sum_{k=1}^{\infty} (a_k - b_k) e_k z^k \right|$$
$$\leq \sum_{k=1}^{\infty} |a_k - b_k| |e_k| |z|^k$$
$$\leq \sum_{k=1}^{\infty} \left| \frac{\left(\frac{l}{l+\lambda k}\right)^m \frac{(\alpha+p)_k}{(1)_k}(k-p)[k-p+\beta\nu]}{\beta p(1-\gamma)(1+\nu)} \right|$$
$$|a_k - b_k| \leq \delta \quad (z \in \mathbb{U}, \delta > 0).$$

Therefore, for any complex number σ such that $|\sigma| = 1$, we have

$$\frac{(g*h)(z)}{z^{-p}} \neq 0,$$

which implies $g \in \mathscr{B}^{\alpha,\nu}_{\lambda,p}(\gamma,\beta)$. So $N_\delta(f) \subset \mathscr{B}^{\alpha,\nu}_{\lambda,p}(\gamma,\beta)$.

Next we prove

Theorem 6.3. Let $f \in \mathscr{T}_p$ be given by (2) and define the partial sums $s_1(z)$ and $s_q(z)$ as

$$s_1(z) = z^{-p}$$

and

$$s_q(z) = z^{-p} + \sum_{k=1}^{q-1} a_k z^{k-p} \quad (q > 1).$$

Suppose that

$$\sum_{k=1}^{\infty} c_k a_k \leq 1$$

where

$$c_k = \frac{\left(\frac{l}{l+\lambda k}\right)^m \frac{(\alpha+p)_k}{(1)_k}(k-p)[k-p+\beta\nu]}{\beta p(1-\gamma)(1+\nu)}. \tag{36}$$

Then

$$(i) \qquad f \in \mathscr{B}^{\alpha,\nu}_{\lambda,p}(\gamma,\beta), \tag{37}$$

$$(ii) \qquad \Re\left\{ \frac{f(z)}{s_q(z)} \right\} > 1 - \frac{1}{c_q}, \tag{38}$$

and

$$(iii) \quad Re\left\{ \frac{s_q(z)}{f(z)} \right\} > 1 - \frac{c_q}{1+c_q} \quad (z \in \mathbb{U}, q > 1). \tag{39}$$

Proof. It follows from (32) that

$$z^{-p} \in \mathscr{B}^{\alpha,\nu}_{\lambda,p}(\gamma,\beta).$$

Thus, from Theorem 6.2 and hypothesis (36) of Theorem 6.3 we have

$$N_1(z^{-p}) \subset \mathscr{B}^{\alpha,\nu}_{\lambda,p}(\gamma,\beta),$$

which shows that $f \in \mathscr{B}^{\alpha,\nu}_{\lambda,p}(\gamma,\beta)$.

(ii) Under the hypothesis in part (ii) of Theorem 6.3, we can see from (36) that

$$c_{k+1} > c_k > 1 \quad (k = 1,2,3,\cdots).$$

Therefore, we have

$$\sum_{k=1}^{q-1} a_k + c_q \sum_{k=q}^{\infty} a_k \leq \sum_{k=1}^{\infty} c_k a_k \leq 1. \tag{40}$$

By setting

$$G_1(z) = c_q \left[\frac{f(z)}{s_q(z)} - \left(1 - \frac{1}{c_q}\right) \right] = \frac{c_q \sum_{k=q}^{\infty} a_k z^k}{1 + \sum_{k=1}^{q-1} a_k z^k} + 1,$$

and applying (40) we find that

$$\left|\frac{G_1(z)-1}{G_1(z)+1}\right| = \left|\frac{c_q\sum_{k=q}^{\infty}a_kz^k}{2+2\sum_{k=1}^{q-1}a_kz^k+c_q\sum_{k=q}^{\infty}a_kz^k}\right| \le \frac{c_q\sum_{k=q}^{\infty}a_k}{2-2\sum_{k=1}^{q-1}a_k-c_q\sum_{k=q}^{\infty}a_k}\le 1,$$

which implies $\Re\{G_1(z)\}>0$. Thus, we obtain

$$\Re\left\{\frac{f(z)}{S_q(z)}\right\} > 1-\frac{1}{c_q},$$

which proof (ii)
Similarly, to prove (39), set

$$\begin{aligned} G_2(z) &= (1+c_q)\left(\frac{s_q(z)}{f(z)}-\frac{c_q}{1+c_q}\right)\\ &= (1+c_q)\frac{s_q(z)}{f(z)}-c_q\\ &= 1-\frac{(1+c_q)\sum_{k=q}^{\infty}a_kz^k}{1+\sum_{k=1}^{\infty}a_kz^k}. \end{aligned}$$

By virtue of (40), we can deduce that

$$\left|\frac{G_2(z)-1}{G_2(z)+1}\right| = \left|\frac{(1+c_q)\sum_{k=q}^{\infty}a_kz^k}{2+2\sum_{k=1}^{\infty}a_kz^k-(1+c_q)\sum_{k=q}^{\infty}a_kz^k}\right| \le \frac{(1+c_q)\sum_{k=q}^{\infty}a_k}{2-2\sum_{k=1}^{\infty}a_k-(1+c_q)\sum_{k=q}^{\infty}a_k} \le 1,$$

which gives the assertion (39) of Theorem 6.3 . Thus, the proof of Theorem 6.3 is completed.

7 Arithmetic Mean

Theorem 7.1. Let $f_1(z),\ f_2(z),\cdots\ f_l(z)$ defined by

$$f_i(z)=\frac{1}{z^p}+\sum_{k=1}^{\infty}a_{k,i}z^{k-p} \quad (a_{k,i}\ge 0,\ i=1,2,3,...,l,\ k\ge 1) \tag{41}$$

be in the class $\mathscr{B}_{\lambda,p}^{\alpha,\nu}(\gamma,\beta)$. Then the arithmetic mean of $f_i(z)\ \ (i=1,2,3,...,l)$ defined by

$$\phi(z)=\frac{1}{l}\sum_{i=1}^{l}f_i(z) \tag{42}$$

is also in the class $\mathscr{B}_{\lambda,p}^{\alpha,\nu}(\gamma,\beta)$.

Proof. Using (41) in (42), we have

$$\begin{aligned}\phi(z) &= \frac{1}{l}\sum_{i=1}^{l}\left[\frac{1}{z^p}+\sum_{k=1}^{\infty}a_{k,i}z^{k-p}\right]\\ &= \frac{1}{z^p}+\sum_{k=1}^{\infty}\left(\frac{1}{l}\sum_{i=1}^{l}a_{k,i}\right)z^{k-p}. \end{aligned}\tag{43}$$

Since $f_i(z)\in\mathscr{B}_{\lambda,p}^{\alpha,\nu}(\gamma,\beta)$ for every $i=1,2,3,...,l$, so by using Theorem 2.1, we prove that

$$\begin{aligned} &\sum_{k=1}^{\infty}\left(\frac{l}{l+\lambda k}\right)^m\frac{(\alpha+p)_k}{(1)_k}(k-p)[k-p+\beta\nu]\left(\frac{1}{l}\sum_{i=1}^{l}a_{k,i}\right)\\ &= \frac{1}{l}\sum_{i=1}^{l}\left[\sum_{k=1}^{\infty}\left(\frac{l}{l+\lambda k}\right)^m\frac{(\alpha+p)_k}{(1)_k}(k-p)[k-p+\beta\nu]a_{k,i}\right]\\ &\le \frac{1}{l}\sum_{i=1}^{l}\beta p(1-\gamma)(1+\nu)\\ &= \beta p(1-\gamma)(1+\nu). \end{aligned}$$

This ends the proof of Theorem 7.1.□

8 Weighted Mean

Definition 8.1 Let $f(z),\ \ g(z)\in\mathscr{T}_p$. The weighted mean $h_j(z)$ of $f(z)$ and $g(z)$ is given by

$$h_j(z)=\frac{1}{2}[(1-j)f(z)+(1+j)g(z)] \quad (0<j<1). \tag{44}$$

In the following theorem, we will show the weighted mean for the class $\mathscr{B}_{\lambda,p}^{\alpha,\nu}(\gamma,\beta)$.

Theorem 8.2. If $f(z)$ and $g(z)$ are in the class $\mathscr{B}_{\lambda,p}^{\alpha,\nu}(\gamma,\beta)$, then the weighted mean $h_j(z)$ of $f(z)$ and $g(z)$ defined as (44) is also in $\mathscr{B}_{\lambda,p}^{\alpha,\nu}(\gamma,\beta)$.

Proof. By Definition 8.1, we have

$$\begin{aligned} h_j(z) &= \frac{1}{2}\left[(1-j)\left(\frac{1}{z^p}+\sum_{k=1}^{\infty}a_kz^{k-p}\right)+(1+j)\left(\frac{1}{z^p}+\sum_{k=1}^{\infty}b_kz^{k-p}\right)\right]\\ &= \frac{1}{z^p}+\frac{1}{2}\sum_{k=1}^{\infty}[(1-j)a_k+(1+j)b_k]z^{k-p}. \end{aligned}\tag{45}$$

To show $h_j(z)\in\mathscr{B}_{\lambda,p}^{\alpha,\nu}(\gamma,\beta)$, by virtue of Theorem 2.1, it is sufficient to show

$$\sum_{k=1}^{\infty}\left(\frac{l}{l+\lambda k}\right)^m\frac{(\alpha+p)_k}{(1)_k}(k-p)[k-p+\beta\nu]\left[\frac{1}{2}(1-j)a_k+\frac{1}{2}(1+j)b_k\right] \le \beta p(1-\gamma)(1+\nu). \tag{46}$$

Now

$$\begin{aligned} &\sum_{k=1}^{\infty}\left(\frac{l}{l+\lambda k}\right)^m\frac{(\alpha+p)_k}{(1)_k}(k-p)[k-p+\beta\nu]\left[\frac{1}{2}(1-j)a_k+\frac{1}{2}(1+j)b_k\right]\\ &= \frac{1}{2}(1-j)\sum_{k=1}^{\infty}\left(\frac{l}{l+\lambda k}\right)^m\frac{(\alpha+p)_k}{(1)_k}(k-p)[k-p+\beta\nu]a_k\\ &\quad+\frac{1}{2}(1+j)\sum_{k=1}^{\infty}\left(\frac{l}{l+\lambda k}\right)^m\frac{(\alpha+p)_k}{(1)_k}(k-p)[k-p+\beta\nu]b_k\\ &\le \frac{1}{2}(1-j)\beta p(1-\gamma)(1+\nu)+\frac{1}{2}(1+j)\beta p(1-\gamma)(1+\nu)\\ &= \beta p(1-\gamma)(1+\nu), \end{aligned}$$

which establish (46).
This ends the proof of Theorem 8.2 .

9 Closure theorem

Theorem 9.1. Let the functions f_i defined by

$$f_i(z)=\frac{1}{z^p}+\sum_{k=1}^{\infty}a_{k,i}z^{k-p} \quad (a_{k,i}\ge 0,\ i=1,2,3,...,l,\ k\ge 1)$$

be in the class $\mathscr{B}^{\alpha,\nu}_{\lambda,p}(\gamma,\beta)$ for every $i=1,2,3,\ldots,l$. Then the function ψ defined by

$$\psi(z)=\frac{1}{z^p}+\sum_{k=1}^{\infty}e_k z^{k-p}\quad (e_k\geq 0)$$

also belong to the class $\mathscr{B}^{\alpha,\nu}_{\lambda,p}(\gamma,\beta)$, where

$$e_k=\frac{1}{l}\sum_{i=1}^{l}a_{k,i}.$$

Proof. Since $f_i(z)\in\mathscr{B}^{\alpha,\nu}_{\lambda,p}(\gamma,\beta)$, so by Theorem 2.1, we have

$$\sum_{k=1}^{\infty}\left(\frac{l}{l+\lambda k}\right)^m\frac{(\alpha+p)_k}{(1)_k}(k-p)[k-p+\beta\nu]a_{k,i}\leq\beta p(1-\gamma)(1+\nu)$$

for every $i=1,2,3,\ldots,l$. Hence

$$\begin{aligned}&\textstyle\sum_{k=1}^{\infty}\left(\frac{l}{l+\lambda k}\right)^m\frac{(\alpha+p)_k}{(1)_k}(k-p)[k-p+\beta\nu]e_k\\ &=\textstyle\sum_{k=1}^{\infty}\left(\frac{l}{l+\lambda k}\right)^m\frac{(\alpha+p)_k}{(1)_k}(k-p)[k-p+\beta\nu]\left(\frac{1}{l}\sum_{i=1}^{l}a_{k,i}\right)\\ &=\textstyle\frac{1}{l}\sum_{i=1}^{l}\left[\sum_{k=1}^{\infty}\left(\frac{l}{l+\lambda k}\right)^m\frac{(\alpha+p)_k}{(1)_k}(k-p)[k-p+\beta\nu]a_{k,i}\right]\\ &\leq\textstyle\frac{1}{l}\sum_{i=1}^{l}\beta p(1-\gamma)(1+\nu)\\ &=\beta p(1-\gamma)(1+\nu).\end{aligned}$$

Hence by Theorem 2.1, it follows that $h\in\mathscr{B}^{\alpha,\nu}_{\lambda,p}(\gamma,\beta)$.

10 Radii of starlikeness and convexity

In this section, we determine the radii of meromorphically p-valent starlikeness of order δ $(0\leq\delta<p)$ and meromorphically convexity of order δ $(0\leq\delta<p)$ for the function in the class $\mathscr{B}^{\alpha,\nu}_{\lambda,p}(\gamma,\beta)$.

Theorem 10.1. Let the function $f(z)$ defined by (2) be in the class $\mathscr{B}^{\alpha,\nu}_{\lambda,p}(\gamma,\beta)$. Then
(i) f is meromorphically p-valent starlike of order δ $(0\leq\delta<p)$ in the disk $|z|<r_1$ where

$$r_1=r_1(l,\lambda,k,p,\delta,\alpha,\beta,\nu,\gamma)=\inf_{k\geq 1}\left[\frac{\left(\frac{l}{l+\lambda k}\right)^m\frac{(\alpha+p)_k}{(1)_k}(k-p)[k-p+\beta\nu](p-\delta)}{\beta p(1-\gamma)(1+\nu)(k+p-\delta)}\right]^{\frac{1}{k}}. \tag{47}$$

(ii) f is meromorphically p-valent convex of order δ $(0\leq\delta<p)$ in the disk $|z|<r_2$ where

$$r_2=r_2(l,\lambda,k,p,\delta,\alpha,\beta,\nu,\gamma)=\inf_{k\geq 1}\left[\frac{(p-\delta)\left(\frac{l}{l+\lambda k}\right)^m\frac{(\alpha+p)_k}{(1)_k}[k-p+\beta\nu]}{\beta(1-\gamma)(1+\nu)(k+p-\delta)}\right]^{\frac{1}{k}}. \tag{48}$$

Proof (i) It is sufficient to show that

$$\left|\frac{zf'(z)}{f(z)}+p\right|\leq p-\delta \tag{49}$$

for $|z|<r_1$.
Replacing $f(z)$ and $zf'(z)$ with their equivalent series expression in left hand side of (49), we obtain

$$\begin{aligned}\left|\frac{zf'(z)}{f(z)}+p\right|&=\left|\frac{\frac{-p}{z^p}+\sum_{k=1}^{\infty}(k-p)a_kz^{k-p}}{\frac{1}{z^p}+\sum_{k=1}^{\infty}a_kz^{k-p}}+p\right|\\ &=\left|\frac{\sum_{k=1}^{\infty}ka_kz^k}{1+\sum_{k=1}^{\infty}a_kz^k}\right|\\ &\leq\frac{\sum_{k=1}^{\infty}ka_k|z|^k}{1-\sum_{k=1}^{\infty}a_k|z|^k}.\end{aligned} \tag{50}$$

Hence (50) holds true if

$$\sum_{k=1}^{\infty}ka_k|z|^k\leq(p-\delta)(1-\sum_{k=1}^{\infty}a_k|z|^k),$$

or

$$\sum_{k=1}^{\infty}\frac{k+p-\delta}{p-\delta}a_k|z|^k\leq 1. \tag{51}$$

With the aid of (9), (51) is true if

$$\frac{k+p-\delta}{p-\delta}|z|^k\leq\frac{\left(\frac{l}{l+\lambda k}\right)^m\frac{(\alpha+p)_k}{(1)_k}(k-p)[k-p+\beta\nu]}{\beta p(1-\gamma)(1+\nu)}\quad(k\geq 1). \tag{52}$$

Solving (52) for $|z|$, we obtain

$$|z|<\left[\frac{\left(\frac{l}{l+\lambda k}\right)^m\frac{(\alpha+p)_k}{(1)_k}(k-p)[k-p+\beta\nu](p-\delta)}{\beta p(1-\gamma)(1+\nu)(k+p-\delta)}\right]^{\frac{1}{k}}\quad(k\geq 1),$$

which proves the assertion (47).
(ii) In order to prove the second assertion of Theorem 10.1, it is sufficient to show that

$$\left|1+\frac{zf''(z)}{f'(z)}+p\right|\leq p-\delta\quad(0\leq\delta<p) \tag{53}$$

for $|z|<r_2$.
Replacing $f'(z)$ and $zf''(z)$ with their equivalent series expression in the left hand side of (53), we get

$$\begin{aligned}&\left|1+\frac{zf''(z)}{f'(z)}+p\right|=\\ &\left|\frac{p(p+1)+\sum_{k=1}^{\infty}(k-p)(k-p-1)a_kz^k}{-p+\sum_{k=1}^{\infty}(k-p)a_kz^k}+\right.\\ &\left.(p+1)\right|=\left|\frac{\sum_{k=1}^{\infty}k(k-p)a_kz^k}{-p+\sum_{k=1}^{\infty}(k-p)a_kz^k}\right|\\ &\leq\frac{\sum_{k=1}^{\infty}k(k-p)a_k|z|^k}{p-\sum_{k=1}^{\infty}(k-p)a_k|z|^k}.\end{aligned} \tag{54}$$

Hence (54) holds true if

$$\sum_{k=1}^{\infty}k(k-p)a_k|z|^k\leq(p-\delta)(p-\sum_{k=1}^{\infty}(k-p)a_k|z|^k),$$

or

$$\sum_{k=1}^{\infty} \frac{(k-p)(k+p-\delta)a_k|z|^k}{p(p-\delta)} \le 1. \tag{55}$$

Hence by application of Theorem 2.1, (55) is true if

$$\frac{(k-p)(k+p-\delta)}{p(p-\delta)}|z|^k \le \frac{(\frac{l}{l+\lambda k})^m \frac{(\alpha+p)_k}{(1)_k}(k-p)[k-p+\beta\nu]}{\beta p(1-\gamma)(1+\nu)}. \tag{56}$$

Solving (56) for $|z|$, we obtain

$$|z| \le \left[\frac{(p-\delta)\left(\frac{l}{l+\lambda k}\right)^m \frac{(\alpha+p)_k}{(1)_k}(k-p+\beta\nu)}{[(k+p-\delta]\beta(1-\gamma)(1+\nu)}\right]^{\frac{1}{k}} \quad (k \ge 1).$$

11 Conclusion

In this paper, we obtain some geometric properties of the function $f \in \mathscr{T}_p$ to be in the function class $\mathscr{B}_{\lambda,p}^{\alpha,\nu}(\gamma,\beta)$. The authors suggest to introduce a new operator instead of $\mathscr{J}_{p,\alpha}^{m}(\lambda,l)$ and study the above results in the context of the modified class.

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