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Quantum Transitions in a Parabolic Intersection

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Abstract: In their seminal work (in 1932), Lev Landau and Clarence Zener derived a non-perturbative prediction for the transition probability between the two energy states. This paper is dedicated to the coupled Schrödinger system :

$$\frac{\hbar}{i}\frac{d\psi(t)}{dt} = \begin{pmatrix} t^2-z^2 & 2zt \\ 2zt & -t^2+z^2 \end{pmatrix} \psi(t) \ ,$$

and we aim to compute a local expansion of the transition coefficient $a(z)$ in the adiabatic limit $z \to 0$.

Keywords: Schrödinger system, 2-state transition, Stokes geometry, phase-integral methods, asymptotic series, Heun triconfluent function

1 General framework

The Landau-Zener model of a quantum mechanical 2-level system driven with a linearly dependent Hamiltonian is a classic paradigm of quantum dynamics. In the so-called conical intersection, the dispersion relation $E(\mathbf{k})$ is actually a linear function with respect to the Bloch wave vector $\mathbf{k}$ - or equivalently the impulsion $\mathbf{p}$.

Nevertheless, for convenience (and consistency with the notations in our previous papers [2]-[5]), the independent variable shall be denoted by the symbol t, and still be referred as ”time”. We are about to explore in the two upcoming sections the case of a quadratic dispersion relation. Hence the terminology ”parabolic intersection” in the title. Let’s get started. Consider the Schödinger equation with the matrix-valued Hamiltonian :

$$H_2(t,z) = \begin{pmatrix} t^2-z^2 & 2zt \\ 2zt & -t^2+z^2 \end{pmatrix} \ , \qquad (1)$$

whose eigenstates are given by $|+\rangle = \exp\left[+\frac{i}{\hbar}\left(\frac{t^3}{3}+z^2t^2\right)\right]$ and $|-\rangle = \exp\left[-\frac{i}{\hbar}\left(\frac{t^3}{3}+z^2t^2\right)\right]$. From the system of coupled differential equations, one obtains :

$$\hbar^2\frac{\partial^2\psi_1(t,z)}{\partial t^2} - \frac{\hbar^2}{t}\frac{\partial\psi_1(t,z)}{\partial t} + \left(t^2+z^2-\frac{i\hbar}{t}\right)(t^2+z^2)\psi_1(t,z) = 0$$

$$\text{and } \hbar^2\frac{\partial^2\psi_2(t,z)}{\partial t^2} - \frac{\hbar^2}{t}\frac{\partial\psi_2(t,z)}{\partial t} + \left(t^2+z^2+\frac{i\hbar}{t}\right)(t^2+z^2)\psi_2(t,z) = 0 \ .$$

Throughout this paper, we shall use extensively the 2 next results :

Lemma 1. *Let T be a fixed positive real. For any $\alpha \in \mathbb{R}_+^*$:*

$$\sum_{k=0}^{+\infty}\frac{\alpha^k}{\Gamma(k+1)}T^k = \int_0^{+\infty}\frac{\alpha^x}{\Gamma(x+1)}T^x dx = e^{\alpha T} \quad \text{as } T \to +\infty \ .$$

Proof. After replacing k by a continuous variable x, then applying the Stirling approximation to $\Gamma(x+1)$, we obtain :

$$\sum_{k=0}^{+\infty}\frac{\alpha^k}{\Gamma(k+1)}T^k \simeq \frac{1}{\sqrt{2\pi x_0}}\int_0^{+\infty}\exp\left[x + x\log(\alpha T) - x\log x\right]dx$$

where x_0 is some value to be determined. The maximum of the auxiliary function $\varphi(x) = x + x\log(\alpha T) - x\log x$ is achieved at $x_0 = \alpha T$. The result follows readily from the Laplace method. The discrepancy between the series and the integral decreases as $T \to +\infty$.

With the above assumptions, the next corollary is immediate :

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Corollary 1. *If P is at most of polynomial growth, then we have :*

$$\sum_{k=0}^{+\infty} \frac{\alpha^k}{\Gamma(k+1)} P(k) T^k = P(\alpha T) e^{\alpha T} \quad as\ T \to +\infty\,.$$

The Borel summation correctly extends the series to the whole complex plane.

2 In the diabatic limit : strong coupling $z \to +\infty$

From [4], we learned that the transition probabilities in the S-matrix are related to the Stokes phenomenon. In the limit of a strong coupling, the SDE framework is relevant. Put it differently, the second-order differential equation is reduced to :

$$\hbar^2 \frac{d^2\phi}{dt^2} + (t^2+z^2)^2 \phi = 0\,.$$

This is also the model for the anharmonic oscillator. Given that $Q(t,z) = (t^2+z^2)^2$, there are 2 second-order zeroes located at $\pm iz$, and infinity is an essential singularity. A closed form of the above equation involves the Heun triconfluent function. More precisely :

$$\begin{aligned}\phi(t) \;=\; & c_1 \,\mathrm{HeunT}\left[0,0,\left(\frac{2}{\hbar}\right)^{2/3} 3^{1/3} e^{i\frac{\pi}{3}} z^2; -\left(\frac{2}{3\hbar}\right)^{1/3} e^{i\frac{\pi}{6}} t\right] \exp\left[\frac{i}{\hbar}\left(\frac{t^3}{3}+z^2t^2\right)\right] \\ & + c_2 \,\mathrm{HeunT}\left[0,0,\left(\frac{2}{\hbar}\right)^{2/3} 3^{1/3} e^{i\frac{\pi}{3}} z^2; \left(\frac{2}{3\hbar}\right)^{1/3} e^{i\frac{\pi}{6}} t\right] \exp\left[-\frac{i}{\hbar}\left(\frac{t^3}{3}+z^2t^2\right)\right]\,.\end{aligned}$$

Edmund T. Whittaker conjectured in 1914 that the Heun functions can not be described in form of contour integrals of elementary functions, even if it is the simplest class of special functions. So any attempt of an integral representation (as in the case $p=1$) would likely fail in view of our current knowledge of these functions. Nevertheless, the asymptotic solutions for ϕ are :

$$\phi_{\pm}(t) \sim \frac{1}{t} \exp\left[\pm \frac{i}{\hbar}\left(\frac{t^3}{3}+z^2t\right)\right]\,,$$

as a direct consequence of the WKB analysis.

2.1 Connecting the asymptotic solutions

From this point, we will use the terminology of "Stokes multiplier" instead of "Stokes constant". As we have seen in [4], this complex-valued quantity may depend of the variable t.

Embed the real line in $\mathbb{C}$. In the Stokes diagram, counterclockwise continuation in the t-plane gives :

1. start with a subdominant solution : $(iz,t)_s$
2. crossing a Stokes line : $(iz,t)_s$
3. crossing an anti-Stokes line : $(iz,t)_d$
4. crossing a Stokes line : $(iz,t)_d + C_{S1}(t,iz)_s$
5. crossing an anti-Stokes line : $(iz,t)_s + C_{S1}(t,iz)_d$
6. crossing a cut emanating from a 2nd-order turning point : $(iz,t)_s + C_{S1}(t,iz)_d$
7. crossing a Stokes line : $(1+C_{S2}C_{S1})(iz,t)_s + C_{S1}(t,iz)_d$
8. crossing an anti-Stokes line : $(1+C_{S2}C_{S1})(iz,t)_d + C_{S1}(t,iz)_s$
   reconnecting iz to $-iz$: $(1+C_{S2}C_{S1})[iz,-iz]_\ell(-iz,t)_d + C_{S1}(t,-iz)_s[-iz,iz]_\ell$
9. rename (formally) t into $-t$: $(1+C_{S2}C_{S1})[iz,-iz]_\ell(-iz,-t)_d + C_{S1}(-t,-iz)_s[-iz,iz]_\ell$

where the Stokes multipliers C_{S1} and C_{S2} are associated with the upper turning point $t_2 = iz$. The value of the action W is defined by $e^W = [-iz,iz]_\ell$:

$$W = -\frac{i}{\hbar}\int_{-iz}^{iz} (t^2+z^2)dt = \frac{4z^3}{3\hbar}\,,$$

as it involves the Wallis integral $I_3 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta d\theta$.

Subsequently, we have :

$$(1+C_{S2}C_{S1})e^{-W}(-iz,-t)_d + C_{S1}e^{W}(-t,-iz)_s \quad \leftrightsquigarrow \quad (iz,t)_s$$

for ϕ. And we can infer the 2 transition probabilities :

$a_{\mathrm{SDE}} = (1+C_{S2}C_{S1})e^{-W}$

and $b_{\mathrm{SDE}} = C_{S1}e^{W}$,

should we are able to compute the Stokes multipliers.

2.2 Compatibility relation

Assume that ϕ can be defined by two different expressions :

$$\begin{aligned}\phi(t) &= [Af_1(t)+Bf_2(t)]\exp\left[-\frac{i}{\hbar}\left(\frac{t^3}{3}-z^2t\right)\right] \\ &= [Cg_1(t)+Dg_2(t)]\exp\left[\frac{i}{\hbar}\left(\frac{t^3}{3}-z^2t\right)\right]\end{aligned}$$

as in the forthcoming subsection. Then the conditions of compatibility for ϕ and its first derivative at $t=0$ yield :

$$\begin{cases} Ac_0^{[f_1]} = Cd_0^{[g_1]} \\ Ac_1^{[f_1]} + \dfrac{iz^2}{\hbar}Ac_0^{[f_1]} + Bc_0^{[f_2]} = Cd_1^{[g_1]} - \dfrac{iz^2}{\hbar}Cd_0^{[g_1]} + Dd_0^{[g_2]} \end{cases}. \tag{2}$$

It seems easier to track one function instead of two, hence the first choice $B=D=0$ - or equivalently $\nu=0$. Taking $A=C$ is natural. If g_1 is set to reproduce the dominant exponential behaviour for $\mathrm{Arg}\,t = \frac{\pi}{6}$, then we have to request :

$$d_0 = 1 \quad , \quad d_1 = d_2 = 0 \quad \Longrightarrow \quad d_3 = -\frac{1}{3!}\frac{2i}{3\hbar}\,.$$

The compatibility relation imposes thus the condition :

$$c_1^{[f_1]} = -\frac{2iz^2}{\hbar}\,.$$

2.3 Power series solutions

Extract the asymptotic behaviour by writing $\phi(t) = f(t)\exp\left[-\frac{i}{\hbar}\left(\frac{t^3}{3} - z^2 t\right)\right]$. Then f satisfies :

$$f'' - \frac{2i}{\hbar}(t^2 - z^2) f' + \left(\frac{4z^2t^2}{\hbar^2} - \frac{2it}{\hbar}\right) f = 0 \,. \quad (3)$$

As before, we can find two independent power series solutions by taking $f(t) = t^\nu \sum_{k=0}^{+\infty} c_k t^k$. Substituting in (3) yields :

$$\frac{1}{t^2}\sum_{k=0}^{+\infty}(\nu+k)(\nu+k-1)c_k t^k + \frac{2iz^2}{\hbar t}\sum_{k=0}^{+\infty}(\nu+k)c_k t^k - \frac{2it}{\hbar}\sum_{k=0}^{+\infty}(\nu+k+1)c_k t^k + \frac{4z^2t^2}{\hbar^2}\sum_{k=0}^{+\infty} c_k t^k = 0 \,.$$

Equating to 0 the respective coefficients of powers of t implies the indicial equation :

$$\nu(\nu-1) = 0 \implies \begin{cases} \nu = 0 \\ \text{or } \nu = 1 \end{cases} ,$$

and for the 3 next coefficients :

- order -1 : $(\nu+1)\nu c_1 + \frac{2iz^2}{\hbar}\nu c_0 = 0 \implies c_1 = -\frac{2iz^2}{\hbar}$ to ensure the compatibility with g_1
- order $+0$:

$$(\nu+2)(\nu+1)c_2 + \frac{2iz^2}{\hbar}(\nu+1)c_1 = 0 \implies c_2 = \frac{1}{2!}\left(-\frac{2iz^2}{\hbar}\right)^2$$

- order $+1$:

$$(\nu+3)(\nu+2)c_3 + \frac{2iz^2}{\hbar}(\nu+2)c_2 - \frac{2i}{\hbar}(\nu+1)c_0 = 0 \implies c_3 = \frac{1}{3!}\left[\left(-\frac{2iz^2}{\hbar}\right)^3 + \frac{2i}{\hbar}\right]$$

as well as the recurrence relation :

$$\begin{aligned} \forall k \geqslant 4,\ c_k &= -\frac{2iz^2}{\hbar}\frac{1}{\nu+k}c_{k-1} + \frac{2i}{\hbar}\frac{\nu+k-2}{(\nu+k)_2}c_{k-3} - \frac{4z^2}{\hbar^2}\frac{1}{(\nu+k)_2}c_{k-4} \\ &= -\frac{2iz^2}{\hbar}\frac{1}{\nu+k}c_{k-1} + \frac{2i}{\hbar}\frac{(\nu+k-2)^2}{(\nu+k)_3}c_{k-3} - \frac{4z^2}{\hbar^2}\frac{(\nu+k-2)_2}{(\nu+k)_4}c_{k-4} \,. \end{aligned}$$

For $k \geqslant 4$, the recurrence relation can be rewritten as :

$$c_k = -\frac{2iz^2}{\hbar}\left[\frac{1}{k}c_{k-1} - \frac{2i}{\hbar}\frac{(k-2)_2}{k_4}c_{k-4}\right] + \frac{2i}{\hbar}\frac{(k-2)^2}{k_3}c_{k-3} \,. \quad (4)$$

The computation of the coefficients c_k (respectively d_k) proves to be horrendous, because they are polynomials. The sequences even depend on the choice of their four initial terms. The reader can grasp the inherent difficulty by examining the excerpts in the appendix. Nonetheless, we may not need to explicit them in their fullness - by performing a truncation after the leading term.

Define $\mathbb{F} = \{0,1,2\}$. Let $j \in \mathbb{F}$. Then for $k \geqslant 1$, we set $c_{3k+j} = \sum_{\ell=0}^{k} \gamma_\ell^{[3k+j]} z^{6\ell+2j}$. By inspection of (4), it is clear that the coefficient of the term of highest degree in z^2 is : $\gamma_k^{[3k+j]} = \frac{1}{(3k+j)!}\left(-\frac{2i}{\hbar}\right)^{3k+j}$. In c_{3k}, the coefficient of the term of lowest degree in z^2 is :

$$\begin{aligned} \gamma_0^{[3k]} &= \left(\frac{2i}{\hbar}\right)^k \frac{(3k-2)^2(3k-5)^2\ldots 1^2}{(3k)!}c_0 \\ &= \left(\frac{2i}{\hbar}\right)^k \frac{3^{2k}}{(3k)!}\left(k-\frac{2}{3}\right)_k^2 = \left(\frac{2i}{\hbar}\right)^k \frac{3^{2k}}{(3k)!}\left[\frac{\Gamma\left(k+\frac{1}{3}\right)}{\Gamma\left(\frac{1}{3}\right)}\right]^2 . \end{aligned} \quad (5)$$

Assuming that k is large enough, let us compare the magnitudes of $\gamma_k^{[3k]}z^{6k}$ and $\gamma_0^{[3k]}$ in the coefficient c_{3k} :

$$\left|\frac{\gamma_0^{[3k]}}{\gamma_k^{[3k]}z^{6k}}\right| = \left(\frac{\hbar}{2}\right)^{2k}\frac{3^{2k}}{z^{6k}}\left[\frac{\Gamma\left(k+\frac{1}{3}\right)}{\Gamma\left(\frac{1}{3}\right)}\right]^2 \simeq \left(\frac{3\hbar}{2z^3}\right)^{2k}\left[\frac{\Gamma\left(k+\frac{4}{3}\right)}{k\Gamma\left(\frac{1}{3}\right)}\right]^2$$

$$\simeq \left(\frac{3\hbar}{2z^3}\right)^{2k}\left[\frac{k^{1/3}\sqrt{2\pi k}\left(\frac{k}{e}\right)^k}{k\Gamma\left(\frac{1}{3}\right)}\right]^2 = \frac{2\pi}{k^{1/3}\left[\Gamma\left(\frac{1}{3}\right)\right]^2}\left(\frac{3\hbar k}{2ez^3}\right)^{2k} \overset{[k\to+\infty]}{\longrightarrow} +\infty \,.$$

Introduce the quantity :

$$\begin{aligned} \chi_0^{[3\ell+1]}(\ell) &= -\frac{2iz^2}{\hbar}\left[\frac{1}{3\ell+1}\gamma_0^{[3\ell]} - \frac{2i}{\hbar}\frac{(3\ell-1)_2}{(3\ell+1)_4}\gamma_0^{[3(\ell-1)]}\right] \\ &= -\frac{2iz^2}{\hbar}\left(\frac{2i}{\hbar}\right)^\ell \frac{3^{2\ell}}{(3\ell+1)!}\left[\frac{\Gamma\left(\ell+\frac{1}{3}\right)}{\Gamma\left(\frac{1}{3}\right)}\right]^2 \cdot \left\{1 - \frac{(3\ell-1)_2}{3^2}\left[\frac{\Gamma\left(\ell-1+\frac{1}{3}\right)}{\Gamma\left(\ell+\frac{1}{3}\right)}\right]^2\right\} \\ &= -\frac{2iz^2}{\hbar}\left(\frac{2i}{\hbar}\right)^\ell \frac{3^{2\ell}}{(3\ell+1)!}\left[\frac{\Gamma\left(\ell+\frac{1}{3}\right)}{\Gamma\left(\frac{1}{3}\right)}\right]^2\left[1 - \frac{(3\ell-1)_2}{(3\ell-2)^2}\right] \\ &= \frac{2iz^2}{\hbar}\left(\frac{2i}{\hbar}\right)^\ell \frac{3^{2\ell}}{(3\ell+1)!}\left[\frac{\Gamma\left(\ell+\frac{1}{3}\right)}{\Gamma\left(\frac{1}{3}\right)}\right]^2\frac{1}{3\ell-2} \,. \end{aligned}$$

Now the coefficient of the term of lowest degree in z^2 in c_{3k+1} can be computed by means of a discrete convolution.

Therefore :

$$\begin{aligned}
\gamma_0^{[3k+1]} &= \chi_0^{[3k+1]}(k)+\sum_{\ell=1}^{k-1}\left(\frac{2i}{\hbar}\right)^{\ell}\frac{3^{2\ell}}{(3k+1)_{3\ell}}\left(k-\frac{1}{3}\right)_{\ell}^{2}\chi_0^{[3(k-\ell)+1]}(k-\ell)\\
&\quad+\left(\frac{2i}{\hbar}\right)^{k}\frac{3^{2k}}{(3k+1)!}\left[\frac{\Gamma\left(k+\frac{2}{3}\right)}{\Gamma\left(\frac{2}{3}\right)}\right]^{2}\gamma_0^{[1]}\\
&= \chi_0^{[3k+1]}(k)+\frac{2iz^2}{\hbar}\left(\frac{2i}{\hbar}\right)^{k}\frac{3^{2k}}{(3k+1)!}\sum_{\ell=1}^{k-1}\left[\frac{\Gamma\left(k+\frac{2}{3}\right)}{\Gamma\left(k-\ell+\frac{2}{3}\right)}\right]^{2}\left[\frac{\Gamma\left(k-\ell+\frac{1}{3}\right)}{\Gamma\left(\frac{1}{3}\right)}\right]^{2}\\
&\quad\cdot\frac{1}{3(k-\ell)-2}+\left(\frac{2i}{\hbar}\right)^{k}\frac{3^{2k}}{(3k+1)!}\left[\frac{\Gamma\left(k+\frac{2}{3}\right)}{\Gamma\left(\frac{2}{3}\right)}\right]^{2}\gamma_0^{[1]}\\
&= \chi_0^{[3k+1]}(k)+\frac{2iz^2}{\hbar}\left(\frac{2i}{\hbar}\right)^{k}\frac{3^{2k}}{(3k+1)!}\left[\frac{\Gamma\left(k+\frac{2}{3}\right)}{\Gamma\left(\frac{2}{3}\right)}\right]^{2}\left\{\sum_{\ell=1}^{k-1}\left[\frac{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\ell+\frac{1}{3}\right)}{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\ell+\frac{2}{3}\right)}\right]^{2}\frac{1}{3\ell-2}-1\right\}\\
&= \frac{2iz^2}{\hbar}\left(\frac{2i}{\hbar}\right)^{k}\frac{3^{2k}}{(3k+1)!}\left[\frac{\Gamma\left(k+\frac{2}{3}\right)}{\Gamma\left(\frac{2}{3}\right)}\right]^{2}\underbrace{\left\{\sum_{\ell=1}^{k}\left[\frac{B\left(\frac{2}{3},\ell+\frac{1}{3}\right)}{B\left(\frac{1}{3},\ell+\frac{2}{3}\right)}\right]^{2}\frac{1}{3\ell-2}-1\right\}}_{F_{z,1}(k)},
\end{aligned}\tag{6}$$

where $B(x,y)=\dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ is the Beta function. Evaluate the quantity :

$$\begin{aligned}
\chi_1^{[3\ell+2]}(\ell) &= -\frac{2iz^2}{\hbar}\left[\frac{1}{3\ell+2}\gamma_0^{[3\ell+1]}-\frac{2i}{\hbar}\frac{(3\ell)_2}{(3\ell+2)_4}\gamma_0^{[3\ell-2)]}\right]\\
&= -\left(\frac{2iz^2}{\hbar}\right)^{2}\left(\frac{2i}{\hbar}\right)^{\ell}\frac{3^{2\ell}}{(3\ell+2)!}\left[\frac{\Gamma\left(\ell+\frac{2}{3}\right)}{\Gamma\left(\frac{2}{3}\right)}\right]^{2}\left[F_{z,1}(\ell)-\frac{(3\ell)_2}{(3\ell-1)^2}F_{z,1}(\ell-1)\right].
\end{aligned}$$

By inserting :

$$\begin{aligned}
F_{z,1}(\ell)-\frac{3\ell}{3\ell-1}F_{z,1}(\ell-1) &= \left\{\sum_{m=1}^{\ell}\left[\frac{B\left(\frac{2}{3},m+\frac{1}{3}\right)}{B\left(\frac{1}{3},m+\frac{2}{3}\right)}\right]^{2}\frac{1}{3m-2}-1\right\}\\
&\quad-\frac{3\ell}{3\ell-1}\left\{\sum_{m=1}^{\ell-1}\left[\frac{B\left(\frac{2}{3},m+\frac{1}{3}\right)}{B\left(\frac{1}{3},m+\frac{2}{3}\right)}\right]^{2}\frac{1}{3m-2}-1\right\}\\
&= \frac{1}{3\ell-1}\left\{1-\sum_{m=1}^{\ell}\left[\frac{B\left(\frac{2}{3},m+\frac{1}{3}\right)}{B\left(\frac{1}{3},m+\frac{2}{3}\right)}\right]^{2}\frac{1}{3m-2}+\frac{3\ell}{3\ell-2}\left[\frac{B\left(\frac{2}{3},\ell+\frac{1}{3}\right)}{B\left(\frac{1}{3},\ell+\frac{2}{3}\right)}\right]^{2}\right\}
\end{aligned}$$

into $\chi_1^{[3\ell+2]}(\ell)$, we finally obtain :

$$\begin{aligned}
\chi_1^{[3\ell+2]}(\ell) &= -\left(\frac{2iz^2}{\hbar}\right)^{2}\left(\frac{2i}{\hbar}\right)^{\ell}\frac{3^{2\ell}}{(3\ell+2)!}\left[\frac{\Gamma\left(\ell+\frac{2}{3}\right)}{\Gamma\left(\frac{2}{3}\right)}\right]^{2}\\
&\quad\cdot\frac{1}{3\ell-1}\left\{1-\sum_{m=1}^{\ell}\left[\frac{B\left(\frac{2}{3},m+\frac{1}{3}\right)}{B\left(\frac{1}{3},m+\frac{2}{3}\right)}\right]^{2}\frac{1}{3m-2}+\frac{3\ell}{3\ell-2}\left[\frac{B\left(\frac{2}{3},\ell+\frac{1}{3}\right)}{B\left(\frac{1}{3},\ell+\frac{2}{3}\right)}\right]^{2}\right\}\\
&= \left(\frac{2iz^2}{\hbar}\right)^{2}\left(\frac{2i}{\hbar}\right)^{\ell}\frac{3^{2\ell}}{(3\ell+2)!}\left[\frac{\Gamma\left(\ell+\frac{2}{3}\right)}{\Gamma\left(\frac{2}{3}\right)}\right]^{2}\\
&\quad\cdot\frac{1}{3\ell-1}\left\{\sum_{m=1}^{\ell}\left[\frac{B\left(\frac{2}{3},m+\frac{1}{3}\right)}{B\left(\frac{1}{3},m+\frac{2}{3}\right)}\right]^{2}\frac{1}{3m-2}-1-\frac{3\ell}{3\ell-2}\left[\frac{B\left(\frac{2}{3},\ell+\frac{1}{3}\right)}{B\left(\frac{1}{3},\ell+\frac{2}{3}\right)}\right]^{2}\right\}.
\end{aligned}$$

Again, the coefficient of the term of lowest degree in z^2 in c_{3k+2} can be computed by :

$$\begin{aligned}
\gamma_0^{[3k+2]} &= \chi_1^{[3k+2]}(k)+\sum_{\ell=1}^{k-1}\left(\frac{2i}{\hbar}\right)^{\ell}\frac{3^{2k}}{(3k+2)_{3\ell}}\left[\frac{\Gamma(k+1)}{\Gamma(k-\ell+1)}\right]^{2}\chi_1^{[3(k-\ell)+2]}(k-\ell)+\left(\frac{2i}{\hbar}\right)^{k}\frac{3^{2k}}{(3k+2)_{3k}}[\Gamma(k+1)]^{2}\gamma_0^{[2]}\\
&= \left(\frac{2i}{\hbar}\right)^{k}\frac{3^{2k}}{(3k+2)_{3k}}[\Gamma(k+1)]^{2}\gamma_0^{[2]}+\chi_1^{[3k+2]}(k)+\left(\frac{2iz^2}{\hbar}\right)^{2}\left(\frac{2i}{\hbar}\right)^{k}\frac{3^{2k}[\Gamma(k+1)]^{2}}{(3k+2)!}\\
&\quad\sum_{\ell=1}^{k-1}\left[\frac{\Gamma\left(k-\ell+\frac{2}{3}\right)}{\Gamma\left(\frac{2}{3}\right)\Gamma(k-\ell+1)}\right]^{2}\frac{1}{3(k-\ell)-1}\left\{\sum_{m=1}^{k-\ell}\left[\frac{B\left(\frac{2}{3},m+\frac{1}{3}\right)}{B\left(\frac{1}{3},m+\frac{2}{3}\right)}\right]^{2}\frac{1}{3m-2}-1-\frac{3(k-\ell)}{3(k-\ell)-2}\left[\frac{B\left(\frac{2}{3},k-\ell+\frac{1}{3}\right)}{B\left(\frac{1}{3},k-\ell+\frac{2}{3}\right)}\right]^{2}\right\}\\
&= \left(-\frac{2iz^2}{\hbar}\right)^{2}\left(\frac{2i}{\hbar}\right)^{k}\frac{3^{2k}}{(3k+2)_{3k}2!}[\Gamma(k+1)]^{2}\\
&\quad+\left(\frac{2iz^2}{\hbar}\right)^{2}\left(\frac{2i}{\hbar}\right)^{k}\frac{3^{2k}[\Gamma(k+1)]^{2}}{(3k+2)!}\cdot\\
&\quad\sum_{\ell=1}^{k}\left[\frac{B\left(1,\ell+\frac{2}{3}\right)}{B\left(\frac{2}{3},\ell+1\right)}\right]^{2}\frac{1}{3\ell-1}\left\{\sum_{m=1}^{\ell}\left[\frac{B\left(\frac{2}{3},m+\frac{1}{3}\right)}{B\left(\frac{1}{3},m+\frac{2}{3}\right)}\right]^{2}\frac{1}{3m-2}-1-\frac{3\ell}{3\ell-2}\left[\frac{B\left(\frac{2}{3},\ell+\frac{1}{3}\right)}{B\left(\frac{1}{3},\ell+\frac{2}{3}\right)}\right]^{2}\right\}\\
&= \left(\frac{2iz^2}{\hbar}\right)^{2}\left(\frac{2i}{\hbar}\right)^{k}\frac{3^{2k}[\Gamma(k+1)]^{2}}{(3k+2)!}F_{z,2}(k),
\end{aligned}\tag{7}$$

where $F_{z,2}:\mathbb{N}^{*}\to\mathbb{R}$ is defined by :

$$F_{z,2}(k)=1+\sum_{\ell=1}^{k}\left[\frac{B\left(1,\ell+\frac{2}{3}\right)}{B\left(\frac{2}{3},\ell+1\right)}\right]^{2}\frac{1}{3\ell-1}\left\{\sum_{m=1}^{\ell}\left[\frac{B\left(\frac{2}{3},m+\frac{1}{3}\right)}{B\left(\frac{1}{3},m+\frac{2}{3}\right)}\right]^{2}\frac{1}{3m-2}-1-\frac{3\ell}{3\ell-2}\left[\frac{B\left(\frac{2}{3},\ell+\frac{1}{3}\right)}{B\left(\frac{1}{3},\ell+\frac{2}{3}\right)}\right]^{2}\right\}.$$

Similarly, if we place $\phi(t)=g(t)\exp\left[\frac{i}{\hbar}\left(\frac{t^3}{3}-z^2t\right)\right]$, we obtain the following ODE for g :

$$g''+\frac{2i}{\hbar}(t^2-z^2)g'+\left(\frac{4z^2t^2}{\hbar^2}+\frac{2it}{\hbar}\right)g=0$$

which looks like the ODE defining the function f, but i being swapped to $-i$. By writing down $g(t)=t^{\nu}\sum_{k=0}^{+\infty}d_kt^k$, we get :

$$\tfrac{1}{t^2}\textstyle\sum_{k=0}^{+\infty}(\nu+k)(\nu+k-1)d_kt^k-\frac{2iz^2}{\hbar t}\sum_{k=0}^{+\infty}(\nu+k)d_kt^k+\frac{2it}{\hbar}\sum_{k=0}^{+\infty}(\nu+k+1)d_kt^k+\frac{4z^2t^2}{\hbar^2}\sum_{k=0}^{+\infty}d_kt^k=0,$$

as well as the conditions :

- order -2 : $\nu(\nu-1)=0$
- order -1 : $(\nu+1)\nu d_1-\dfrac{2iz^2}{\hbar}\nu d_0=0$

$$\Longrightarrow\begin{cases}\text{if } \nu=0: d_1 \text{ is arbitrary}\\ \text{otherwise if } \nu=1: d_1=\dfrac{1}{(\nu+1)}\dfrac{2iz^2}{\hbar}\end{cases},$$

- order $+0$: $(\nu+2)(\nu+1)d_2-\dfrac{2iz^2}{\hbar}(\nu+1)d_1=0$

$$\Longrightarrow\begin{cases}\text{if } \nu=0: d_2 \text{ depends on the choice of } d_1\\ \text{otherwise if } \nu=1: d_2=\dfrac{1}{(\nu+2)_2}\left(\dfrac{2iz^2}{\hbar}\right)^{2}\end{cases},$$

- order $+1$: $(\nu+3)(\nu+2)d_3-\dfrac{2iz^2}{\hbar}(\nu+2)d_2+\dfrac{2i}{\hbar}(\nu+1)d_0=0$

$$\Longrightarrow\begin{cases}\text{if } \nu=0: d_3 \text{ depends on the choice of } d_1\\ \text{otherwise if } \nu=1: d_3=\dfrac{1}{(\nu+3)_3}\left(\dfrac{2iz^2}{\hbar}\right)^{3}-\dfrac{2i}{\hbar}\dfrac{(\nu+1)^2}{(\nu+3)_3}\end{cases}.$$

And for $k \geqslant 4$, the recurrence relation is given by :

$$d_k = \frac{2iz^2}{\hbar}\frac{1}{\nu+k}d_{k-1} - \frac{2i}{\hbar}\frac{(\nu+k-2)^2}{(\nu+k)_3}d_{k-3} - \frac{4z^2}{\hbar^2}\frac{(\nu+k-2)_2}{(\nu+k)_4}d_{k-4}\,.$$

For $k \geqslant 1$, we set $d_{3k+j} = \sum_{\ell=0}^{k-1}\delta_\ell^{[3k+j]}z^{6\ell+2j}$. The coefficient of the term of highest degree in z^2 can be easily deduced :

$$\delta_{k-1}^{[3k+j]} = \frac{1}{(3k+j)!}\left(\frac{2i}{\hbar}\right)^{3k-3+j}.$$

The above recurrence relation can be rewritten as :

$$d_k = \frac{2iz^2}{\hbar}\left[\frac{1}{k}d_{k-1} + \frac{2i}{\hbar}\frac{(k-2)_2}{k_4}d_{k-4}\right] - \frac{2i}{\hbar}\frac{(k-2)^2}{k_3}d_{k-3}\,. \tag{8}$$

In d_{3k}, the coefficient of the term of lowest degree in z^2 is :

$$\begin{aligned}\delta_0^{[3k]} &= \left(-\frac{2i}{\hbar}\right)^k\frac{(3k-2)^2(3k-5)^2\dots 1^2}{(3k)!}d_0\\ &= \left(-\frac{2i}{\hbar}\right)^k\frac{3^{2k}}{(3k)!}\left(k-\frac{2}{3}\right)_k^2 = \left(-\frac{2i}{\hbar}\right)^k\frac{3^{2k}}{(3k)!}\left[\frac{\Gamma\left(k+\frac{1}{3}\right)}{\Gamma\left(\frac{1}{3}\right)}\right]^2.\end{aligned} \tag{9}$$

Assuming that k is large enough, let us compare the magnitudes of $\delta_{k-1}^{[3k]}z^{6k-6}$ and $\delta_0^{[3k]}$ in the coefficient d_{3k} :

$$\begin{aligned}\left|\frac{\delta_0^{[3k]}}{\delta_{k-1}^{[3k]}z^{6k-6}}\right| &= \left(\frac{2}{\hbar}\right)^3\left(\frac{\hbar}{2}\right)^{2k}\frac{3^{2k}}{z^{6k-6}}\left[\frac{\Gamma\left(k+\frac{1}{3}\right)}{\Gamma\left(\frac{1}{3}\right)}\right]^2 \simeq \left(\frac{2}{\hbar}\right)^3\left(\frac{\hbar}{2}\right)^{2k}\frac{3^{2k}}{z^{6k-6}}\left[\frac{\Gamma\left(k+\frac{4}{3}\right)}{k\Gamma\left(\frac{1}{3}\right)}\right]^2\\ &\simeq \left(\frac{2}{\hbar}\right)^3\left(\frac{\hbar}{2}\right)^{2k}\frac{3^{2k}}{z^{6k-6}}\left[\frac{k^{1/3}\sqrt{2\pi k}\left(\frac{k}{e}\right)^k}{k\Gamma\left(\frac{1}{3}\right)}\right]^2\\ &= \left(\frac{2z^2}{\hbar}\right)^3\frac{2\pi}{k^{1/3}\left[\Gamma\left(\frac{1}{3}\right)\right]^2}\left(\frac{3\hbar k}{2ez^3}\right)^{2k}\ [k\to+\infty]\longrightarrow+\infty\,.\end{aligned}$$

Introduce the quantity :

$$\begin{aligned}\chi_0^{[3\ell+1]}(\ell) &= \frac{2iz^2}{\hbar}\left[\frac{1}{3\ell+1}\delta_0^{[3\ell]} + \frac{2i}{\hbar}\frac{(3\ell-1)_2}{(3\ell+1)_4}\delta_0^{[3(\ell-1)]}\right]\\ &= \frac{2iz^2}{\hbar}\left(-\frac{2i}{\hbar}\right)^\ell\frac{3^{2\ell}}{(3\ell+1)!}\left[\frac{\Gamma\left(\ell+\frac{1}{3}\right)}{\Gamma\left(\frac{1}{3}\right)}\right]^2\left\{1-\frac{(3\ell-1)_2}{3^2}\left[\frac{\Gamma\left(\ell-1+\frac{1}{3}\right)}{\Gamma\left(\ell+\frac{1}{3}\right)}\right]^2\right\}\\ &= \frac{2iz^2}{\hbar}\left(-\frac{2i}{\hbar}\right)^\ell\frac{3^{2\ell}}{(3\ell+1)!}\left[\frac{\Gamma\left(\ell+\frac{1}{3}\right)}{\Gamma\left(\frac{1}{3}\right)}\right]^2\left[1-\frac{(3\ell-1)_2}{(3\ell-2)^2}\right]\\ &= -\frac{2iz^2}{\hbar}\left(-\frac{2i}{\hbar}\right)^\ell\frac{3^{2\ell}}{(3\ell+1)!}\left[\frac{\Gamma\left(\ell+\frac{1}{3}\right)}{\Gamma\left(\frac{1}{3}\right)}\right]^2\frac{1}{3\ell-2}\,.\end{aligned}$$

Now the coefficient of the term of lowest degree in z^2 in d_{3k+1} can be computed by means of a discrete convolution. Therefore :

$$\begin{aligned}\delta_0^{[3k+1]} &= \chi_0^{[3k+1]}(k) + \sum_{\ell=1}^{k-1}\left(-\frac{2i}{\hbar}\right)^\ell\frac{3^{2\ell}}{(3k+1)_{3\ell}}\left(k-\frac{1}{3}\right)_\ell^2\chi_0^{[3(k-\ell)+1]}(k-\ell)\\ &= \chi_0^{[3k+1]}(k) - \frac{2iz^2}{\hbar}\left(-\frac{2i}{\hbar}\right)^k\frac{3^{2k}}{(3k+1)!}\sum_{\ell=1}^{k-1}\left[\frac{\Gamma\left(k+\frac{2}{3}\right)}{\Gamma\left(k-\ell+\frac{2}{3}\right)}\right]^2\left[\frac{\Gamma\left(k-\ell+\frac{1}{3}\right)}{\Gamma\left(\frac{1}{3}\right)}\right]^2\frac{1}{3(k-\ell)-2}\\ &= \chi_0^{[3k+1]}(k) - \frac{2iz^2}{\hbar}\left(-\frac{2i}{\hbar}\right)^k\frac{3^{2k}}{(3k+1)!}\left[\frac{\Gamma\left(k+\frac{2}{3}\right)}{\Gamma\left(\frac{2}{3}\right)}\right]^2\sum_{\ell=1}^{k-1}\left[\frac{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\ell+\frac{1}{3}\right)}{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\ell+\frac{2}{3}\right)}\right]^2\frac{1}{3\ell-2}\\ &= -\frac{2iz^2}{\hbar}\left(-\frac{2i}{\hbar}\right)^k\frac{3^{2k}}{(3k+1)!}\left[\frac{\Gamma\left(k+\frac{2}{3}\right)}{\Gamma\left(\frac{2}{3}\right)}\right]^2\underbrace{\sum_{\ell=1}^{k}\left[\frac{B\left(\frac{2}{3},\ell+\frac{1}{3}\right)}{B\left(\frac{1}{3},\ell+\frac{2}{3}\right)}\right]^2\frac{1}{3\ell-2}}_{G_{z,1}(k)=1+F_{z,1}(k)}\,.\end{aligned} \tag{10}$$

When x is fixed, for large values of y : $B(x,y) \simeq \frac{\Gamma(x)}{y^x}$. Consequently, the general term in the sum is a $\mathscr{O}\left(\frac{1}{\ell^{5/3}}\right)$ for ℓ large enough.

Evaluate the quantity :

$$\begin{aligned}\chi_1^{[3\ell+2]}(\ell) &= \frac{2iz^2}{\hbar}\left[\frac{1}{3\ell+2}\delta_0^{[3\ell+1]} + \frac{2i}{\hbar}\frac{(3\ell)_2}{(3\ell+2)_4}\delta_0^{[3\ell-2)]}\right]\\ &= -\left(\frac{2iz^2}{\hbar}\right)^2\left(-\frac{2i}{\hbar}\right)^\ell\frac{3^{2\ell}}{(3\ell+2)!}\left[\frac{\Gamma\left(\ell+\frac{2}{3}\right)}{\Gamma\left(\frac{2}{3}\right)}\right]^2\\ &\quad\cdot\left\{\sum_{m=1}^{\ell}\left[\frac{B\left(\frac{2}{3},m+\frac{1}{3}\right)}{B\left(\frac{1}{3},m+\frac{2}{3}\right)}\right]^2\frac{1}{3m-2} - \frac{(3\ell)_2}{(3\ell-1)^2}\sum_{m=1}^{\ell-1}\left[\frac{B\left(\frac{2}{3},m+\frac{1}{3}\right)}{B\left(\frac{1}{3},m+\frac{2}{3}\right)}\right]^2\frac{1}{3m-2}\right\}\\ &= \left(\frac{2iz^2}{\hbar}\right)^2\left(-\frac{2i}{\hbar}\right)^\ell\frac{3^{2\ell}}{(3\ell+2)!}\left[\frac{\Gamma\left(\ell+\frac{2}{3}\right)}{\Gamma\left(\frac{2}{3}\right)}\right]^2\frac{1}{3\ell-1}\\ &\quad\cdot\left\{\sum_{m=1}^{\ell}\left[\frac{B\left(\frac{2}{3},m+\frac{1}{3}\right)}{B\left(\frac{1}{3},m+\frac{2}{3}\right)}\right]^2\frac{1}{3m-2} - \frac{3\ell}{3\ell-2}\left[\frac{B\left(\frac{2}{3},\ell+\frac{1}{3}\right)}{B\left(\frac{1}{3},\ell+\frac{2}{3}\right)}\right]^2\right\}.\end{aligned}$$

Again, the coefficient of the term of lowest degree in z^2 in d_{3k+2} can be computed by :

$$\begin{aligned}\delta_0^{[3k+2]} &= \chi_1^{[3k+2]}(k) + \sum_{\ell=1}^{k-1}\left(-\frac{2i}{\hbar}\right)^\ell\frac{3^{2\ell}}{(3k+2)_{3\ell}}k_\ell^2\chi_1^{[3(k-\ell)+2]}(k-\ell)\\ &= \chi_1^{[3k+2]}(k) + \left(\frac{2iz^2}{\hbar}\right)^2\left(-\frac{2i}{\hbar}\right)^k\frac{3^{2k}}{(3k+2)!}\\ &\quad\sum_{\ell=1}^{k-1}\left[\frac{\Gamma\left(k-\ell+\frac{2}{3}\right)}{\Gamma\left(\frac{2}{3}\right)}\right]^2\frac{k_\ell^2}{3(k-\ell)-1}\left\{\sum_{m=1}^{k-\ell}\left[\frac{B\left(\frac{2}{3},m+\frac{1}{3}\right)}{B\left(\frac{1}{3},m+\frac{2}{3}\right)}\right]^2\frac{1}{3m-2} - \frac{3(k-\ell)}{3(k-\ell)-2}\left[\frac{B\left(\frac{2}{3},k-\ell+\frac{1}{3}\right)}{B\left(\frac{1}{3},k-\ell+\frac{2}{3}\right)}\right]^2\right\}\\ &= \chi_1^{[3k+2]}(k) + \left(\frac{2iz^2}{\hbar}\right)^2\left(-\frac{2i}{\hbar}\right)^k\frac{3^{2k}}{(3k+2)!}\\ &\quad\sum_{\ell=1}^{k-1}\left[\frac{\Gamma\left(k-\ell+\frac{2}{3}\right)\Gamma(k+1)}{\Gamma\left(\frac{2}{3}\right)\Gamma(k-\ell+1)}\right]^2\frac{1}{3(k-\ell)-1}\\ &\quad\cdot\left\{\sum_{m=1}^{k-\ell}\left[\frac{B\left(\frac{2}{3},m+\frac{1}{3}\right)}{B\left(\frac{1}{3},m+\frac{2}{3}\right)}\right]^2\frac{1}{3m-2} - \frac{3(k-\ell)}{3(k-\ell)-2}\left[\frac{B\left(\frac{2}{3},k-\ell+\frac{1}{3}\right)}{B\left(\frac{1}{3},k-\ell+\frac{2}{3}\right)}\right]^2\right\}\\ &= \left(\frac{2iz^2}{\hbar}\right)^2\left(-\frac{2i}{\hbar}\right)^k\frac{3^{2k}}{(3k+2)!}\left[\frac{\Gamma\left(k+\frac{2}{3}\right)}{\Gamma\left(\frac{2}{3}\right)}\right]^2\frac{1}{3k-1}\\ &\quad\cdot\left\{\sum_{m=1}^{k}\left[\frac{B\left(\frac{2}{3},m+\frac{1}{3}\right)}{B\left(\frac{1}{3},m+\frac{2}{3}\right)}\right]^2\frac{1}{3m-2} - \frac{3k}{3k-2}\left[\frac{B\left(\frac{2}{3},k+\frac{1}{3}\right)}{B\left(\frac{1}{3},k+\frac{2}{3}\right)}\right]^2\right\}\\ &\quad+\left(\frac{2iz^2}{\hbar}\right)^2\left(-\frac{2i}{\hbar}\right)^k\frac{3^{2k}\Gamma(k+1)^2}{(3k+2)!}\sum_{\ell=1}^{k-1}\left[\frac{\Gamma\left(\ell+\frac{2}{3}\right)}{\Gamma\left(\frac{2}{3}\right)\Gamma(\ell+1)}\right]^2\frac{1}{3\ell-1}\\ &\quad\cdot\left\{\sum_{m=1}^{\ell}\left[\frac{B\left(\frac{2}{3},m+\frac{1}{3}\right)}{B\left(\frac{1}{3},m+\frac{2}{3}\right)}\right]^2\frac{1}{3m-2} - \frac{3\ell}{3\ell-2}\left[\frac{B\left(\frac{2}{3},\ell+\frac{1}{3}\right)}{B\left(\frac{1}{3},\ell+\frac{2}{3}\right)}\right]^2\right\}\\ &= \left(\frac{2iz^2}{\hbar}\right)^2\left(-\frac{2i}{\hbar}\right)^k\frac{3^{2k}\Gamma(k+1)^2}{(3k+2)!}\\ &\quad\cdot\underbrace{\sum_{\ell=1}^{k}\left[\frac{B\left(1,\ell+\frac{2}{3}\right)}{B\left(\frac{2}{3},\ell+1\right)}\right]^2\frac{1}{3\ell-1}\left\{\sum_{m=1}^{\ell}\left[\frac{B\left(\frac{2}{3},m+\frac{1}{3}\right)}{B\left(\frac{1}{3},m+\frac{2}{3}\right)}\right]^2\frac{1}{3m-2} - \frac{3\ell}{3\ell-2}\left[\frac{B\left(\frac{2}{3},\ell+\frac{1}{3}\right)}{B\left(\frac{1}{3},\ell+\frac{2}{3}\right)}\right]^2\right\}}_{G_{z,2}(k)}.\end{aligned} \tag{11}$$

The general term in the sum is a $\mathscr{O}\left(\frac{1}{\ell^{4/3}}\right)$ for ℓ large enough.

2.4 Dominant exponential solutions in the $\frac{\pi}{6}$ and $\frac{5\pi}{6}$ directions

For the series $g_1=\sum_{k=0}^{+\infty} d_k t^k$, when k becomes large, the summands in the polynomial expansion of d_k are negligible before the term $\delta_0^{[k]} z^{6\{k/3\}}$ of lowest degree. Moreover :

$$\sum_{k=0}^{+\infty} d_k t^k = 1+\sum_{k=1}^{+\infty}\left(d_{3k}+d_{3k+1}t+d_{3k+2}t^2\right)t^{3k} \simeq \sum_{k=1}^{+\infty}\left(\delta_0^{[3k]}+\delta_0^{[3k+1]}t+\delta_0^{[3k+2]}t^2\right)t^{3k}$$

$$=\sum_{k=1}^{+\infty}\left\{1-\frac{2iz^2}{\hbar}\frac{G_{z,1}(k)}{3k+1}\left[\frac{\Gamma\left(\frac13\right)\Gamma\left(k+\frac23\right)}{\Gamma\left(\frac23\right)\Gamma\left(k+\frac13\right)}\right]^2 t+\left(\frac{2iz^2}{\hbar}\right)^2\frac{G_{z,2}(k)}{(3k+2)_2}\left[\frac{\Gamma\left(\frac13\right)\Gamma(k+1)}{\Gamma\left(k+\frac13\right)}\right]^2 t^2\right\}$$
$$\cdot\left(-\frac{2i}{\hbar}\right)^k\frac{3^{2k}}{(3k)!}\left[\frac{\Gamma\left(k+\frac13\right)}{\Gamma\left(\frac13\right)}\right]^2 t^{3k}$$

$$\simeq\sum_{k=1}^{+\infty}\left\{1-\frac{2iz^2}{\hbar}\frac{G_{z,1}^\infty}{3k+1}\left[\frac{B\left(\frac13,k+\frac23\right)}{B\left(\frac23,k+\frac13\right)}\right]^2 t+\left(\frac{2iz^2}{\hbar}\right)^2\frac{G_{z,2}^\infty}{(3k+2)_2}\left[\frac{B\left(\frac13,k+1\right)}{B\left(1,k+\frac13\right)}\right]^2 t^2\right\}$$
$$\cdot\left(-\frac{2i}{\hbar}\right)^k\frac{3^{2k}}{(3k)!}\left[\frac{\Gamma\left(k+\frac13\right)}{\Gamma\left(\frac13\right)}\right]^2 t^{3k},$$

with the numerical values :

$$\begin{cases} G_{z,1}^\infty=\lim\limits_{k\to+\infty}G_{z,1}(k)\simeq 3,67.10^{-1}\\ G_{z,2}^\infty=\lim\limits_{k\to+\infty}G_{z,2}(k)\simeq -7,1.10^{-2}\end{cases}.$$

For $\operatorname{Arg} t=\frac{\pi}{6}$ i.e. $t=Te^{i\frac{\pi}{6}}$ (allowing $T\to+\infty$ as usual) :

$$\sum_{k=0}^{+\infty} d_k t^k \simeq$$
$$\sum_{k=1}^{+\infty}\left\{1-\frac{2iz^2}{\hbar}\frac{G_{z,1}^\infty}{3k+1}\left[\frac{B\left(\frac13,k+\frac23\right)}{B\left(\frac23,k+\frac13\right)}\right]^2 Te^{i\frac{\pi}{6}}+\left(\frac{2iz^2}{\hbar}\right)^2\frac{G_{z,2}^\infty}{(3k+2)_2}\left[\frac{B\left(\frac13,k+1\right)}{B\left(1,k+\frac13\right)}\right]^2 T^2e^{i\frac{\pi}{3}}\right\}$$
$$\cdot\left(\frac{2}{\hbar}\right)^k\frac{3^{2k}}{(3k)!}\left[\frac{\Gamma\left(k+\frac13\right)}{\Gamma\left(\frac13\right)}\right]^2 T^{3k}$$

$$\simeq\sum_{k=1}^{+\infty}\left\{1-\frac{2iz^2}{\hbar}\left[\frac{\Gamma\left(\frac13\right)}{\Gamma\left(\frac23\right)}\right]^2\frac{G_{z,1}^\infty Te^{i\frac{\pi}{6}}}{3k^{1/3}}+\left(\frac{2iz^2}{\hbar}\right)^2\left[\Gamma\left(\frac13\right)\right]^2\frac{G_{z,2}^\infty T^2e^{i\frac{\pi}{3}}}{9k^{2/3}}\right\}$$
$$\cdot\left(\frac{2}{\hbar}\right)^k\frac{3^{2k}}{(3k)!}\left[\frac{\Gamma\left(k+\frac13\right)}{\Gamma\left(\frac13\right)}\right]^2 T^{3k}$$

$$\simeq\sum_{k=1}^{+\infty}\left\{\frac{1}{\left[\Gamma\left(\frac13\right)\right]^2}-\frac{2iz^2}{\hbar}\frac{1}{\left[\Gamma\left(\frac23\right)\right]^2}\frac{G_{z,1}^\infty Te^{i\frac{\pi}{6}}}{3k^{1/3}}+\left(\frac{2iz^2}{\hbar}\right)^2\frac{G_{z,2}^\infty T^2e^{i\frac{\pi}{3}}}{9k^{2/3}}\right\}$$
$$\cdot\left(\frac{2}{\hbar}\right)^k\frac{3^{2k}}{\Gamma(3k+1)}\frac{[\Gamma(k+1)]^2}{k^{4/3}}T^{3k}$$

$$\simeq\left\{\frac{1}{\left[\Gamma\left(\frac13\right)\right]^2}-\frac{2iz^2}{\hbar}\frac{1}{\left[\Gamma\left(\frac23\right)\right]^2}\frac{G_{z,1}^\infty Te^{i\frac{\pi}{6}}}{3x_0^{1/3}}+\left(\frac{2iz^2}{\hbar}\right)^2\frac{G_{z,2}^\infty T^2e^{i\frac{\pi}{3}}}{9x_0^{2/3}}\right\}$$
$$\cdot\frac{2\pi x_0}{\sqrt3}\frac{1}{x_0^{4/3}}\int_0^{+\infty}\frac{1}{\sqrt{2\pi x}}\left(\frac{2}{\hbar}\right)^x 3^{2x}\left(\frac{e}{3x}\right)^{3x}\left(\frac{x}{e}\right)^{2x}T^{3x}dx$$

$$\simeq\left\{\frac{1}{\left[\Gamma\left(\frac13\right)\right]^2}-\frac{2iz^2}{\hbar}\frac{1}{\left[\Gamma\left(\frac23\right)\right]^2}\frac{G_{z,1}^\infty Te^{i\frac{\pi}{6}}}{3x_0^{1/3}}+\left(\frac{2iz^2}{\hbar}\right)^2\frac{G_{z,2}^\infty T^2e^{i\frac{\pi}{3}}}{9x_0^{2/3}}\right\}$$
$$\cdot\frac{2\pi}{\sqrt3}\frac{1}{x_0^{1/3}}\int_0^{+\infty}\frac{1}{\sqrt{2\pi x}}\left(\frac{2T^3}{3\hbar}\right)^x\left(\frac{e}{x}\right)^x dx$$

$$\simeq\left\{\frac{1}{\left[\Gamma\left(\frac13\right)\right]^2}-\frac{2iz^2}{\hbar}\frac{1}{\left[\Gamma\left(\frac23\right)\right]^2}\frac{G_{z,1}^\infty Te^{i\frac{\pi}{6}}}{3x_0^{1/3}}+\left(\frac{2iz^2}{\hbar}\right)^2\frac{G_{z,2}^\infty T^2e^{i\frac{\pi}{3}}}{9x_0^{2/3}}\right\}\frac{2\pi}{\sqrt3}\frac{1}{x_0^{1/3}}e^{x_0},$$

where $x_0=\frac{2T^3}{3\hbar}$ by using Corollary 1. So in the $\frac{\pi}{6}$ direction, we have asymptotically :

$$g_1(t)=\left\{\frac{1}{\left[\Gamma\left(\frac13\right)\right]^2}-\left(\frac{2}{3\hbar}\right)^{2/3}\frac{ie^{i\frac{\pi}{6}}G_{z,1}^\infty}{\left[\Gamma\left(\frac23\right)\right]^2}z^2-\left(\frac{2}{3\hbar}\right)^{4/3}e^{i\frac{\pi}{3}}G_{z,2}^\infty z^4\right\}\frac{2\pi}{\sqrt3}\left(\frac{3\hbar}{2}\right)^{1/3}\frac{1}{T}\exp\left(\frac{2T^3}{3\hbar}\right)$$

$$=\frac{2\pi}{\sqrt3}\left(\frac{3\hbar}{2}\right)^{1/3}\left\{\frac{e^{i\frac{\pi}{6}}}{\left[\Gamma\left(\frac13\right)\right]^2}-\left(\frac{2}{3\hbar}\right)^{2/3}\frac{ie^{i\frac{\pi}{3}}G_{z,1}^\infty}{\left[\Gamma\left(\frac23\right)\right]^2}z^2-\left(\frac{2}{3\hbar}\right)^{4/3}iG_{z,2}^\infty z^4\right\}\frac{1}{t}\exp\left(-\frac{2it^3}{3\hbar}\right)$$

$$\simeq -i\frac{4\pi G_{z,2}^\infty z^4}{3\sqrt3\hbar}\left\{1+\left(\frac{3\hbar}{2}\right)^{2/3}\frac{e^{i\frac{\pi}{3}}}{\left[\Gamma\left(\frac23\right)\right]^2}\frac{G_{z,1}^\infty}{G_{z,2}^\infty}\frac{1}{z^2}\right\}\frac{1}{t}\exp\left(-\frac{2it^3}{3\hbar}\right)$$

by recalling that z is large. Similarly in the $\frac{5\pi}{6}$ direction, we have asymptotically :

$$g_1(t)=\frac{2\pi}{\sqrt3}\left(\frac{3\hbar}{2}\right)^{1/3}\left\{\frac{e^{i\frac{5\pi}{6}}}{\left[\Gamma\left(\frac13\right)\right]^2}-\left(\frac{2}{3\hbar}\right)^{2/3}\frac{ie^{i\frac{5\pi}{3}}G_{z,1}^\infty}{\left[\Gamma\left(\frac23\right)\right]^2}z^2-\left(\frac{2}{3\hbar}\right)^{4/3}e^{i\frac{5\pi}{2}}G_{z,2}^\infty z^4\right\}\frac{1}{t}\exp\left(-\frac{2it^3}{3\hbar}\right)$$

$$\simeq -i\frac{4\pi G_{z,2}^\infty z^4}{3\sqrt3\hbar}\left\{1+\left(\frac{3\hbar}{2}\right)^{2/3}\frac{e^{-i\frac{\pi}{3}}}{\left[\Gamma\left(\frac23\right)\right]^2}\frac{G_{z,1}^\infty}{G_{z,2}^\infty}\frac{1}{z^2}\right\}\frac{1}{t}\exp\left(-\frac{2it^3}{3\hbar}\right).$$

2.5 Dominant exponential solutions in the $\frac{\pi}{2}$ and $\frac{7\pi}{6}$ directions

For the series $f_1=\sum_{k=0}^{+\infty} c_k t^k$, when k becomes large, we shall neglect every summand in the polynomial expansion of c_k but $\gamma_0^{[k]} z^{6\{k/3\}}$. Hence :

$$\sum_{k=0}^{+\infty} c_k t^k = 1-\frac{2iz^2}{\hbar}t+\frac{1}{2!}\left(-\frac{2iz^2}{\hbar}\right)^2t^2+\sum_{k=1}^{+\infty}\left(c_{3k}+d_{3k+1}t+d_{3k+2}t^2\right)t^{3k}\simeq\sum_{k=1}^{+\infty}\left(\gamma_0^{[3k]}+\gamma_0^{[3k+1]}t+\gamma_0^{[3k+2]}t^2\right)t^{3k}$$

$$=\sum_{k=1}^{+\infty}\left\{1+\frac{2iz^2}{\hbar}\frac{F_{z,1}(k)}{3k+1}\left[\frac{\Gamma\left(\frac13\right)\Gamma\left(k+\frac23\right)}{\Gamma\left(\frac23\right)\Gamma\left(k+\frac13\right)}\right]^2 t+\left(\frac{2iz^2}{\hbar}\right)^2\frac{F_{z,2}^\infty}{(3k+2)_2}\left[\frac{\Gamma\left(\frac13\right)\Gamma(k+1)}{\Gamma\left(k+\frac13\right)}\right]^2 t^2\right\}$$
$$\cdot\left(\frac{2i}{\hbar}\right)^k\frac{3^{2k}}{(3k)!}\left[\frac{\Gamma\left(k+\frac13\right)}{\Gamma\left(\frac13\right)}\right]^2 t^{3k}$$

$$\simeq\sum_{k=1}^{+\infty}\left\{1+\frac{2iz^2}{\hbar}\frac{F_{z,1}^\infty}{3k+1}\left[\frac{B\left(\frac13,k+\frac23\right)}{B\left(\frac23,k+\frac13\right)}\right]^2 t+\left(\frac{2iz^2}{\hbar}\right)^2\frac{F_{z,2}^\infty}{(3k+2)_2}\left[\frac{B\left(\frac13,k+1\right)}{B\left(1,k+\frac13\right)}\right]^2 t^2\right\}$$
$$\cdot\left(\frac{2i}{\hbar}\right)^k\frac{3^{2k}}{(3k)!}\left[\frac{\Gamma\left(k+\frac13\right)}{\Gamma\left(\frac13\right)}\right]^2 t^{3k},$$

with the numerical values :

$$\begin{cases} F_{z,1}^\infty=\lim\limits_{k\to+\infty}F_{z,1}(k)=G_{z,1}^\infty-1\\ F_{z,2}^\infty=\lim\limits_{k\to+\infty}F_{z,2}^\infty\simeq 4,9.10^{-1}\end{cases}.$$

For $\operatorname{Arg} t=\frac{\pi}{2}$ i.e. $t=iT$:

$$\sum_{k=0}^{+\infty} c_k t^k \simeq\sum_{k=1}^{+\infty}\left\{1-\frac{2z^2}{\hbar}\frac{F_{z,1}^\infty}{3k+1}\left[\frac{B\left(\frac13,k+\frac23\right)}{B\left(\frac23,k+\frac13\right)}\right]^2 T+\left(\frac{2z^2}{\hbar}\right)^2\frac{F_{z,2}^\infty}{(3k+2)_2}\left[\frac{B\left(\frac13,k+1\right)}{B\left(1,k+\frac13\right)}\right]^2 T^2\right\}\left(\frac{2}{\hbar}\right)^k\frac{3^{2k}}{(3k)!}\left[\frac{\Gamma\left(k+\frac13\right)}{\Gamma\left(\frac13\right)}\right]^2 T^{3k}$$

$$\simeq\left\{\frac{1}{\left[\Gamma\left(\frac13\right)\right]^2}-\frac{2z^2}{\hbar}\frac{1}{\left[\Gamma\left(\frac23\right)\right]^2}\frac{F_{z,1}^\infty Te^{i\frac{\pi}{6}}}{3x_0^{1/3}}+\left(\frac{2z^2}{\hbar}\right)^2\frac{F_{z,2}^\infty T^2e^{i\frac{\pi}{3}}}{9x_0^{2/3}}\right\}\frac{2\pi}{\sqrt3}\frac{1}{x_0^{1/3}}e^{x_0},$$

where $x_0 = \frac{2T^3}{3\hbar}$ by using Corollary 1. So in the $\frac{\pi}{2}$ direction, we have asymptotically :

$$f_1(t) = \left\{ \frac{1}{\left[\Gamma\left(\frac{1}{3}\right)\right]^2} - \left(\frac{2}{3\hbar}\right)^{2/3} \frac{F^\infty_{z,1}}{\left[\Gamma\left(\frac{2}{3}\right)\right]^2} z^2 + \left(\frac{2}{3\hbar}\right)^{4/3} F^\infty_{z,2} z^4 \right\} \frac{2\pi}{\sqrt{3}} \left(\frac{3\hbar}{2}\right)^{1/3} \frac{1}{T} \exp\left(\frac{2T^3}{3\hbar}\right)$$
$$\simeq i\frac{4\pi F^\infty_{z,2} z^4}{3\sqrt{3}\hbar} \left\{ 1 - \left(\frac{3\hbar}{2}\right)^{2/3} \frac{1}{\left[\Gamma\left(\frac{2}{3}\right)\right]^2} \frac{F^\infty_{z,1}}{F^\infty_{z,2}} \frac{1}{z^2} \right\} \frac{1}{t} \exp\left(\frac{2it^3}{3\hbar}\right)$$

since z is large. Similarly for $\operatorname{Arg} t = \frac{7\pi}{6}$ i.e. $t = Te^{i\frac{7\pi}{6}}$:

$$\sum_{k=0}^{+\infty} c_k t^k \simeq \sum_{k=1}^{+\infty} \left\{ 1 - \frac{2iz^2}{\hbar} \frac{F^\infty_{z,1}}{3k+1} \left[\frac{B\left(\frac{1}{3}, k+\frac{2}{3}\right)}{B\left(\frac{2}{3}, k+\frac{1}{3}\right)} \right]^2 Te^{i\frac{\pi}{6}} + \left(\frac{2iz^2}{\hbar}\right)^2 \frac{F^\infty_{z,2}}{(3k+2)_2} \left[\frac{B\left(\frac{1}{3}, k+1\right)}{B\left(1, k+\frac{1}{3}\right)} \right]^2 T^2 e^{i\frac{\pi}{3}} \right\}$$
$$\left(\frac{2}{\hbar}\right)^k \frac{3^{2k}}{(3k)!} \left[\frac{\Gamma\left(k+\frac{1}{3}\right)}{\Gamma\left(\frac{1}{3}\right)} \right]^2 T^{3k}$$
$$\simeq \left\{ \frac{1}{\left[\Gamma\left(\frac{1}{3}\right)\right]^2} - \frac{2z^2}{\hbar} \frac{1}{\left[\Gamma\left(\frac{2}{3}\right)\right]^2} \frac{F^\infty_{z,1} Te^{i\frac{2\pi}{3}}}{3x_0^{1/3}} - \left(\frac{2z^2}{\hbar}\right)^2 \frac{F^\infty_{z,2} T^2 e^{i\frac{\pi}{3}}}{9x_0^{2/3}} \right\} \frac{2\pi}{\sqrt{3}} \frac{1}{x_0^{1/3}} e^{x_0}$$

by using Corollary 1. In the $\frac{7\pi}{6}$ direction, we have asymptotically :

$$f_1(t) = \frac{2\pi}{\sqrt{3}} \left(\frac{3\hbar}{2}\right)^{1/3} \left\{ -\frac{e^{i\frac{\pi}{6}}}{\left[\Gamma\left(\frac{1}{3}\right)\right]^2} - \left(\frac{2}{3\hbar}\right)^{2/3} \frac{e^{-i\frac{\pi}{6}} F^\infty_{z,1}}{\left[\Gamma\left(\frac{2}{3}\right)\right]^2} z^2 - \left(\frac{2}{3\hbar}\right)^{4/3} e^{-i\frac{\pi}{2}} F^\infty_{z,2} z^4 \right\} \frac{1}{t} \exp\left(\frac{2it^3}{3\hbar}\right)$$
$$\simeq i\frac{4\pi F^\infty_{z,2} z^4}{3\sqrt{3}\hbar} \left\{ 1 + \left(\frac{3\hbar}{2}\right)^{2/3} \frac{e^{i\frac{\pi}{3}}}{\left[\Gamma\left(\frac{2}{3}\right)\right]^2} \frac{F^\infty_{z,1}}{F^\infty_{z,2}} \frac{1}{z^2} \right\} \frac{1}{t} \exp\left(\frac{2it^3}{3\hbar}\right)$$

since z is large.

2.6 Stokes multipliers

Recall that $\phi(t) = f_1(t) \exp\left[-\frac{i}{\hbar}\left(\frac{t^3}{3} - z^2 t\right)\right] = g_1(t) \exp\left[\frac{i}{\hbar}\left(\frac{t^3}{3} - z^2 t\right)\right]$.
Let us recap our results :

- for $\operatorname{Arg} t = \frac{\pi}{6} : \phi(t) = -i\frac{4\pi G^\infty_{z,2} z^4}{3\sqrt{3}\hbar} \left\{ 1 + \left(\frac{3\hbar}{2}\right)^{2/3} \frac{e^{i\frac{\pi}{3}}}{\left[\Gamma\left(\frac{2}{3}\right)\right]^2} \frac{G^\infty_{z,1}}{G^\infty_{z,2}} \frac{1}{z^2} \right\} \frac{1}{t} \exp\left[-\frac{i}{\hbar}\left(\frac{t^3}{3} + z^2 t\right)\right]$
- for $\operatorname{Arg} t = \frac{\pi}{2} : \phi(t) = i\frac{4\pi F^\infty_{z,2} z^4}{3\sqrt{3}\hbar} \left\{ 1 - \left(\frac{3\hbar}{2}\right)^{2/3} \frac{1}{\left[\Gamma\left(\frac{2}{3}\right)\right]^2} \frac{F^\infty_{z,1}}{F^\infty_{z,2}} \frac{1}{z^2} \right\} \frac{1}{t} \exp\left[\frac{i}{\hbar}\left(\frac{t^3}{3} + z^2 t\right)\right]$
- for $\operatorname{Arg} t = \frac{5\pi}{6} : \phi(t) = -i\frac{4\pi G^\infty_{z,2} z^4}{3\sqrt{3}\hbar} \left\{ 1 + \left(\frac{3\hbar}{2}\right)^{2/3} \frac{e^{-i\frac{\pi}{3}}}{\left[\Gamma\left(\frac{2}{3}\right)\right]^2} \frac{G^\infty_{z,1}}{G^\infty_{z,2}} \frac{1}{z^2} \right\} \frac{1}{t} \exp\left[-\frac{i}{\hbar}\left(\frac{t^3}{3} + z^2 t\right)\right]$
- for $\operatorname{Arg} t = \frac{7\pi}{6} : \phi(t) = i\frac{4\pi F^\infty_{z,2} z^4}{3\sqrt{3}\hbar} \left\{ 1 + \left(\frac{3\hbar}{2}\right)^{2/3} \frac{e^{i\frac{\pi}{3}}}{\left[\Gamma\left(\frac{2}{3}\right)\right]^2} \frac{F^\infty_{z,1}}{F^\infty_{z,2}} \frac{1}{z^2} \right\} \frac{1}{t} \exp\left[\frac{i}{\hbar}\left(\frac{t^3}{3} + z^2 t\right)\right]$

in the sense of asymptotic series. So the Stokes multipliers are approximated by :

$$\widetilde{C_{S1}} = -\frac{G^\infty_{z,2}}{F^\infty_{z,2}} \left(\frac{3\hbar}{2}\right)^{2/3} \frac{e^{-i\frac{\pi}{3}} - e^{i\frac{\pi}{3}}}{\left[\Gamma\left(\frac{2}{3}\right)\right]^2} \frac{G^\infty_{z,1}}{G^\infty_{z,2}} \frac{1}{z^2}$$
$$= \left(\frac{3\hbar}{2}\right)^{2/3} \frac{i\sqrt{3}}{\left[\Gamma\left(\frac{2}{3}\right)\right]^2} \frac{G^\infty_{z,1}}{F^\infty_{z,2}} \frac{1}{z^2}$$

and $$\widetilde{C_{S2}} = -\frac{F^\infty_{z,2}}{G^\infty_{z,2}} \left(\frac{3\hbar}{2}\right)^{2/3} \frac{e^{i\frac{\pi}{3}} + 1}{\left[\Gamma\left(\frac{2}{3}\right)\right]^2} \frac{F^\infty_{z,1}}{F^\infty_{z,2}} \frac{1}{z^2} = -\left(\frac{3\hbar}{2}\right)^{2/3} \frac{\sqrt{3}e^{i\frac{\pi}{6}}}{\left[\Gamma\left(\frac{2}{3}\right)\right]^2} \frac{G^\infty_{z,1}}{F^\infty_{z,2}} \frac{1}{z^2}$$

while $$\widetilde{C_{S1}}\widetilde{C_{S2}} = -\left(\frac{3\hbar}{2}\right)^{4/3} \frac{3ie^{i\frac{\pi}{6}}}{\left[\Gamma\left(\frac{2}{3}\right)\right]^4} \left(\frac{G^\infty_{z,1}}{F^\infty_{z,2}}\right)^2 \frac{1}{z^4} .$$

2.7 Scattering coefficients

Once the Stokes multipliers obtained, it follows that the S-matrix components can be expressed as :

$$a_{\mathrm{SDE}}(z)^* = \left[1 - \left(\frac{3\hbar}{2}\right)^{4/3} \frac{3ie^{i\frac{\pi}{6}}}{\left[\Gamma\left(\frac{2}{3}\right)\right]^4} \left(\frac{G^\infty_{z,1}}{F^\infty_{z,2}}\right)^2 \frac{1}{z^4} \right] \exp\left(-\frac{4z^3}{3\hbar}\right)$$

and $$b_{\mathrm{SDE}}(z) = \left(\frac{3\hbar}{2}\right)^{2/3} \frac{i\sqrt{3}}{\left[\Gamma\left(\frac{2}{3}\right)\right]^2} \frac{G^\infty_{z,1}}{F^\infty_{z,2}} \frac{1}{z^2} \exp\left(\frac{4z^3}{3\hbar}\right) .$$

Recall the validity domain of our computations, that is $z \gg \hbar$. The absence of exact expressions for the series coefficients c_k (respectively d_k) ends up in the discrepancy of $b_{\mathrm{SDE}}(z)$: the factor $\left(\frac{\hbar}{z^3}\right)^{2/3}$ clearly fails to prevent the exponential growth. This phenomenon was previously observed in Section 5 of [4].

Define the adimensional parameter $\mu_2 = \frac{z^3}{\hbar}$. The next figure displays the graphs of a_{SDE} and b_{SDE} with respect to μ_2.
Anyway : $\lim_{z \to +\infty} a_{\mathrm{SDE}}(z) = 0^+$. So we can reasonably conclude that :

$$\lim_{z \to +\infty} S(z) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \tag{12}$$

3 In the adiabatic limit : weak coupling $z \to 0$

From this point, we shall assume that $0 < z \ll \hbar$. All turning points are collapsing towards the origin, though with different paces. Let us return to the original second-order ODE :

$$\hbar^2 \frac{d^2\psi_1}{dt^2} - \frac{\hbar^2}{t}\frac{d\psi_1}{dt} + \left[(t^2+z^2) - \frac{i\hbar}{t}\right](t^2+z^2)\psi_1 = 0 . \tag{13}$$

Our main result is :

Proposition 1.*Let us consider the following vector-valued Schrödinger equation :*

$$\frac{\hbar}{i}\frac{d\psi(t)}{dt} = \left[\mathrm{Re}(t+iz)^2\sigma_3 + \mathrm{Im}(t+iz)^2\sigma_1\right]\psi(t)\ .$$

When $0 < z \ll \hbar$, the two transition probabilities in a parabolic intersection are :

$$\begin{cases} a(z) = \exp\left[-\frac{3\sqrt{3}}{\pi}\Gamma\left(\frac{2}{3}\right)^3\left(\frac{z^3}{2\hbar}\right)^{2/3}\right] + 3e^{-i\frac{2\pi}{3}}G^{\infty}_{0,1}\left(\frac{3}{2}\right)^{1/3}\Gamma\left(\frac{4}{3}\right)\cosh\left[\frac{3\sqrt{3}}{\pi}\Gamma\left(\frac{2}{3}\right)^3\left(\frac{z^3}{2\hbar}\right)^{2/3}\right]\left(\frac{z^3}{\hbar}\right)^{2/3} \\ b(z) = \cosh\left[\frac{3\sqrt{3}}{\pi}\Gamma\left(\frac{2}{3}\right)^3\left(\frac{z^3}{2\hbar}\right)^{2/3}\right]\frac{\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{5}{3}\right)}\sqrt{3}G^{\infty}_{0,1}\left(\frac{2z^3}{3\hbar}\right)^{1/3} \end{cases}, \tag{14}$$

where $G^{\infty}_{0,1}$ is the constant defined by :

$$G^{\infty}_{0,1} = \lim_{k\to+\infty}\left[1 - \frac{1}{4}\sum_{\ell=1}^{k}\frac{B\left(2, \frac{3\ell}{2}-\frac{1}{2}\right)}{B\left(\frac{4}{3},\ell\right)}\frac{1}{\ell}\right]\ .$$

Corollary 2.*The S-matrix is given by :*

$$S(z)\left[0^{+}\right]\sim$$
$$\left\{\exp\left[-\frac{3\sqrt{3}}{\pi}\Gamma\left(\frac{2}{3}\right)^3\left(\frac{z^3}{2\hbar}\right)^{2/3}\right] - G^{\infty}_{0,1}\left(\frac{3}{2}\right)^{4/3}\Gamma\left(\frac{4}{3}\right)\cosh\left[\frac{3\sqrt{3}}{\pi}\Gamma\left(\frac{2}{3}\right)^3\left(\frac{z^3}{2\hbar}\right)^{2/3}\right]\left(\frac{z^3}{\hbar}\right)^{2/3}\right\}1_2$$
$$-\sqrt{3}G^{\infty}_{0,1}\left(\frac{3}{2}\right)^{4/3}\Gamma\left(\frac{4}{3}\right)\cosh\left[\frac{3\sqrt{3}}{\pi}\Gamma\left(\frac{2}{3}\right)^3\left(\frac{z^3}{2\hbar}\right)^{2/3}\right]\left(\frac{z^3}{\hbar}\right)^{2/3}\sigma_3$$
$$+i\sqrt{3}G^{\infty}_{0,1}\cosh\left[\frac{3\sqrt{3}}{\pi}\Gamma\left(\frac{2}{3}\right)^3\left(\frac{z^3}{2\hbar}\right)^{2/3}\right]\frac{\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{5}{3}\right)}\left(\frac{2z^3}{3\hbar}\right)^{1/3}\sigma_2\ .$$

This expression holds in a small neighbourhood of 0*, yielding :* $\lim_{z\to 0} S(z) = 1_2$.

The rest of this paper is devoted to the proof of Proposition 1. We will follow the same strategy as in Section 2. Let us start in determining the singularities of the differential equation (13) in the complex plane. After a suitable change of variables, it is always possible to eliminate the first derivative in (13). The Wronskian is :

$$\mathrm{Wr}(t) = \exp\left(-\int^t -\frac{ds}{s}\right) = t\ .$$

Introduce a new variable :

$$u = \int^t \mathrm{Wr}(s)ds = \frac{t^2}{2} \quad \Longleftrightarrow \quad t = \sqrt{2u}\ .$$

Set $\psi_1(t) = y(u)$. Then the ODE transforms as :

$$\hbar^2(2uy'' + y') - \hbar^2 y' + \left[(2u+z^2) - \frac{i\hbar}{\sqrt{2u}}\right](2u+z^2)y = 0$$
$$\Longleftrightarrow y'' + \frac{1}{\hbar^2}\frac{(2u+z^2)\sqrt{2u} - i\hbar}{2u\sqrt{2u}}(2u+z^2)y = 0\ .$$

The singularities in the u-plane are the poles and zeroes of the function $\frac{(2u+z^2)\sqrt{2u}-i\hbar}{2u\sqrt{2u}}(2u+z^2)$. Thus we shall discover that :

- 0 is a pole
- $2u = -z^2$ is a first-order zero
- 3 new first-order zeroes appear approximately at :

$$\sqrt{2u} \simeq e^{i\left(\frac{\pi}{6}+\frac{2k\pi}{3}\right)}\hbar^{1/3} - \frac{1}{3}e^{-i\left(\frac{\pi}{6}+\frac{2k\pi}{3}\right)}\frac{z^2}{\hbar^{1/3}}$$

by performing a Kruskal-Newton iteration

Going back in the t-plane, aside the "inner" zeroes $\pm iz$, we find 3 extra "outer" first-order zeroes located at $t \simeq e^{i\left(\frac{\pi}{6}+\frac{2k\pi}{3}\right)}\hbar^{1/3}$ where $k \in \mathbb{F}$. Moreover :

$$\left[\frac{(2u+z^2)\sqrt{2u}-i\hbar}{2u\sqrt{2u}}(2u+z^2)\right]^{1/2}du$$
$$= \left[\frac{(t^2+z^2)t - i\hbar}{t}(t^2+z^2)\right]^{1/2}dt = Q(t)^{1/2}dt\ .$$

The above figure, viewed on a very large scale, is not distinguishable from a simplified Stokes plot obtained by using asymptotically large $|t|$, as shown in Figure **??**. The local structure near the origin is not relevant for the global continuation at infinity.

3.1 Connecting the asymptotic solutions

Continuation through the upper half-plane gives :

1. start on the positive real axis with the subdominant solution : $(t_2,t)_s$
2. crossing a Stokes line : $(t_2,t)_s$
3. crossing an anti-Stokes line : $(t_2,t)_d$
4. stepping on a Stokes line : $(t_2,t)_d + \frac{C_{S1}}{2}(t,t_2)_s$ reconnecting t_2 to t_3 : $e^{-i\frac{\pi}{3}}[t_2,0]_\ell[0,t_3]_r(t_3,t)_d + \frac{C_{S1}}{2}(t,t_3)_s[t_3,0]_r[0,t_2]_\ell e^{i\frac{\pi}{3}}$
5. stepping the Stoke line : $e^{-i\frac{\pi}{3}}[t_2,0]_\ell[0,t_3]_r(t_3,t)_d + \frac{C_{S1}}{2}\left\{[t_3,0]_r[0,t_2]_\ell e^{i\frac{\pi}{3}} + e^{-i\frac{\pi}{3}}[t_2,0]_\ell[0,t_3]_r\right\}(t,t_3)_s$
6. crossing an anti-Stokes line :$e^{-i\frac{\pi}{3}}[t_2,0]_\ell[0,t_3]_r(t_3,t)_s + \frac{C_{S1}}{2}\left\{[t_3,0]_r[0,t_2]_\ell e^{i\frac{\pi}{3}} + e^{-i\frac{\pi}{3}}[t_2,0]_\ell[0,t_3]_r\right\}(t,t_3)_d$ crossing a Riemannian cut : $e^{-i\frac{\pi}{3}}[t_2,0]_\ell[0,t_3]_r(t,t_3)_s$ $\frac{C_{S1}}{2}\left\{[t_3,0]_r[0,t_2]_\ell e^{i\frac{\pi}{3}} + e^{-i\frac{\pi}{3}}[t_2,0]_\ell[0,t_3]_r\right\}(t_3,t)_d$
7. crossing a Stokes line :

$$\left[e^{-i\frac{\pi}{3}}[t_2,0]_\ell[0,t_3]_r + \frac{C_{S2}C_{S1}}{2}\left\{[t_3,0]_r[0,t_2]_\ell e^{i\frac{\pi}{3}} + e^{-i\frac{\pi}{3}}[t_2,0]_\ell[0,t_3]_r\right\}\right](t,t_3)_s$$
$$+\frac{C_{S1}}{2}\left\{[t_3,0]_r[0,t_2]_\ell e^{i\frac{\pi}{3}} + e^{-i\frac{\pi}{3}}[t_2,0]_\ell[0,t_3]_r\right\}(t_3,t)_d$$

8. crossing an anti-Stokes line :

$$\left[e^{-i\frac{\pi}{3}}[t_2,0]_\ell[0,t_3]_r + \frac{C_{S2}C_{S1}}{2}\left\{[t_3,0]_r[0,t_2]_\ell e^{i\frac{\pi}{3}} + e^{-i\frac{\pi}{3}}[t_2,0]_\ell[0,t_3]_r\right\}\right](-t,t_3)_d$$
$$+\frac{C_{S1}}{2}\left\{[t_3,0]_r[0,t_2]_\ell e^{i\frac{\pi}{3}} + e^{-i\frac{\pi}{3}}[t_2,0]_\ell[0,t_3]_r\right\}(t_3,-t)_s$$

with the Stokes multipliers $C_{S1} = C_S\left(\frac{\pi}{2}\right)$ and $C_{S2} = C_S\left(\frac{5\pi}{6}\right)$. As in Section 5 of [4], we also add a phase $e^{\mp i\frac{\pi}{3}} = e^{\mp i\frac{5\pi}{12}}e^{\pm i\frac{\pi}{12}}$ to compensate the additional presence of a square root $t^{\pm 1/2}$ before $\exp\left[\pm\frac{i}{\hbar}\left(\frac{t^3}{3}+z^2t\right)\right]$.

3.2 Computing the left action W

Besides, the action W is defined by $e^W = [t_3,0]_r[0,t_2]_\ell$:

$$\begin{aligned} W &= -\frac{i}{\hbar}\left\{\int_{t_3}^{iz}\left[\frac{(t^2+z^2)t-i\hbar}{t}(t^2+z^2)\right]^{1/2}dt + \int_{iz}^{t_2}\left[\frac{(t^2+z^2)t-i\hbar}{t}(t^2+z^2)\right]^{1/2}dt\right\} \\ &= W_1+W_2\,. \end{aligned}$$

Write down :

$$Q(t) = -i\hbar t\left(1+\frac{z^2}{t^2}\right)\left[1-\frac{t^3}{i\hbar}\left(1+\frac{z^2}{t^2}\right)\right].$$

Since $z \leqslant |t| \leqslant \hbar^{1/3}$, let us try a Taylor expansion of the square root :

$$\begin{aligned} Q(t)^{1/2} &= \pm i e^{i\frac{\pi}{4}}\hbar^{1/2}\sqrt{t}\left[\sum_{k=0}^{+\infty}\left(\frac{1}{2}\right)_k\frac{1}{k!}\left(\frac{z^2}{t^2}\right)^k\right]\left[\sum_{\ell=0}^{+\infty}\left(\frac{1}{2}\right)_\ell\frac{1}{\ell!}\left(\frac{it^3}{\hbar}\right)^\ell\sum_{0\leqslant m\leqslant\ell}\binom{\ell}{m}\left(\frac{z^2}{t^2}\right)^m\right] \\ &= \pm i e^{i\frac{\pi}{4}}\hbar^{1/2}\sum_{k,\ell}\sum_{0\leqslant m\leqslant\ell}\left(\frac{1}{2}\right)_k\left(\frac{1}{2}\right)_\ell\binom{\ell}{m}\frac{1}{k!\ell!}\left(\frac{i}{\hbar}\right)^\ell z^{2(k+m)}t^{\frac{1}{2}+3\ell-2(k+m)} \\ &= \pm i e^{i\frac{\pi}{4}}\hbar^{1/2}\sum_{k,\ell}\sum_{0\leqslant m\leqslant\ell}\binom{\ell}{m}\frac{(-1)^{k+\ell}}{2^{2(k+\ell)}(2k-1)(2\ell-1)}\frac{(2k)!(2\ell)!}{k!^2\ell!^2}\left(\frac{i}{\hbar}\right)^\ell z^{2(k+m)}t^{\frac{1}{2}+3\ell-2(k+m)}\,, \end{aligned}$$

by using the identity $\left(\frac{1}{2}\right)_k = -\frac{(-1)^k}{2^{2k}}\frac{(2k)!}{(2k-1)k!}$.

Therefore :

$$\begin{aligned} W_2 = {} & -\frac{e^{i\frac{\pi}{4}}}{\hbar^{1/2}}\sum_{k,\ell}\sum_{0\leqslant m\leqslant\ell}\frac{1}{\frac{3}{2}+3\ell-2(k+m)}\binom{\ell}{m}\frac{(-1)^{k+\ell}}{2^{2(k+\ell)}(2k-1)(2\ell-1)}\frac{(2k)!(2\ell)!}{k!^2\ell!^2} \\ & \left(\frac{i}{\hbar}\right)^\ell z^{2(k+m)}t^{\frac{3}{2}+3\ell-2(k+m)}\Big|_{iz}^{t_3} \\ = {} & -\frac{e^{i\frac{\pi}{4}}}{\hbar^{1/2}}\sum_{k,\ell}\sum_{0\leqslant m\leqslant\ell}\frac{1}{\frac{3}{2}+3\ell-2(k+m)}\binom{\ell}{m}\frac{(-1)^{k+\ell}}{2^{2(k+\ell)}(2k-1)(2\ell-1)}\frac{(2k)!(2\ell)!}{k!^2\ell!^2}\left(\frac{i}{\hbar}\right)^\ell \\ & z^{2(k+m)}\left[e^{i\frac{\pi}{4}}\hbar^{1/2}\left(e^{i\frac{\pi}{6}}\hbar^{1/3}\right)^{3\ell-2(k+m)}\left(1-\frac{1}{3}e^{-i\frac{\pi}{3}}\frac{z^2}{\hbar^{2/3}}\right)^{\frac{3}{2}+3\ell-2(k+m)} - (iz)^{\frac{3}{2}+3\ell-2(k+m)}\right] \\ \simeq {} & -\frac{e^{i\frac{\pi}{4}}}{\hbar^{1/2}}\sum_{k,\ell}\sum_{0\leqslant m\leqslant\ell}\frac{1}{\frac{3}{2}+3\ell-2(k+m)}\binom{\ell}{m}\frac{(-1)^{k+\ell}}{2^{2(k+\ell)}(2k-1)(2\ell-1)}\frac{(2k)!(2\ell)!}{k!^2\ell!^2}\left(\frac{i}{\hbar}\right)^\ell \\ & z^{2(k+m)}\left[e^{i\frac{\pi}{4}}\hbar^{1/2}\left(e^{i\frac{\pi}{6}}\hbar^{1/3}\right)^{3\ell-2(k+m)}\left(1-\frac{1}{3}\left[\frac{3}{2}+3\ell-2(k+m)\right]e^{-i\frac{\pi}{3}}\frac{z^2}{\hbar^{2/3}}\right) - (iz)^{\frac{3}{2}+3\ell-2(k+m)}\right]. \end{aligned}$$

The dominant term in W_2 is obtained for $k = m = 0$:

$$\begin{aligned} & \frac{2i}{3}\sum_{\ell=0}^{+\infty}\frac{1}{2\ell+1}\frac{(-1)^\ell(2\ell)!}{2^{2\ell}(2\ell-1)\ell!^2}\left(\frac{i}{\hbar}\right)^\ell\left(e^{i\frac{\pi}{6}}\hbar^{1/3}\right)^{3\ell} \\ & = \frac{2i}{3}\sum_{\ell=0}^{+\infty}\frac{(2\ell)!}{2^{2\ell}(2\ell+1)(2\ell-1)\ell!^2} \\ & = -\frac{i}{3}\sum_{\ell=0}^{+\infty}\binom{\ell-\frac{3}{2}}{\ell}\frac{1}{\ell+\frac{1}{2}} = -\frac{i}{3}B\left(\frac{1}{2},\frac{3}{2}\right) = -\frac{i\pi}{6}\,. \end{aligned}$$

The next one is when $\ell = 0$:

$$\frac{e^{i\frac{\pi}{4}}}{\hbar^{1/2}}\sum_{k=0}^{+\infty}\frac{2}{4k-3}\frac{(-1)^k(2k)!}{2^{2k}(2k-1)k!^2}(iz)^{3/2} = 2\left(\frac{z^3}{\hbar}\right)^{1/2}\sum_{k=0}^{+\infty}\frac{(-1)^k(2k)!}{2^{2k}(4k-3)(2k-1)k!^2}\,.$$

The next one is the sum of three contributions :

- when $k=0,\ m=0$:

$$\begin{aligned} & -\frac{i}{3}\sum_{\ell=0}^{+\infty}\frac{(-1)^\ell}{2^{2\ell}(2\ell-1)}\frac{(2\ell)!}{\ell!^2}\left(\frac{i}{\hbar}\right)^\ell\left(e^{i\frac{\pi}{6}}\hbar^{1/3}\right)^{3\ell}e^{-i\frac{\pi}{3}}\frac{z^2}{\hbar^{2/3}} \\ & = -\frac{i}{3}e^{-i\frac{\pi}{3}}\left(\frac{z^3}{\hbar}\right)^{2/3}\sum_{\ell=0}^{+\infty}\frac{(2\ell)!}{2^{2\ell}(2\ell-1)\ell!^2} \end{aligned}$$

- when $k=1,\ m=0$:

$$\begin{aligned} & i\sum_{\ell=0}^{+\infty}\frac{1}{6\ell-1}\frac{(-1)^\ell}{2^{2\ell}(2\ell-1)}\frac{(2\ell)!}{\ell!^2}\left(\frac{i}{\hbar}\right)^\ell\left(e^{i\frac{\pi}{6}}\hbar^{1/3}\right)^{3\ell-2}z^2 \\ & = ie^{-i\frac{\pi}{3}}\left(\frac{z^3}{\hbar}\right)^{2/3}\sum_{\ell=0}^{+\infty}\frac{(2\ell)!}{2^{2\ell}(6\ell-1)(2\ell-1)\ell!^2} \end{aligned}$$

- when $k=0,\ m=1$:

$$\begin{aligned} & i\sum_{\ell=1}^{+\infty}\frac{2\ell}{6\ell-1}\frac{(-1)^\ell}{2^{2(k+\ell)}(2\ell-1)}\frac{(2\ell)!}{\ell!^2}\left(\frac{i}{\hbar}\right)^\ell\left(e^{i\frac{\pi}{6}}\hbar^{1/3}\right)^{3\ell-2}z^2 \\ & = 2ie^{-i\frac{\pi}{3}}\left(\frac{z^3}{\hbar}\right)^{2/3}\sum_{\ell=0}^{+\infty}\frac{\ell(2\ell)!}{2^{2\ell}(6\ell-1)(2\ell-1)\ell!^2}\,, \end{aligned}$$

yielding :

$$ie^{-i\frac{\pi}{3}}\left(\frac{z^3}{\hbar}\right)^{2/3}\left[\frac{4}{3}-\sum_{\ell=1}^{+\infty}\frac{(2\ell)!}{2^{2\ell}(2\ell-1)\ell!^2}\left(\frac{1}{3}-\frac{1}{6\ell-1}-\frac{2\ell}{6\ell-1}\right)\right] = \frac{4i}{3}e^{-i\frac{\pi}{3}}\left(\frac{z^3}{\hbar}\right)^{2/3}\sum_{\ell=1}^{+\infty}\frac{(2\ell)!}{2^{2\ell}(6\ell-1)(2\ell-1)\ell!^2}\,.$$

It is left to the reader (as an exercise on the hypergeometric function ${}_2F_1$) to show that :

$$W_2 = -\frac{i\pi}{6}+\frac{2}{3}{}_2F_1\left(-\frac{3}{4},-\frac{1}{2},\frac{1}{4};-1\right)\left(\frac{z^3}{\hbar}\right)^{1/2}+\frac{3}{\pi}e^{i\frac{\pi}{6}}\Gamma\left(\frac{2}{3}\right)^3\left(\frac{z^3}{2\hbar}\right)^{2/3}+\mathscr{O}\left(z^4\right).$$

For the W_1 part :

$$\begin{aligned} -W_1 \simeq {} & -\frac{e^{i\frac{\pi}{4}}}{\hbar^{1/2}}\sum_{k,\ell}\sum_{0\leqslant m\leqslant\ell}\frac{1}{\frac{3}{2}+3\ell-2(k+m)}\binom{\ell}{m}\frac{(-1)^{k+\ell}}{2^{2(k+\ell)}(2k-1)(2\ell-1)}\frac{(2k)!(2\ell)!}{k!^2\ell!^2}\left(\frac{i}{\hbar}\right)^\ell \\ & z^{2(k+m)}\left[e^{i\frac{5\pi}{4}}\hbar^{1/2}\left(e^{i\frac{5\pi}{6}}\hbar^{1/3}\right)^{3\ell-2(k+m)}\left(1-\frac{1}{3}\left[\frac{3}{2}+3\ell-2(k+m)\right]e^{-i\frac{5\pi}{3}}\frac{z^2}{\hbar^{2/3}}\right)-(iz)^{\frac{3}{2}+3\ell-2(k+m)}\right] \\ = {} & \frac{i\pi}{6}+\frac{2}{3}{}_2F_1\left(-\frac{3}{4},-\frac{1}{2},\frac{1}{4};-1\right)\left(\frac{z^3}{\hbar}\right)^{1/2}-\frac{3}{\pi}e^{-i\frac{\pi}{6}}\Gamma\left(\frac{2}{3}\right)^3\left(\frac{z^3}{2\hbar}\right)^{2/3}+\mathscr{O}\left(z^4\right). \end{aligned}$$

Finally, we find that :

$$W = -\frac{i\pi}{3}+\frac{6}{\pi}\cos\frac{\pi}{6}\Gamma\left(\frac{2}{3}\right)^3\frac{z^2}{(2\hbar)^{2/3}} = -\frac{i\pi}{3}+\frac{3\sqrt{3}}{\pi}\Gamma\left(\frac{2}{3}\right)^3\frac{z^2}{(2\hbar)^{2/3}} \quad (15)$$

is almost an imaginary complex number as expected. Subsequently, we have :

$$\begin{aligned} & \left[e^{-W}e^{-i\frac{\pi}{3}}+\frac{C_{S2}C_{S1}}{2}\left(e^We^{i\frac{\pi}{3}}+e^{-i\frac{\pi}{3}}e^{-W}\right)\right](-t,t_3)_d \\ & +\frac{C_{S1}}{2}\left(e^We^{i\frac{\pi}{3}}+e^{-i\frac{\pi}{3}}e^{-W}\right)(t_3,-t)_s \quad \rightsquigarrow \quad (t_2,t)_s\,. \end{aligned}$$

Take the reference point for the phase to be $\tau = 0$. Formally divide both sides by $[t_2,0]_r$, then :

$$\begin{aligned} & \left[e^{-W}e^{-i\frac{\pi}{3}}+\frac{C_{S2}C_{S1}}{2}\left(e^We^{i\frac{\pi}{3}}+e^{-i\frac{\pi}{3}}e^{-W}\right)\right](-t,0)_d \\ & +\frac{C_{S1}}{2}\left(e^We^{i\frac{\pi}{3}}+e^{-i\frac{\pi}{3}}e^{-W}\right)(0,-t)_s \quad \rightsquigarrow \quad (0,t)_s \end{aligned}$$

since $[t_2,0]_r[0,t_3]_r = 1+\mathcal{O}(z^2)$. Therefore, we can infer the 2 transition probabilities :

$$\begin{aligned} a(z)^* &= \exp\left[-\frac{3\sqrt{3}}{\pi}\Gamma\left(\frac{2}{3}\right)^3\left(\frac{z^3}{2\hbar}\right)^{2/3}\right] \\ &\quad + C_{S2}C_{S1}\cosh\left[\frac{3\sqrt{3}}{\pi}\Gamma\left(\frac{2}{3}\right)^3\left(\frac{z^3}{2\hbar}\right)^{2/3}\right] \\ \text{and } b(z) &= -\frac{t}{z}C_{S1}\cosh\left[\frac{3\sqrt{3}}{\pi}\Gamma\left(\frac{2}{3}\right)^3\left(\frac{z^3}{2\hbar}\right)^{2/3}\right] \end{aligned}$$

accordingly to Rule 1 of [5].

3.3 Compatibility relation

Assume that ψ_1 can be defined by two different expressions :

$$\begin{aligned} \psi_1(t) &= [Af_1(t)+Bf_2(t)]\exp\left[-\frac{i}{\hbar}\left(\frac{t^3}{3}-z^2t\right)\right] \\ &= [Cg_1(t)+Dg_2(t)]\exp\left[\frac{i}{\hbar}\left(\frac{t^3}{3}-z^2t\right)\right] \end{aligned}$$

as in the forthcoming subsection. Then :

$$\begin{aligned} \psi_1(t) &= \left(A\sum_{k=0}^{+\infty}c_k^{[f_1]}t^k + B\sum_{k=0}^{+\infty}c_k^{[f_2]}t^{k+2}\right)\exp\left[-\frac{i}{\hbar}\left(\frac{t^3}{3}-z^2t\right)\right] \\ \psi_1'(t) &= \left\{A\left[\sum_{k=1}^{+\infty}kc_k^{[f_1]}t^{k-1} - \frac{i(t^2-z^2)}{\hbar}\sum_{k=0}^{+\infty}c_k^{[f_1]}t^k\right] + B\left[\sum_{k=0}^{+\infty}(k+2)c_k^{[f_2]}t^{k+1} - \frac{i(t^2-z^2)}{\hbar}\sum_{k=0}^{+\infty}c_k^{[f_2]}t^{k+2}\right]\right\} \\ &\quad \exp\left[-\frac{i}{\hbar}\left(\frac{t^3}{3}-z^2t\right)\right] \\ \psi_1''(t) &= A\left[\sum_{k=2}^{+\infty}k_2c_k^{[f_1]}t^{k-2} - \frac{2i(t^2-z^2)}{\hbar}\sum_{k=1}^{+\infty}kc_k^{[f_1]}t^{k-1} - \frac{(t^2-z^2)^2}{\hbar^2}\sum_{k=0}^{+\infty}c_k^{[f_1]}t^k\right]\exp\left[-\frac{i}{\hbar}\left(\frac{t^3}{3}-z^2t\right)\right] \\ &\quad + B\left[\sum_{k=0}^{+\infty}(k+2)_2c_k^{[f_2]}t^k - \frac{2i(t^2-z^2)}{\hbar}\sum_{k=0}^{+\infty}(k+2)c_k^{[f_2]}t^{k+1} - \frac{(t^2-z^2)^2}{\hbar^2}\sum_{k=0}^{+\infty}c_k^{[f_2]}t^{k+2}\right] \\ &\quad .\exp\left[-\frac{i}{\hbar}\left(\frac{t^3}{3}-z^2t\right)\right] \end{aligned}$$

by Liebniz formula. The conditions of compatibility for ψ_1 and its first two derivatives at $t=0$ yield :

$$\begin{cases} Ac_0^{[f_1]} = Cd_0^{[g_1]} \\ A\left(c_1^{[f_1]} + \frac{iz^2}{\hbar}c_0^{[f_1]}\right) = C\left(d_1^{[g_1]} - \frac{iz^2}{\hbar}d_0^{[g_1]}\right) \\ A\left(2c_2^{[f_1]} + \frac{2iz^2}{\hbar}c_1^{[f_1]} - \frac{z^4}{\hbar^2}c_0^{[f_1]}\right) + 2Bc_0^{[f_2]} = C\left(2d_2^{[g_1]} - \frac{2iz^2}{\hbar}d_1^{[g_1]} - \frac{z^4}{\hbar^2}d_0^{[g_1]}\right) + 2Dd_0^{[g_2]} \end{cases} . \quad (16)$$

For convenience, we choose $A=C=0$ and $B=D=1$ (or equivalently $\nu=2$).

3.4 Power series solutions

By setting $\psi_1(t) = f(t)\exp\left[-\frac{i}{\hbar}\left(\frac{t^3}{3}-z^2t\right)\right]$, we find that f satisfies :

$$f'' - \left[\frac{2i}{\hbar}(t^2-z^2)+\frac{1}{t}\right]f' + \left[\frac{4z^2t^2}{\hbar^2} - \frac{2i}{\hbar t}(t^2+z^2)\right]f = 0 . \quad (17)$$

Write f as a normalized power series : $f(t) = t^\nu\sum_{k=0}^{+\infty}c_kt^k$, with $c_0=1$. Substituting in (17) :

$$\begin{aligned} &\frac{1}{t^2}\sum_{k=0}^{+\infty}(\nu+k)_2c_kt^k - \left(\frac{2it}{\hbar} - \frac{2iz^2}{\hbar t} + \frac{1}{t^2}\right)\sum_{k=0}^{+\infty}(\nu+k)c_kt^k + \left(\frac{4z^2t^2}{\hbar^2} - \frac{2it}{\hbar} - \frac{2iz^2}{\hbar t}\right)\sum_{k=0}^{+\infty}c_kt^k = 0 \\ \iff &\frac{1}{t^2}\sum_{k=0}^{+\infty}(\nu+k)(\nu+k-2)c_kt^k + \frac{2iz^2}{\hbar t}\sum_{k=0}^{+\infty}(\nu+k-1)c_kt^k - \frac{2it}{\hbar}\sum_{k=0}^{+\infty}(\nu+k+1)c_kt^k + \frac{4z^2t^2}{\hbar^2}\sum_{k=0}^{+\infty}c_kt^k = 0 , \end{aligned}$$

that is a recurrence relation over 5 ranks. Aside c_0, let us find the 3 next coefficients :

- order -2 : the necessary condition $\nu(\nu-2)=0$ implies $\begin{cases} \nu = 0 \\ \text{or } \nu = 2 \end{cases}$
- order -1 : $(\nu+1)(\nu-1)c_1 + \frac{2iz^2}{\hbar}(\nu-1)c_0 = 0$
$$\implies \quad c_1 = \frac{1}{\nu+1}\left(-\frac{2iz^2}{\hbar}\right) ,$$
- order $+0$: $(\nu+2)\nu c_2 + \frac{2iz^2}{\hbar}\nu c_1 = 0$
$$\implies \begin{cases} \text{if } \nu=0 : c_2 \text{ is arbitrary} \\ \text{otherwise if } \nu=2 : c_2 = \frac{1}{(\nu+2)_2}\left(-\frac{2iz^2}{\hbar}\right)^2 \end{cases} ,$$
- order $+1$: $(\nu+3)(\nu+1)c_3 + \frac{2iz^2}{\hbar}(\nu+1)c_2 - \frac{2i}{\hbar}(\nu+1)c_0 = 0$
$$\implies \begin{cases} \text{if } \nu=0 : c_3 \text{ depends on the choice of } c_2 \\ \text{otherwise if } \nu=2 : c_3 = \frac{1}{(\nu+3)_3}\left(-\frac{2iz^2}{\hbar}\right)^3 + \frac{1}{\nu+3}\frac{2i}{\hbar} \end{cases} .$$

For $k \geqslant 4$, the recurrence relation between the coefficients c_k reads as :

$$(\nu+k)(\nu+k-2)c_k + \frac{2iz^2}{\hbar}(\nu+k-2)c_{k-1} - \frac{2i}{\hbar}(\nu+k-2)c_{k-3} + \frac{4z^2}{\hbar^2}c_{k-4} = 0 .$$

The whole sequence is inductively obtained by :

$$c_k = -\frac{2iz^2}{\hbar}\left[\frac{1}{\nu+k}c_{k-1} - \frac{2i}{\hbar}\frac{1}{(\nu+k)(\nu+k-2)}c_{k-4}\right] + \frac{2i}{\hbar}\frac{1}{\nu+k}c_{k-3} .$$

Since we decide to choose $\nu=2$:

$$c_k = -\frac{2iz^2}{\hbar}\left[\frac{1}{k+2}c_{k-1} - \frac{2i}{\hbar}\frac{1}{(k+2)k}c_{k-4}\right] + \frac{2i}{\hbar}\frac{1}{k+2}c_{k-3} . \quad (18)$$

Let $j \in \mathbb{F}$. Then for $k \geqslant 1$, we set $c_{3k+j} = \sum_{\ell=0}^{k}\gamma_\ell^{[3k+j]}z^{6\ell+2j}$. By inspection of (18), it is clear that the coefficient of the term of highest degree in z^2 is : $\gamma_k^{[3k+j]} = \frac{2}{(3k+j)!}\left(-\frac{2i}{\hbar}\right)^{3k+j}$. In c_{3k}, the coefficient of the term of lowest degree in z^2 is :

$$\gamma_0^{[3k]} = \left(\frac{2i}{\hbar}\right)^k\frac{1}{(3k+2)(3k-1)\ldots2}c_0 = \left(\frac{2i}{3\hbar}\right)^k\frac{\Gamma\left(\frac{5}{3}\right)}{\Gamma\left(k+\frac{5}{3}\right)} . \quad (19)$$

Introduce the quantity :

$$\chi_0^{[3\ell+1]}(\ell) = -\frac{2iz^2}{\hbar}\left[\frac{1}{3\ell+3}\gamma_0^{[3\ell]} - \frac{2i}{\hbar}\frac{1}{(3\ell+3)(3\ell+1)}\gamma_0^{[3(\ell-1)]}\right]$$
$$= -\frac{2iz^2}{\hbar}\left(\frac{2i}{3\hbar}\right)^{\ell}\frac{\Gamma\left(\frac{5}{3}\right)}{\Gamma\left(\ell+\frac{5}{3}\right)}\frac{1}{3\ell+3}\left[1-\frac{3\ell+2}{3\ell+1}\right]$$
$$= \frac{2iz^2}{\hbar}\left(\frac{2i}{3\hbar}\right)^{\ell}\frac{1}{(3\ell+3)(3\ell+1)}\frac{\Gamma\left(\frac{5}{3}\right)}{\Gamma\left(\ell+\frac{5}{3}\right)} .$$

Now the coefficient of the term of lowest degree in z^2 in c_{3k+1} can be computed by means of a discrete convolution. Therefore :

$$\begin{aligned}
\gamma_0^{[3k+1]} &= \chi_0^{[3k+1]}(k) + \sum_{\ell=1}^{k-1}\left(\frac{2i}{3\hbar}\right)^{\ell}\frac{1}{(k+1)_{\ell}}\chi_0^{[3(k-\ell)+1]}(k-\ell) + \left(\frac{2i}{3\hbar}\right)^{k}\frac{1}{(k+1)_k}\gamma_0^{[1]} \\
&= \chi_0^{[3k+1]}(k) + \frac{2iz^2}{\hbar}\left(\frac{2i}{3\hbar}\right)^{k}\frac{\Gamma\left(\frac{5}{3}\right)}{\Gamma(k+2)}\sum_{\ell=1}^{k-1}\frac{\Gamma(k-\ell+2)}{[3(k-\ell)+3][3(k-\ell)+1]\Gamma\left(k-\ell+\frac{5}{3}\right)} \\
&\quad -\frac{2iz^2}{\hbar}\frac{1}{3(k+1)!}\left(\frac{2i}{3\hbar}\right)^{k} \\
&= \chi_0^{[3k+1]}(k) + \frac{2iz^2}{\hbar}\left(\frac{2i}{3\hbar}\right)^{k}\frac{\Gamma\left(\frac{5}{3}\right)}{\Gamma(k+2)}\sum_{\ell=1}^{k-1}\frac{\Gamma(\ell+2)}{(3\ell+3)(3\ell+1)\Gamma\left(\ell+\frac{5}{3}\right)} - \\
&\quad \frac{2iz^2}{\hbar}\frac{1}{3(k+1)!}\left(\frac{2i}{3\hbar}\right)^{k} \\
&= \frac{2iz^2}{\hbar}\left(\frac{2i}{3\hbar}\right)^{k}\frac{1}{\Gamma(k+2)}\left[\Gamma\left(\frac{5}{3}\right)\sum_{\ell=1}^{k}\frac{\Gamma(\ell+1)}{3(3\ell+1)\Gamma\left(\ell+\frac{5}{3}\right)} - \frac{1}{3}\right] \\
&= \frac{2iz^2}{\hbar}\left(\frac{2i}{3\hbar}\right)^{k}\frac{1}{\Gamma(k+2)}\underbrace{\frac{1}{3}\left[-1+\frac{2}{3}\sum_{\ell=1}^{k}\frac{1}{3\ell+1}B\left(\frac{2}{3},\ell+1\right)\right]}_{F_{0,1}(k)} .
\end{aligned} \tag{20}$$

Evaluate the quantity :

$$\begin{aligned}
\chi_1^{[3\ell+2]}(\ell) &= -\frac{2iz^2}{\hbar}\left[\frac{1}{3\ell+4}\gamma_0^{[3\ell+1]} - \frac{2i}{\hbar}\frac{1}{(3\ell+4)(3\ell+2)}\gamma_0^{[3\ell-2)]}\right] \\
&= -\left(\frac{2iz^2}{\hbar}\right)^2\left(\frac{2i}{3\hbar}\right)^{\ell}\frac{1}{3\Gamma(\ell+2)}\frac{1}{3\ell+4} \\
&\quad \left\{\left[-1+\frac{2}{3}\sum_{m=1}^{\ell}\frac{1}{3m+1}B\left(\frac{2}{3},m+1\right)\right] - \frac{3(\ell+1)}{3\ell+2}\left[-1+\frac{2}{3}\sum_{m=1}^{\ell-1}\frac{1}{3m+1}B\left(\frac{2}{3},m+1\right)\right]\right\} \\
&= -\left(\frac{2iz^2}{\hbar}\right)^2\left(\frac{2i}{3\hbar}\right)^{\ell}\frac{1}{\Gamma(\ell+2)}\frac{1}{3(3\ell+4)(3\ell+2)} \\
&\quad \left[1-\frac{2}{3}\sum_{m=1}^{\ell}\frac{1}{3m+1}B\left(\frac{2}{3},m+1\right) + \frac{2(\ell+1)}{3\ell+1}B\left(\frac{2}{3},\ell+1\right)\right] .
\end{aligned}$$

Again, the coefficient of the term of lowest degree in z^2 in c_{3k+2} can be computed by :

$$\begin{aligned}
\gamma_0^{[3k+2]} &= \chi_1^{[3k+2]}(k) + \sum_{\ell=1}^{k-1}\left(\frac{2i}{3\hbar}\right)^{\ell}\frac{1}{\left(k+\frac{4}{3}\right)_{\ell}}\chi_1^{[3(k-\ell)+2]}(k-\ell) + \left(\frac{2i}{3\hbar}\right)^{k}\frac{1}{\left(k+\frac{4}{3}\right)_k}\gamma_0^{[2]} \\
&= \left(\frac{2i}{3\hbar}\right)^{k}\frac{\Gamma\left(\frac{7}{3}\right)}{\Gamma\left(k+\frac{7}{3}\right)}\frac{1}{12}\left(-\frac{2iz^2}{\hbar}\right)^2 + \chi_1^{[3k+2]}(k) - \left(\frac{2iz^2}{\hbar}\right)^2\left(\frac{2i}{3\hbar}\right)^{k}\frac{1}{\Gamma\left(k+\frac{7}{3}\right)} \\
&\quad \sum_{\ell=1}^{k-1}\frac{\Gamma\left(k-\ell+\frac{7}{3}\right)}{\Gamma(k-\ell+2)}\frac{1}{3[3(k-\ell)+4][3(k-\ell)+2]}\left[1-\frac{2}{3}\sum_{m=1}^{k-\ell}\frac{1}{3m+1}B\left(\frac{2}{3},m+1\right)+\frac{2(k-\ell+1)}{3(k-\ell)+1}B\left(\frac{2}{3},k-\ell+1\right)\right] \\
&= \left(\frac{2i}{3\hbar}\right)^{k}\frac{\Gamma\left(\frac{7}{3}\right)}{\Gamma\left(k+\frac{7}{3}\right)}\frac{1}{12}\left(-\frac{2iz^2}{\hbar}\right)^2 + \chi_1^{3k+2]}(k) \\
&\quad -\left(\frac{2iz^2}{\hbar}\right)^2\left(\frac{2i}{3\hbar}\right)^{k}\frac{1}{\Gamma\left(k+\frac{7}{3}\right)} \\
&\quad \cdot\sum_{\ell=1}^{k-1}\frac{\Gamma\left(\ell+\frac{7}{3}\right)}{\Gamma(\ell+2)}\frac{1}{3(3\ell+4)(3\ell+2)}\left[1-\frac{2}{3}\sum_{m=1}^{\ell}\frac{1}{3m+1}B\left(\frac{2}{3},m+1\right)+\frac{2(\ell+1)}{3\ell+1}B\left(\frac{2}{3},\ell+1\right)\right] \\
&= \frac{1}{12}\left(\frac{2iz^2}{\hbar}\right)^2\left(\frac{2i}{3\hbar}\right)^{k}\frac{\Gamma\left(\frac{7}{3}\right)}{\Gamma\left(k+\frac{7}{3}\right)} \\
&\quad -\left(\frac{2iz^2}{\hbar}\right)^2\left(\frac{2i}{3\hbar}\right)^{k}\frac{\Gamma\left(\frac{7}{3}\right)}{\Gamma\left(k+\frac{7}{3}\right)} \\
&\quad \cdot\sum_{\ell=1}^{k}\frac{\Gamma\left(\ell+\frac{4}{3}\right)}{\Gamma\left(\frac{7}{3}\right)\Gamma(\ell)}\frac{1}{3(3\ell+3)2\ell}\left[1-\frac{2}{3}\sum_{m=1}^{\ell}\frac{1}{3m+1}B\left(\frac{2}{3},m+1\right)+\frac{2(\ell+1)}{3\ell+1}B\left(\frac{2}{3},\ell+1\right)\right] \\
&= \left(\frac{2iz^2}{\hbar}\right)^2\left(\frac{2i}{3\hbar}\right)^{k}\frac{\Gamma\left(\frac{7}{3}\right)}{\Gamma\left(k+\frac{7}{3}\right)}F_{0,2}(k) ,
\end{aligned} \tag{21}$$

with :

$$F_{0,2}(k) = \frac{1}{3}\left\{\frac{1}{4} - \sum_{\ell=1}^{k}\frac{B\left(1,\ell+\frac{4}{3}\right)}{B\left(\frac{7}{3},\ell\right)}\frac{1}{(3\ell+3)2\ell}\left[1-\frac{2}{3}\sum_{m=1}^{\ell}\frac{1}{3m+1}B\left(\frac{2}{3},m+1\right)+\frac{2(\ell+1)}{3\ell+1}B\left(\frac{2}{3},\ell+1\right)\right]\right\} .$$

Similarly if we place $\psi_1(t) = g(t)\exp\left[\frac{i}{\hbar}\left(\frac{t^3}{3}-z^2t\right)\right]$, then g satisfies :

$$g'' + \left[\frac{2i}{\hbar}(t^2-z^2) - \frac{1}{t}\right]g' + \frac{4z^2t^2}{\hbar^2}g = 0 . \tag{22}$$

The slight asymmetry between the ODEs of f and g lies in the presence of the term $-\frac{i\hbar}{t}$ in the Schrödinger equation (13). Write g as a normalized power series : $g(t) = t^{\nu}\sum_{k=0}^{+\infty}d_k(z)t^k$, with $d_0 = 1$. Substituting in (22) :

$$\frac{1}{t^2}\sum_{k=0}^{+\infty}(\nu+k)_2 d_k t^k + \left(\frac{2it}{\hbar} - \frac{2iz^2}{\hbar t} - \frac{1}{t^2}\right)\sum_{k=0}^{+\infty}(\nu+k)d_k t^k + \frac{4z^2t^2}{\hbar^2}\sum_{k=0}^{+\infty}d_k t^k = 0$$
$$\Longleftrightarrow \frac{1}{t^2}\sum_{k=0}^{+\infty}(\nu+k)(\nu+k-2)d_k t^k - \frac{2iz^2}{\hbar t}\sum_{k=0}^{+\infty}(\nu+k)d_k t^k + \frac{2it}{\hbar}\sum_{k=0}^{+\infty}(\nu+k)d_k t^k +$$
$$\frac{4z^2t^2}{\hbar^2}\sum_{k=0}^{+\infty}d_k t^k = 0 .$$

Let us find the 4 first coefficients :

- order -2 : the same necessary condition $\nu(\nu-2) = 0$ implies $\begin{cases} \nu = 0 \\ \text{or } \nu = 2 \end{cases}$

•order -1 : $(\nu+1)(\nu-1)d_1 - \frac{2iz^2}{\hbar}\nu d_0 = 0$

$$\Longrightarrow \begin{cases} \text{if } \nu=0:\ d_1=0 \\ \text{otherwise if } \nu=2:\ d_1 = \frac{\nu}{(\nu+1)(\nu-1)}\frac{2iz^2}{\hbar} \end{cases},$$

•order $+0$: $(\nu+2)\nu d_2 - \frac{2iz^2}{\hbar}(\nu+1)d_1 = 0$

$$\Longrightarrow \begin{cases} \text{if } \nu=0:\ d_2 \text{ is arbitrary} \\ \text{otherwise if } \nu=2:\ d_2 = \frac{1}{(\nu+2)(\nu-1)}\left(\frac{2iz^2}{\hbar}\right)^2 \end{cases},$$

•order $+1$: $(\nu+3)(\nu+1)d_3 - \frac{2iz^2}{\hbar}(\nu+2)d_2 + \frac{2i}{\hbar}\nu d_0 = 0$

$$\Longrightarrow \begin{cases} \text{if } \nu=0:\ d_3 \text{ depends on the choice of } c_2 \\ \text{otherwise if } \nu=2:\ d_3 = \frac{1}{(\nu+3)(\nu+1)(\nu-1)}\left(\frac{2iz^2}{\hbar}\right)^3 - \frac{\nu}{(\nu+3)(\nu+1)}\frac{2i}{\hbar} \end{cases}$$

For $k \geqslant 4$, the recurrence relation between the coefficients d_k reads as :

$$(\nu+k)(\nu+k-2)d_k - \frac{2iz^2}{\hbar}(\nu+k-1)d_{k-1} + \frac{2i}{\hbar}(\nu+k-3)d_{k-3} + \frac{4z^2}{\hbar^2}d_{k-4} = 0\,.$$

The whole sequence is then obtained by :

$$d_k = \frac{2iz^2}{\hbar}\left[\frac{\nu+k-1}{(\nu+k)(\nu+k-2)}d_{k-1} + \frac{2i}{\hbar}\frac{1}{(\nu+k)(\nu+k-2)}d_{k-4}\right] - \frac{2i}{\hbar}\frac{\nu+k-3}{(\nu+k)(\nu+k-2)}d_{k-3}\,.$$

As $\nu = 2$, we shall deal with the recurrence relation :

$$d_k = \frac{2iz^2}{\hbar}\left[\frac{k+1}{(k+2)k}d_{k-1} + \frac{2i}{\hbar}\frac{1}{(k+2)k}d_{k-4}\right] - \frac{2i}{\hbar}\frac{k-1}{(k+2)k}d_{k-3}\,.$$

For $k \geqslant 1$, we set $d_{3k+j} = \sum_{\ell=0}^{k}\delta_\ell^{[3k+j]}z^{6\ell+2j}$. The coefficient of the term of highest degree in z^2 can be easily deduced :

$$\delta_k^{[3k+j]} = \frac{2}{(3k+j+2)(3k+j)!}\left(\frac{2i}{\hbar}\right)^{3k+j}.$$

In d_{3k}, the coefficient of the term of lowest degree in z^2 is :

$$\delta_0^{[3k]} = \left(-\frac{2i}{\hbar}\right)^k \frac{2}{3k+2}\frac{1}{3k(3k-3)\ldots 3}d_0 = \left(-\frac{2i}{3\hbar}\right)^k\frac{2}{(3k+2)k!}\,. \tag{23}$$

Introduce the quantity :

$$\begin{aligned}\chi_0^{[3\ell+1]}(\ell) &= \frac{2iz^2}{\hbar}\left[\frac{3\ell+2}{(3\ell+3)(3\ell+1)}\delta_0^{[3\ell]} + \frac{2i}{\hbar}\frac{1}{(3\ell+3)(3\ell+1)}\delta_0^{[3(\ell-1)]}\right] \\ &= \frac{2iz^2}{\hbar}\left(-\frac{2i}{3\hbar}\right)^\ell\frac{2}{\ell!(3\ell+3)(3\ell+1)}\left[1-\frac{3\ell}{3\ell-1}\right] = -\frac{2iz^2}{\hbar}\left(-\frac{2i}{3\hbar}\right)^\ell\frac{2}{\ell!(3\ell+3)(3\ell+1)(3\ell-1)} \\ &= -\frac{2iz^2}{\hbar}\left(-\frac{2i}{3\hbar}\right)^\ell\frac{1}{4\ell!}\frac{\Gamma\left(\frac{3\ell}{2}-\frac{1}{2}\right)}{\Gamma\left(\frac{3\ell}{2}+\frac{5}{2}\right)}\,.\end{aligned}$$

The coefficient of the term of lowest degree in z^2 in d_{3k+1} can be computed by means of a discrete convolution. Therefore :

$$\begin{aligned}\delta_0^{[3k+1]} &= \chi_0^{[3k+1]}(k) + \\ &\quad \sum_{\ell=1}^{k-1}\left(-\frac{2i}{3\hbar}\right)^\ell\frac{3(k-\ell+1)}{3k+3}\frac{\Gamma\left(k-\ell+\frac{4}{3}\right)}{\Gamma\left(k+\frac{4}{3}\right)}\chi_0^{[3(k-\ell)+1]}(k-\ell) + \\ &\quad \left(-\frac{2i}{3\hbar}\right)^k\frac{3}{3k+3}\frac{\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(k+\frac{4}{3}\right)}\delta_0^{[1]} \\ &= \frac{2}{3}\frac{2iz^2}{\hbar}\left(-\frac{2i}{3\hbar}\right)^k\frac{3}{3k+3}\frac{\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(k+\frac{4}{3}\right)} + \chi_0^{[3k+1]}(k) \\ &\quad -\frac{2iz^2}{\hbar}\left(-\frac{2i}{3\hbar}\right)^k\frac{1}{4(3k+3)\Gamma\left(k+\frac{4}{3}\right)}\sum_{\ell=1}^{k-1}\frac{3(k-\ell)+3}{(k-\ell)!}\frac{\Gamma\left(k-\ell+\frac{4}{3}\right)\Gamma\left(\frac{3(k-\ell)}{2}-\frac{1}{2}\right)}{\Gamma\left(\frac{3(k-\ell)}{2}+\frac{5}{2}\right)} \\ &= \chi_0^{[3k+1]}(k) - \frac{2iz^2}{\hbar}\left(-\frac{2i}{3\hbar}\right)^k\frac{1}{4(3k+3)\Gamma\left(k+\frac{4}{3}\right)}\sum_{\ell=1}^{k-1}\frac{3(\ell+1)}{\ell!}\frac{\Gamma\left(\ell+\frac{4}{3}\right)\Gamma\left(\frac{3\ell}{2}-\frac{1}{2}\right)}{\Gamma\left(\frac{3\ell}{2}+\frac{5}{2}\right)} + \\ &\quad \frac{2iz^2}{\hbar}\left(-\frac{2i}{3\hbar}\right)^k\frac{2}{3k+3}\frac{\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(k+\frac{4}{3}\right)} \\ &= \frac{2iz^2}{\hbar}\left(-\frac{2i}{3\hbar}\right)^k\frac{\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(k+\frac{4}{3}\right)}\frac{2}{3k+3}\left[1-\frac{1}{8}\sum_{\ell=1}^{k}\frac{3(\ell+1)\Gamma\left(\ell+\frac{4}{3}\right)\Gamma\left(\frac{3\ell}{2}-\frac{1}{2}\right)}{\Gamma(\ell+1)\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{3\ell}{2}+\frac{5}{2}\right)}\right] \\ &= \frac{2iz^2}{\hbar}\left(-\frac{2i}{3\hbar}\right)^k\frac{\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(k+\frac{4}{3}\right)}\frac{2}{3k+3}\underbrace{\left[1-\frac{1}{4}\sum_{\ell=1}^{k}\frac{B\left(2,\frac{3\ell}{2}-\frac{1}{2}\right)}{B\left(\frac{4}{3},\ell\right)}\frac{1}{\ell}\right]}_{G_{0,1}(k)}\,.\end{aligned} \tag{24}$$

The general term in the sum is a $\mathscr{O}\left(\frac{1}{\ell^{5/3}}\right)$ for ℓ large enough. Evaluate the quantity :

$$\begin{aligned}\chi_1^{[3\ell+2]}(\ell) &= \frac{2iz^2}{\hbar}\left[\frac{3\ell+3}{(3\ell+4)(3\ell+2)}\delta_0^{[3\ell+1]} + \frac{2i}{\hbar}\frac{1}{(3\ell+4)(3\ell+2)}\delta_0^{[3\ell-2)]}\right] \\ &= \left(\frac{2iz^2}{\hbar}\right)^2\left(-\frac{2i}{3\hbar}\right)^\ell\frac{\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\ell+\frac{4}{3}\right)}\frac{2}{(3\ell+4)(3\ell+2)} \\ &\quad .\left\{\left[1-\frac{1}{4}\sum_{m=1}^{\ell}\frac{B\left(2,\frac{3m}{2}-\frac{1}{2}\right)}{B\left(\frac{4}{3},m\right)}\frac{1}{m}\right] - \frac{3\left(\ell+\frac{1}{3}\right)}{3\ell}\left[1-\frac{1}{4}\sum_{m=1}^{\ell-1}\frac{B\left(2,\frac{3m}{2}-\frac{1}{2}\right)}{B\left(\frac{4}{3},m\right)}\frac{1}{m}\right]\right\} \\ &= -\left(\frac{2iz^2}{\hbar}\right)^2\left(-\frac{2i}{3\hbar}\right)^\ell\frac{\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\ell+\frac{4}{3}\right)}\frac{2}{(3\ell+4)(3\ell+2)3\ell} \\ &\quad \left[1-\frac{1}{4}\sum_{m=1}^{\ell}\frac{B\left(2,\frac{3m}{2}-\frac{1}{2}\right)}{B\left(\frac{4}{3},m\right)}\frac{1}{m} + \frac{3\ell+1}{4\ell}\frac{B\left(2,\frac{3\ell}{2}-\frac{1}{2}\right)}{B\left(\frac{4}{3},\ell\right)}\right] \\ &= -\left(\frac{2iz^2}{\hbar}\right)^2\left(-\frac{2i}{3\hbar}\right)^\ell\frac{\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\ell+\frac{4}{3}\right)}\frac{\Gamma\left(\frac{3\ell}{2}\right)}{\Gamma\left(\frac{3\ell}{2}+3\right)} \\ &\quad .\frac{1}{4}\left[1-\frac{1}{4}\sum_{m=1}^{\ell}\frac{B\left(2,\frac{3m}{2}-\frac{1}{2}\right)}{B\left(\frac{4}{3},m\right)}\frac{1}{m} + \frac{3\ell+1}{4\ell}\frac{B\left(2,\frac{3\ell}{2}-\frac{1}{2}\right)}{B\left(\frac{4}{3},\ell\right)}\right]\,.\end{aligned}$$

The coefficient of the term of lowest degree in z^2 in d_{3k+2} can be computed by :

$$\begin{aligned}
\delta_0^{[3k+2]} &= \chi_1^{[3k+2]}(k) + \\
&\quad \sum_{\ell=1}^{k-1} \left(-\frac{2i}{3\hbar}\right)^{\ell} \frac{3(k-\ell)+4}{3k+4} \frac{\Gamma\left(k-\ell+\frac{5}{3}\right)}{\Gamma\left(k+\frac{5}{3}\right)} \chi_1^{[3(k-\ell)+2]}(k-\ell) \\
&\quad + \left(-\frac{2i}{3\hbar}\right)^{k} \frac{4}{3k+4} \frac{\Gamma\left(\frac{5}{3}\right)}{\Gamma\left(k+\frac{5}{3}\right)} \delta_0^{[2]} \\
&= \left(\frac{2iz^2}{\hbar}\right)^2 \left(-\frac{2i}{3\hbar}\right)^{k} \frac{\Gamma\left(\frac{5}{3}\right)}{\Gamma\left(k+\frac{5}{3}\right)} \frac{1}{3k+4} \\
&\quad + \chi_1^{[3k+2]}(k) - \left(\frac{2iz^2}{\hbar}\right)^2 \left(-\frac{2i}{3\hbar}\right)^{k} \frac{1}{3k+4} \\
&\quad + \sum_{\ell=1}^{k-1} \frac{3(k-\ell)+4}{4} \frac{\Gamma\left(k-\ell+\frac{5}{3}\right)}{\Gamma\left(k+\frac{5}{3}\right)} \frac{\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(k-\ell+\frac{4}{3}\right)} \frac{\Gamma\left(\frac{3(k-\ell)}{2}\right)}{\Gamma\left(\frac{3(k-\ell)}{2}+3\right)} \\
&\quad \left[1-\frac{1}{4}\sum_{m=1}^{k-\ell} \frac{B\left(2,\frac{3m}{2}-\frac{1}{2}\right)}{B\left(\frac{4}{3},m\right)} \frac{1}{m} + \frac{3(k-\ell)+1}{4(k-\ell)} \frac{B\left(2,\frac{3(k-\ell)}{2}-\frac{1}{2}\right)}{B\left(\frac{4}{3},k-\ell\right)}\right] \\
&= \left(\frac{2iz^2}{\hbar}\right)^2 \left(-\frac{2i}{3\hbar}\right)^{k} \frac{\Gamma\left(\frac{5}{3}\right)}{\Gamma\left(k+\frac{5}{3}\right)} \frac{1}{3k+4} \\
&\quad + \chi_1^{[3k+2]}(k) - \left(\frac{2iz^2}{\hbar}\right)^2 \left(-\frac{2i}{3\hbar}\right)^{k} \frac{\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(k+\frac{5}{3}\right)} \frac{1}{3k+4} \\
&\quad \sum_{\ell=1}^{k-1} \frac{3\ell+4}{4} \frac{\Gamma\left(\ell+\frac{5}{3}\right)}{\Gamma\left(\ell+\frac{4}{3}\right)} \frac{\Gamma\left(\frac{3\ell}{2}\right)}{\Gamma\left(\frac{3\ell}{2}+3\right)} \\
&\quad \left[1-\frac{1}{4}\sum_{m=1}^{\ell} \frac{B\left(2,\frac{3m}{2}-\frac{1}{2}\right)}{B\left(\frac{4}{3},m\right)} \frac{1}{m} + \frac{3\ell+1}{4\ell} \frac{B\left(2,\frac{3\ell}{2}-\frac{1}{2}\right)}{B\left(\frac{4}{3},\ell\right)}\right] \\
&= \left(\frac{2iz^2}{\hbar}\right)^2 \left(-\frac{2i}{3\hbar}\right)^{k} \frac{\Gamma\left(\frac{5}{3}\right)}{\Gamma\left(k+\frac{5}{3}\right)} \frac{2}{3k+4} G_{0,2}(k)\,, \qquad (25)
\end{aligned}$$

where $G_{0,2} : \mathbb{N}^* \to \mathbb{R}$ is the arithmetic function defined by

$$\begin{aligned}
&G_{0,2}(k) = \\
&\frac{1}{2}\left\{1-\sum_{\ell=1}^{k} \frac{3\ell+4}{4} \frac{\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{5}{3}\right)} \frac{\Gamma\left(\ell+\frac{5}{3}\right)}{\Gamma\left(\ell+\frac{4}{3}\right)} \frac{\Gamma\left(\frac{3\ell}{2}\right)}{\Gamma\left(\frac{3\ell}{2}+3\right)} \left[1-\frac{1}{4}\sum_{m=1}^{\ell} \frac{B\left(2,\frac{3m}{2}-\frac{1}{2}\right)}{B\left(\frac{4}{3},m\right)} \frac{1}{m} + \frac{3\ell+1}{4\ell} \frac{B\left(2,\frac{3\ell}{2}-\frac{1}{2}\right)}{B\left(\frac{4}{3},\ell\right)}\right]\right\} \\
&= \frac{1}{2}\left\{1-\sum_{\ell=1}^{k} \frac{3\ell+4}{4} \frac{B\left(\frac{4}{3},\ell+\frac{5}{3}\right)}{B\left(\frac{5}{3},\ell+\frac{4}{3}\right)} \frac{\Gamma\left(\frac{3\ell}{2}\right)}{\Gamma\left(\frac{3\ell}{2}+3\right)} \left[1-\frac{1}{4}\sum_{m=1}^{\ell} \frac{B\left(2,\frac{3m}{2}-\frac{1}{2}\right)}{B\left(\frac{4}{3},m\right)} \frac{1}{m} + \frac{3\ell+1}{4\ell} \frac{B\left(2,\frac{3\ell}{2}-\frac{1}{2}\right)}{B\left(\frac{4}{3},\ell\right)}\right]\right\}.
\end{aligned}$$

The general term in the sum is a $\mathcal{O}\left(\frac{1}{\ell^{5/3}}\right)$ for ℓ large enough.

3.5 Dominant exponential solutions in the $\frac{\pi}{6}$ and $\frac{5\pi}{6}$ directions

For both series, since $z \ll \hbar$, every summand in the polynomial expansion are negligible before the leading term i.e. the one of lowest degree in z^2. Moreover, for g_1 :

$$\begin{aligned}
\sum_{k=2}^{+\infty} d_k t^k &= t^2 \sum_{k=0}^{+\infty} \left(d_{3k}+d_{3k+1}t+d_{3k+2}t^2\right) t^{3k} \simeq t^2 \sum_{k=0}^{+\infty} \left(\delta_0^{[3k]}+\delta_0^{[3k+1]}t+\delta_0^{[3k+2]}t^2\right) t^{3k} \\
&= t^2 \sum_{k=0}^{+\infty} \left[1+\frac{2iz^2}{\hbar}\frac{3k+2}{3k+3}\frac{\Gamma\left(\frac{4}{3}\right)\Gamma(k+1)}{\Gamma\left(k+\frac{4}{3}\right)} G_{0,1}(k)t + \left(\frac{2iz^2}{\hbar}\right)^2 \frac{3k+2}{3k+4}\frac{\Gamma\left(\frac{5}{3}\right)\Gamma(k+1)}{\Gamma\left(k+\frac{5}{3}\right)} G_{0,2}(k)t^2\right] \\
&\quad \cdot \left(-\frac{2i}{3\hbar}\right)^k \frac{2}{(3k+2)k!} t^{3k} \\
&= t^2 \sum_{k=0}^{+\infty} \left[1+\frac{2iz^2}{\hbar}\frac{3k+2}{3k+3}\frac{\Gamma\left(\frac{4}{3}\right)\Gamma(k+1)}{\Gamma\left(k+\frac{4}{3}\right)} G_{0,1}^{\infty}t + \left(\frac{2iz^2}{\hbar}\right)^2 \frac{3k+2}{3k+4}\frac{\Gamma\left(\frac{5}{3}\right)\Gamma(k+1)}{\Gamma\left(k+\frac{5}{3}\right)} G_{0,2}^{\infty}t^2\right] \\
&\quad \cdot \left(-\frac{2i}{3\hbar}\right)^k \frac{2}{(3k+2)k!} t^{3k},
\end{aligned}$$

with the numerical values :

$$\begin{cases} G_{0,1}^{\infty} = \lim_{k\to+\infty} G_{0,1}(k) \simeq 6,84.10^{-1} \\ G_{0,2}^{\infty} = \lim_{k\to+\infty} G_{0,2}(k) \simeq 2,72.10^{-1} \end{cases}.$$

For $\mathrm{Arg}\,t = \frac{\pi}{6}$ i.e. $t = Te^{i\frac{\pi}{6}}$:

$$\begin{aligned}
\sum_{k=2}^{+\infty} d_k t^k &\simeq T^2 e^{i\frac{\pi}{3}} \sum_{k=0}^{+\infty} \left[1+\frac{2iz^2}{\hbar}\frac{3k+2}{3k+3}\frac{\Gamma\left(\frac{4}{3}\right)\Gamma(k+1)}{\Gamma\left(k+\frac{4}{3}\right)} G_{0,1}^{\infty} Te^{i\frac{\pi}{6}} + \left(\frac{2iz^2}{\hbar}\right)^2 \frac{3k+2}{3k+4}\frac{\Gamma\left(\frac{5}{3}\right)\Gamma(k+1)}{\Gamma\left(k+\frac{5}{3}\right)} G_{0,2}^{\infty} T^2 e^{i\frac{\pi}{3}}\right] \\
&\quad \cdot \left(\frac{2}{3\hbar}\right)^k \frac{2}{(3k+2)k!} T^{3k} \\
&\simeq T^2 e^{i\frac{\pi}{3}} \left[1+\frac{2iz^2}{\hbar}\Gamma\left(\frac{4}{3}\right)\frac{G_{0,1}^{\infty} Te^{i\frac{\pi}{6}}}{x_0^{1/3}} + \left(\frac{2iz^2}{\hbar}\right)^2 \Gamma\left(\frac{5}{3}\right)\frac{G_{0,2}^{\infty} T^2 e^{i\frac{\pi}{3}}}{x_0^{2/3}}\right] \frac{2}{3x_0} e^{x_0},
\end{aligned}$$

where $x_0 = \frac{2T^3}{3\hbar}$ by using Corollary 1. So in the $\frac{\pi}{6}$ direction, we have asymptotically :

$$\begin{aligned}
g_1(t) &= e^{i\frac{\pi}{3}} \left[1+\frac{2iz^2}{\hbar}\Gamma\left(\frac{4}{3}\right)\frac{G_{0,1}^{\infty} Te^{i\frac{\pi}{6}}}{x_0^{1/3}} + \left(\frac{2iz^2}{\hbar}\right)^2 \Gamma\left(\frac{5}{3}\right)\frac{G_{0,2}^{\infty} T^2 e^{i\frac{\pi}{3}}}{x_0^{2/3}}\right] \frac{\hbar}{T}\exp\left(\frac{2T^3}{3\hbar}\right) \\
&= i\left[1+\left(\frac{2}{\hbar}\right)^{2/3}\Gamma\left(\frac{4}{3}\right)3^{1/3}G_{0,1}^{\infty} ie^{i\frac{\pi}{6}}z^2 - \left(\frac{2}{\hbar}\right)^{4/3}\Gamma\left(\frac{5}{3}\right)3^{2/3}G_{0,2}^{\infty} e^{i\frac{\pi}{3}}z^4\right] \frac{\hbar}{t}\exp\left(-\frac{2it^3}{3\hbar}\right) \\
&\simeq i\left[1+\left(\frac{2}{\hbar}\right)^{2/3}\Gamma\left(\frac{4}{3}\right)3^{1/3}G_{0,1}^{\infty} e^{i\frac{2\pi}{3}}z^2\right] \frac{\hbar}{t}\exp\left(-\frac{2it^3}{3\hbar}\right)
\end{aligned}$$

by recalling that $z \to 0$. Similarly in the $\frac{5\pi}{6}$ direction, we have asymptotically :

$$\begin{aligned}
g_1(t) &= e^{i\frac{5\pi}{6}} e^{i\frac{5\pi}{3}} \left[1+\left(\frac{2}{\hbar}\right)^{2/3}\Gamma\left(\frac{4}{3}\right)3^{1/3}G_{0,1}^{\infty} ie^{i\frac{5\pi}{6}}z^2 - \left(\frac{2}{\hbar}\right)^{4/3}\Gamma\left(\frac{5}{3}\right)3^{2/3}G_{0,2}^{\infty} e^{i\frac{5\pi}{3}}z^4\right] \\
&\quad \cdot \frac{\hbar}{t}\exp\left(-\frac{2it^3}{3\hbar}\right) \\
&\simeq i\left[1+\left(\frac{2}{\hbar}\right)^{2/3}\Gamma\left(\frac{4}{3}\right)3^{1/3}G_{0,1}^{\infty} e^{-i\frac{2\pi}{3}}z^2\right] \frac{\hbar}{t}\exp\left(-\frac{2it^3}{3\hbar}\right).
\end{aligned}$$

3.6 Dominant exponential solutions in the $\frac{\pi}{2}$ and $\frac{7\pi}{6}$ directions

By keeping only the leading term $\gamma_0^{[k]} z^{6\{k/3\}}$ in the polynomial expansion of c_k :

$$
\begin{aligned}
\sum_{k=2}^{+\infty} c_k t^k &= t^2 \sum_{k=0}^{+\infty}\left(c_{3k}+d_{3k+1}t+d_{3k+2}t^2\right)t^{3k} \simeq t^2 \sum_{k=1}^{+\infty}\left(\gamma_0^{[3k]}+\gamma_0^{[3k+1]}t+\gamma_0^{[3k+2]}t^2\right)t^{3k} \\
&= t^2 \sum_{k=1}^{+\infty}\left[1+\frac{2iz^2}{\hbar}\frac{\Gamma\left(k+\frac{5}{3}\right)}{\Gamma(k+2)\Gamma\left(\frac{5}{3}\right)}F_{0,1}(k)t+\left(\frac{2iz^2}{\hbar}\right)^2\frac{\Gamma\left(\frac{7}{3}\right)\Gamma\left(k+\frac{5}{3}\right)}{\Gamma\left(k+\frac{7}{3}\right)\Gamma\left(\frac{5}{3}\right)}F_{0,2}(k)t^2\right]\left(\frac{2i}{3\hbar}\right)^k\frac{\Gamma\left(\frac{5}{3}\right)}{\Gamma\left(k+\frac{5}{3}\right)}t^{3k} \\
&= t^2 \sum_{k=1}^{+\infty}\left[1+\frac{2iz^2}{\hbar}\frac{B\left(2,k+\frac{5}{3}\right)}{B\left(\frac{5}{3},k+2\right)}F_{0,1}^{\infty}t+\left(\frac{2iz^2}{\hbar}\right)^2\frac{B\left(\frac{7}{3},k+\frac{5}{3}\right)}{B\left(\frac{5}{3},k+\frac{7}{3}\right)}F_{0,2}^{\infty}t^2\right]\left(\frac{2i}{3\hbar}\right)^k\frac{\Gamma\left(\frac{5}{3}\right)}{\Gamma\left(k+\frac{5}{3}\right)}t^{3k},
\end{aligned}
$$

with the numerical values :

$$
\begin{cases}
F_{0,1}^{\infty} = \lim\limits_{k\to+\infty} F_{0,1}(k) \simeq -2,08.10^{-1} \\
F_{0,2}^{\infty} = \lim\limits_{k\to+\infty} F_{0,2}^{\infty} \simeq 3,8.10^{-2}
\end{cases}.
$$

For $\mathrm{Arg}\, t = \dfrac{\pi}{2}$ i.e. $t = iT$:

$$
\begin{aligned}
\sum_{k=2}^{+\infty} c_k t^k &\simeq -T^2 \sum_{k=1}^{+\infty}\left[\Gamma\left(\frac{5}{3}\right)-\frac{2z^2}{\hbar}\frac{F_{0,1}^{\infty}T}{k^{1/3}}+\left(\frac{2z^2}{\hbar}\right)^2\Gamma\left(\frac{7}{3}\right)\frac{F_{0,2}^{\infty}T^2}{k^{2/3}}\right]\left(\frac{2}{3\hbar}\right)^k\frac{1}{k^{2/3}k!}T^{3k} \\
&\simeq -T^2\left[\Gamma\left(\frac{5}{3}\right)-\frac{2z^2}{\hbar}\frac{F_{0,1}^{\infty}T}{x_0^{1/3}}+\left(\frac{2z^2}{\hbar}\right)^2\Gamma\left(\frac{7}{3}\right)\frac{F_{0,2}^{\infty}T^2}{x_0^{2/3}}\right]\frac{1}{x_0^{2/3}}e^{x_0},
\end{aligned}
$$

where $x_0 = \dfrac{2T^3}{3\hbar}$ by using Corollary 1. So in the $\dfrac{\pi}{2}$ direction, we have asymptotically :

$$
\begin{aligned}
f_1(t) &= -\left(\frac{3\hbar}{2}\right)^{2/3}\left[\Gamma\left(\frac{5}{3}\right)-\left(\frac{2}{\hbar}\right)^{2/3}F_{0,1}^{\infty}z^2+\left(\frac{2}{\hbar}\right)^{4/3}\Gamma\left(\frac{7}{3}\right)F_{0,2}^{\infty}z^4\right]\exp\left(\frac{2T^3}{3\hbar}\right) \\
&\simeq -\left(\frac{3\hbar}{2}\right)^{2/3}\left[\Gamma\left(\frac{5}{3}\right)-\left(\frac{2}{\hbar}\right)^{2/3}F_{0,1}^{\infty}z^2\right]\exp\left(\frac{2it^3}{3\hbar}\right)
\end{aligned}
$$

since $z \to 0$. Similarly for $\mathrm{Arg}\, t = \dfrac{7\pi}{6}$ i.e. $t = Te^{i\frac{7\pi}{6}}$:

$$
\begin{aligned}
\sum_{k=2}^{+\infty} c_k t^k &\simeq T^2 e^{i\frac{\pi}{3}} \sum_{k=1}^{+\infty}\left[\Gamma\left(\frac{5}{3}\right)+\frac{2z^2}{\hbar}\frac{F_{0,1}^{\infty}Tie^{i\frac{7\pi}{6}}}{k^{1/3}}-\left(\frac{2z^2}{\hbar}\right)^2\Gamma\left(\frac{7}{3}\right)\frac{F_{0,2}^{\infty}T^2e^{i\frac{\pi}{3}}}{k^{2/3}}\right]\left(\frac{2}{3\hbar}\right)^k\frac{1}{k^{2/3}k!}T^{3k} \\
&\simeq T^2 e^{i\frac{\pi}{3}}\left[\Gamma\left(\frac{5}{3}\right)+\frac{2z^2}{\hbar}\frac{F_{0,1}^{\infty}Te^{-i\frac{\pi}{3}}}{x_0^{1/3}}-\left(\frac{2z^2}{\hbar}\right)^2\Gamma\left(\frac{7}{3}\right)\frac{F_{0,2}^{\infty}T^2e^{i\frac{\pi}{3}}}{x_0^{2/3}}\right]\frac{1}{x_0^{2/3}}e^{x_0}
\end{aligned}
$$

by using Corollary 1. In the $\dfrac{7\pi}{6}$ direction, we have asymptotically :

$$
\begin{aligned}
f_1(t) &= e^{i\frac{\pi}{3}}\left(\frac{3\hbar}{2}\right)^{2/3}\left[\Gamma\left(\frac{5}{3}\right)+\left(\frac{2}{\hbar}\right)^{2/3}F_{0,1}^{\infty}e^{-i\frac{\pi}{3}}z^2-\left(\frac{2}{\hbar}\right)^{4/3}\Gamma\left(\frac{7}{3}\right)F_{0,2}^{\infty}e^{i\frac{\pi}{3}}z^4\right]\exp\left(\frac{2it^3}{3\hbar}\right) \\
&\simeq e^{i\frac{\pi}{3}}\left(\frac{3\hbar}{2}\right)^{2/3}\left[\Gamma\left(\frac{5}{3}\right)+\left(\frac{2}{\hbar}\right)^{2/3}F_{0,1}^{\infty}e^{-i\frac{\pi}{3}}z^2\right]\exp\left(\frac{2it^3}{3\hbar}\right)
\end{aligned}
$$

since $z \to 0$.

3.7 Stokes multipliers

Recall that $\psi_1(t) = f_1(t)\exp\left[-\dfrac{i}{\hbar}\left(\dfrac{t^3}{3}-z^2t\right)\right] = g_1(t)\exp\left[\dfrac{i}{\hbar}\left(\dfrac{t^3}{3}-z^2t\right)\right]$.
Let us recollect our results :

- for $\mathrm{Arg}\, t = \frac{\pi}{6}$: $\psi_1(t) = i\left[1+\left(\frac{2}{\hbar}\right)^{2/3}\Gamma\left(\frac{4}{3}\right)3^{1/3}G_{0,1}^{\infty}e^{i\frac{2\pi}{3}}z^2\right]\frac{\hbar}{t}\exp\left[-\frac{i}{\hbar}\left(\frac{t^3}{3}+z^2t\right)\right]$
- for $\mathrm{Arg}\, t = \frac{\pi}{2}$: $\psi_1(t) = -\left(\frac{3\hbar}{2}\right)^{2/3}\left[\Gamma\left(\frac{5}{3}\right)-\left(\frac{2}{\hbar}\right)^{2/3}F_{0,1}^{\infty}z^2\right]\exp\left[\frac{i}{\hbar}\left(\frac{t^3}{3}+z^2t\right)\right]$
- for $\mathrm{Arg}\, t = \frac{5\pi}{6}$: $\psi_1(t) = i\left[1+\left(\frac{2}{\hbar}\right)^{2/3}\Gamma\left(\frac{4}{3}\right)3^{1/3}G_{0,1}^{\infty}e^{-i\frac{2\pi}{3}}z^2\right]\frac{\hbar}{t}\exp\left[-\frac{i}{\hbar}\left(\frac{t^3}{3}+z^2t\right)\right]$
- for $\mathrm{Arg}\, t = \frac{7\pi}{6}$: $\psi_1(t) = e^{i\frac{\pi}{3}}\left(\frac{3\hbar}{2}\right)^{2/3}\left[\Gamma\left(\frac{5}{3}\right)+\left(\frac{2}{\hbar}\right)^{2/3}F_{0,1}^{\infty}e^{-i\frac{\pi}{3}}z^2\right]\exp\left[\frac{i}{\hbar}\left(\frac{t^3}{3}+z^2t\right)\right]$

in the sense of asymptotic series. So the Stokes multipliers are approximated by :

$$
\begin{aligned}
\widetilde{C_{S1}} &= -\frac{i\hbar}{t}\left(\frac{2}{\hbar}\right)^{4/3}\frac{\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{5}{3}\right)}\left(e^{-i\frac{2\pi}{3}}-e^{i\frac{2\pi}{3}}\right)\frac{G_{0,1}^{\infty}}{3^{1/3}}z^2 = -\frac{1}{t}\left(\frac{2}{3\hbar}\right)^{1/3}\frac{\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{5}{3}\right)}\sqrt{3}G_{0,1}^{\infty}z^2 \\
\text{and}\ \widetilde{C_{S2}} &= -\frac{it}{\hbar}\left(\frac{3\hbar}{2}\right)^{2/3}\Gamma\left(\frac{5}{3}\right)\left(e^{i\frac{\pi}{3}}+1\right) = -\frac{t}{\hbar}e^{i\frac{2\pi}{3}}\left(\frac{3\hbar}{2}\right)^{2/3}\Gamma\left(\frac{5}{3}\right)\sqrt{3} \\
\text{while}\ \widetilde{C_{S1}}\widetilde{C_{S2}} &= \frac{3}{\hbar}e^{i\frac{2\pi}{3}}\left(\frac{3\hbar}{2}\right)^{1/3}\Gamma\left(\frac{4}{3}\right)G_{0,1}^{\infty}z^2 .
\end{aligned}
$$

3.8 Scattering coefficients

Recall the formula (16). Finally, we find that :

$$
\begin{aligned}
a(z)^* &= \exp\left[-\frac{3\sqrt{3}}{\pi}\Gamma\left(\frac{2}{3}\right)^3\left(\frac{z^3}{2\hbar}\right)^{2/3}\right]+ \\
&\quad C_{S2}C_{S1}\cosh\left[\frac{3\sqrt{3}}{\pi}\Gamma\left(\frac{2}{3}\right)^3\left(\frac{z^3}{2\hbar}\right)^{2/3}\right] \\
&\simeq \exp\left[-\frac{3\sqrt{3}}{\pi}\Gamma\left(\frac{2}{3}\right)^3\left(\frac{z^3}{2\hbar}\right)^{2/3}\right] \\
&\quad +3e^{i\frac{2\pi}{3}}G_{0,1}^{\infty}\left(\frac{3}{2}\right)^{1/3}\Gamma\left(\frac{4}{3}\right)\cosh\left[\frac{3\sqrt{3}}{\pi}\Gamma\left(\frac{2}{3}\right)^3\left(\frac{z^3}{2\hbar}\right)^{2/3}\right]\left(\frac{z^3}{\hbar}\right)^{2/3} \\
\Longrightarrow\quad a(z) &\simeq \exp\left[-\frac{3\sqrt{3}}{\pi}\Gamma\left(\frac{2}{3}\right)^3\left(\frac{z^3}{2\hbar}\right)^{2/3}\right] \\
&\quad +3e^{-i\frac{2\pi}{3}}G_{0,1}^{\infty}\left(\frac{3}{2}\right)^{1/3}\Gamma\left(\frac{4}{3}\right)\cosh\left[\frac{3\sqrt{3}}{\pi}\Gamma\left(\frac{2}{3}\right)^3\left(\frac{z^3}{2\hbar}\right)^{2/3}\right]\left(\frac{z^3}{\hbar}\right)^{2/3},
\end{aligned}
\tag{26}
$$

and :

$$
\begin{aligned}
b(z) &= -\frac{t}{z}C_{S1}\cosh\left[\frac{3\sqrt{3}}{\pi}\Gamma\left(\frac{2}{3}\right)^3\left(\frac{z^3}{2\hbar}\right)^{2/3}\right] \\
&\simeq \cosh\left[\frac{3\sqrt{3}}{\pi}\Gamma\left(\frac{2}{3}\right)^3\left(\frac{z^3}{2\hbar}\right)^{2/3}\right]\frac{\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{5}{3}\right)}\sqrt{3}G_{0,1}^{\infty}\left(\frac{2z^3}{3\hbar}\right)^{1/3}.
\end{aligned}
$$

Both expressions are plotted with respect to the adimensional parameter $\mu_2 = \dfrac{z^3}{\hbar}$.
To conclude this section, the S-matrix is trivial in the adiabatic limit, i.e. : $\lim\limits_{z\to 0} S(z) = 1_2$.

4 Empirical Discussion

The Landau-Zener effect (occurring in a conical intersection) is characterized by the transition probability $a_1(\mu_1) = \exp\left(-\dfrac{\pi\mu_1}{2}\right)$, with a slope $-\dfrac{\pi}{2}$ at the adiabatic

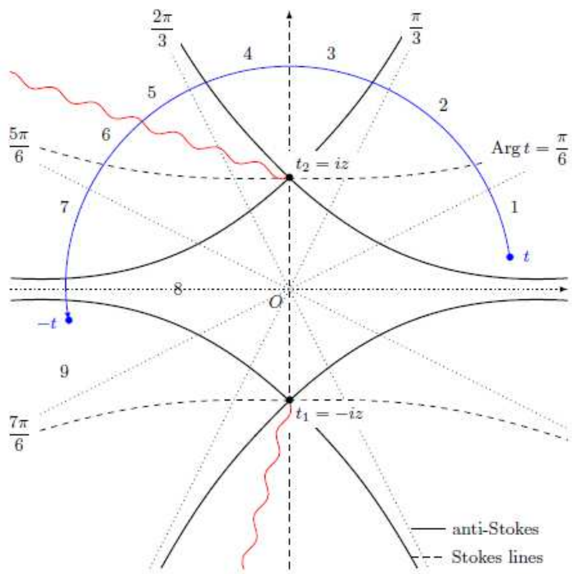

Fig. 1: Stokes structure for 2 second-order turning points on the imaginary axis.

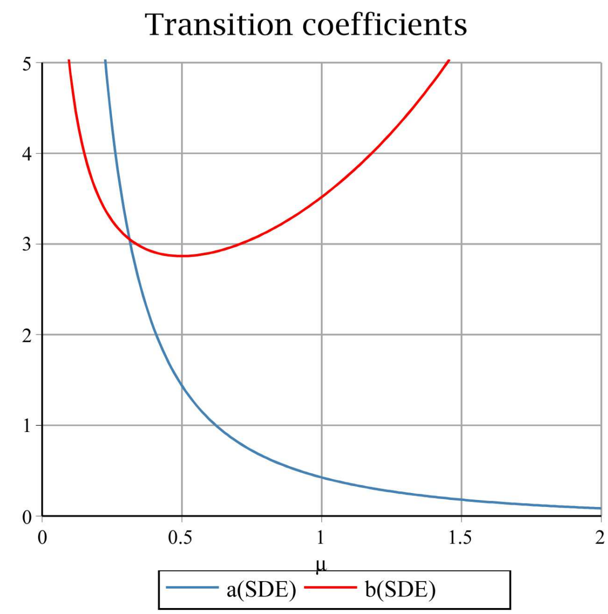

Fig. 2

limit. In the case of a parabolic intersection :

$$\begin{aligned}
a_2(\mu_2) &= \exp\left[-\frac{3\sqrt{3}}{2^{2/3}\pi}\Gamma\left(\frac{2}{3}\right)^3 \mu_2{}^{2/3}\right] \\
&\quad +3e^{-i\frac{2\pi}{3}} G^{\infty}_{0,1}\left(\frac{3}{2}\right)^{1/3}\Gamma\left(\frac{4}{3}\right)\cosh\left[\frac{3\sqrt{3}}{2^{2/3}\pi}\Gamma\left(\frac{2}{3}\right)^3 \mu_2{}^{2/3}\right]\mu_2{}^{2/3} \\
&\simeq 1-\frac{3\sqrt{3}}{2^{2/3}\pi}\Gamma\left(\frac{2}{3}\right)^3 \mu_2{}^{2/3} + 3e^{-i\frac{2\pi}{3}} G^{\infty}_{0,1}\left(\frac{3}{2}\right)^{1/3}\Gamma\left(\frac{4}{3}\right)\mu_2{}^{2/3} \\
\Longrightarrow \quad |a_2(\mu_2)| &\simeq \left[1-\frac{3.2^{1/3}\sqrt{3}}{\pi}\Gamma\left(\frac{2}{3}\right)^3\mu_2{}^{2/3} - 3G^{\infty}_{0,1}\left(\frac{3}{2}\right)^{1/3}\Gamma\left(\frac{4}{3}\right)\mu_2{}^{2/3}\right]^{1/2} \\
&\simeq 1-\frac{3}{2}\left[\frac{2^{1/3}\sqrt{3}}{\pi}\Gamma\left(\frac{2}{3}\right)^3 + G^{\infty}_{0,1}\left(\frac{3}{2}\right)^{1/3}\Gamma\left(\frac{4}{3}\right)\right]\mu_2{}^{2/3} \\
&= 1-3,64\mu_2{}^{2/3}\,. \qquad (27)
\end{aligned}$$

The slope at the origin is $-\infty$, with :

$$\frac{d|a_2|}{d\mu_2}\left[0^+\right] \sim -\left[\frac{2^{1/3}\sqrt{3}}{\pi}\Gamma\left(\frac{2}{3}\right)^3 + G^{\infty}_{0,1}\left(\frac{3}{2}\right)^{1/3}\Gamma\left(\frac{4}{3}\right)\right]\frac{1}{\mu_2{}^{1/3}} = -\frac{2,42}{\mu_2{}^{1/3}}\,. \qquad (28)$$

For a given value of z, the energy gap in the vicinity of the avoided level crossing, that is $\Delta E = \dfrac{2z^p}{\hbar}$ between the conduction and valence bands, is lowered in the case $p = 2$. We recover the qualitative conclusion that the quantum tunnelling becomes thus easier. But the new element in the transition probability a_2 is the appearance of a small phase Ω such that :

$$\Omega = -\arctan\frac{\sqrt{3}G^{\infty}_{0,1}\left(\frac{3}{2}\right)^{4/3}\Gamma\left(\frac{4}{3}\right)\mu_2{}^{2/3}}{1-\frac{3}{2}\left[\frac{2^{1/3}\sqrt{3}}{\pi}\Gamma\left(\frac{2}{3}\right)^3 + G^{\infty}_{0,1}\left(\frac{3}{2}\right)^{1/3}\Gamma\left(\frac{4}{3}\right)\right]\mu_2{}^{2/3}} \simeq -1,82\mu_2{}^{2/3}\,.$$

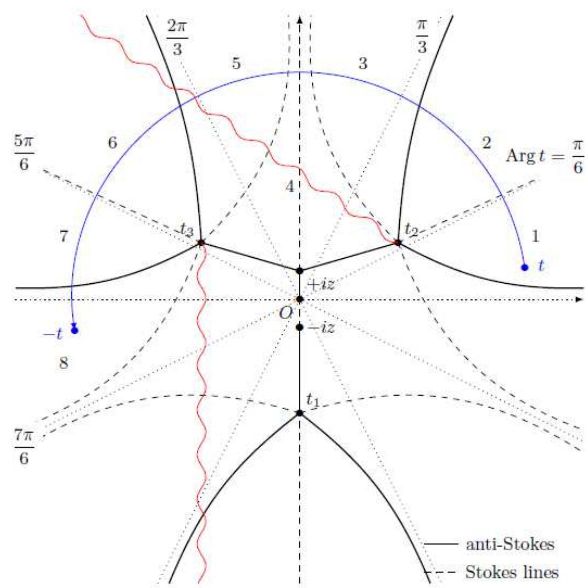

Fig. 3: Stokes structure for 5 first-order zeroes and 1 pole.

Appendix : Tables of 12 first coefficients c_k and d_k

Before every polynomial, there is a prefactor $\dfrac{1}{k!}\left(\varepsilon\dfrac{2iz^2}{\hbar}\right)^{3\{k/3\}}$ with $\varepsilon = \begin{cases} -1 \text{ for } c_k \\ +1 \text{ for } d_k \end{cases}$.

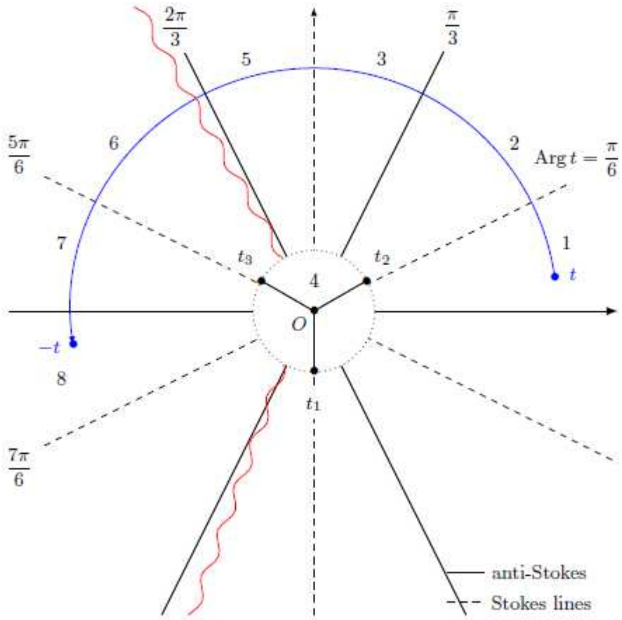

Fig. 4: Dezooming.

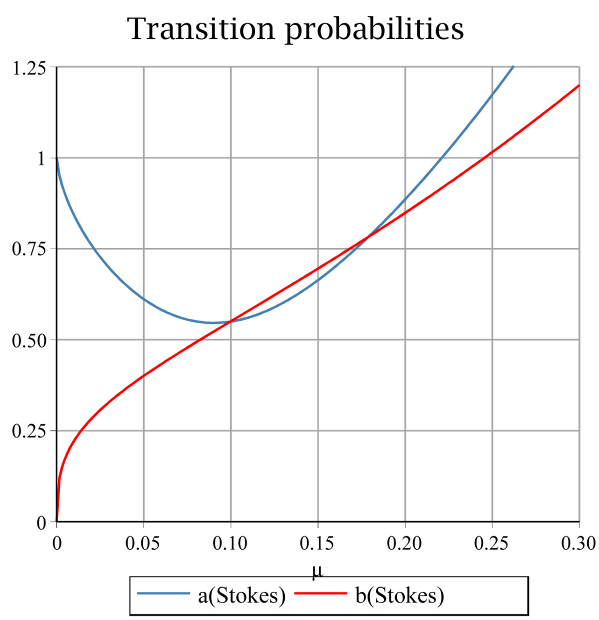

Fig. 5

Table 1: Strong coupling $\hbar \ll z$.

k	$(\nu=0)$ polynomial part of c_k	$(\nu=0)$ polynomial part of d_k
0	1	1
1	1	0
2	1	0
3	$\frac{8i}{\hbar^3}z^6+\frac{2i}{\hbar}$	$-\frac{2i}{\hbar}$
4	$\frac{8i}{\hbar^3}z^6+\frac{6i}{\hbar}$	$\frac{2i}{\hbar}$
5	$\frac{8i}{\hbar^3}z^6+\frac{12i}{\hbar}$	$\frac{2i}{\hbar}$
6	$-\frac{64}{\hbar^6}z^{12}-\frac{160}{\hbar^4}z^6-\frac{64}{\hbar^2}$	$\left(\frac{16}{\hbar^3}z^6-\frac{64}{\hbar}\right)\frac{1}{\hbar}$
7	$-\frac{64}{\hbar^6}z^{12}-\frac{240}{\hbar^4}z^6-\frac{284}{\hbar^2}$	$\frac{16}{\hbar^3}z^6+\frac{116}{\hbar}$
8	$-\frac{64}{\hbar^6}z^{12}-\frac{336}{\hbar^4}z^6-\frac{788}{\hbar^2}$	$\frac{16}{\hbar^3}z^6+\frac{140}{\hbar}$
9	$-\frac{512i}{\hbar^9}z^{18}-\frac{3584i}{\hbar^7}z^{12}-\frac{13920i}{\hbar^5}z^6-\frac{6272i}{\hbar^3}$	$\left(-\frac{128i}{\hbar^6}z^{12}-\frac{1344i}{\hbar^4}z^6+\frac{6272i}{\hbar^2}\right)\frac{1}{\hbar}$
10	$-\frac{512i}{\hbar^9}z^{18}-\frac{4608i}{\hbar^7}z^{12}-\frac{26720i}{\hbar^5}z^6-\frac{35456i}{\hbar^3}$	$-\frac{128i}{\hbar^6}z^{12}-\frac{1600i}{\hbar^4}z^6-\frac{15744i}{\hbar^2}$
11	$-\frac{512i}{\hbar^9}z^{18}-\frac{5760i}{\hbar^7}z^{12}-\frac{46592i}{\hbar^5}z^6-\frac{122216i}{\hbar^3}$	$-\frac{128i}{\hbar^6}z^{12}-\frac{1888i}{\hbar^4}z^6-\frac{21720i}{\hbar^2}$

Observe also that odd index coefficients of both series are pure imaginary, while even index coefficients are real.

Table 2: Weak coupling $0 < z \ll \hbar$.

k	$(\nu=2)$ polynomial part of c_k	$(\nu=2)$ polynomial part of d_k
0	1	1
1	$\frac{1}{3}$	$\frac{2}{3}$
2	$\frac{1}{12}$	$\frac{1}{4}$
3	$\frac{2i}{15\hbar^3}z^6+\frac{2i}{5\hbar}$	$-\frac{8i}{15\hbar^3}z^6-\frac{4i}{15\hbar}$
4	$\frac{i}{45\hbar^3}z^6+\frac{17i}{180\hbar}$	$-\frac{i}{9\hbar^3}z^6-\frac{5i}{36\hbar}$
5	$\frac{i}{315\hbar^3}z^6+\frac{23i}{1260\hbar}$	$-\frac{2i}{105\hbar^3}z^6-\frac{3i}{70\hbar}$
6	$-\frac{1}{315\hbar^6}z^{12}-\frac{1}{42\hbar^4}z^6-\frac{1}{10\hbar^2}$	$-\frac{1}{45\hbar^6}z^{12}-\frac{7}{90\hbar^4}z^6-\frac{1}{18\hbar^2}$
7	$-\frac{1}{2835\hbar^6}z^{12}-\frac{19}{5670\hbar^4}z^6-\frac{11}{567\hbar^2}$	$-\frac{8}{2835\hbar^6}z^{12}-\frac{8}{567\hbar^4}z^6-\frac{71}{2835\hbar^2}$
8	$-\frac{1}{28350\hbar^6}z^{12}-\frac{47}{113400\hbar^4}z^6-\frac{293}{90720\hbar^2}$	$-\frac{1}{3150\hbar^6}z^{12}-\frac{3}{1400\hbar^4}z^6+\frac{23}{3360\hbar^2}$
9	$-\frac{4i}{155925\hbar^9}z^{18}-\frac{19i}{51975\hbar^7}z^{12}-\frac{31i}{8316\hbar^5}z^6-\frac{i}{55\hbar^3}$	$\frac{8i}{31185\hbar^9}z^{18}+\frac{2i}{891\hbar^7}z^{12}+\frac{697i}{62370\hbar^5}z^6+\frac{8i}{891\hbar^3}$
10	$-\frac{i}{467775\hbar^9}z^{18}-\frac{17i}{467775\hbar^7}z^{12}-\frac{101i}{213840\hbar^5}z^6-\frac{11533i}{3742200\hbar^3}$	$\frac{i}{42525\hbar^9}z^{18}+\frac{11i}{42525\hbar^7}z^{12}+\frac{251i}{136080\hbar^5}z^6+\frac{1243i}{340200\hbar^3}$
11	$-\frac{i}{6081075\hbar^9}z^{18}-\frac{4i}{1216215\hbar^7}z^{12}-\frac{5179i}{97297200\hbar^5}z^6-\frac{45011i}{97297200\hbar^3}$	$\frac{4i}{2027025\hbar^9}z^{18}+\frac{2i}{75075\hbar^7}z^{12}+\frac{139i}{540540\hbar^5}z^6+\frac{449i}{491400\hbar^3}$

Before every polynomial, there is a prefactor $\left(\varepsilon\frac{2iz^2}{\hbar}\right)^{3\{k/3\}}$ with $\varepsilon = \begin{cases} -1 \text{ for } c_k \\ +1 \text{ for } d_k \end{cases}$. No factorial.

References

[1] Roscoe B. White. Asymptotic analysis of differential equations. Imperial College Press (2010).
[2] Chieh-Lei Wong. The Landau-Zener formula. SOP Transactions on Theoretical Physics : 2204-1709 (2016).
[3] Chieh-Lei Wong. Symmetrization of the Landau-Zener problem. SOP Transactions on Theoretical Physics : 2204-1710 (2016).
[4] Chieh-Lei Wong. The Landau-Zener problem in the light of the Stokes geometry (2016).
[5] Chieh-Lei Wong. A class of higher Landau-Zener -type problems (2016).