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Non-Instantaneous Impulsive Fractional Neutral Differential Equations with State-Dependent Delay

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Abstract: We prove the existence and uniqueness of fractional neutral differential equations with state-dependent delay subject to non-instantaneous impulsive conditions. The results are proved by using fixed point theorem for condensing map and justified by a suitable application.

Keywords: Fractional neutral differential equations, non-instantaneous impulse, state-dependent delay, resolvent operator, fixed point theorem.

1 Introduction

Fractional impulsive differential equations originate in several field of applied mathematics and science. Abstract fractional functional differential equations have recently proved to be valuable tools for the description of memory processes and hereditary properties of various materials and systems. For detailed study of fractional differential equations we refer the books [1,2,3], and the papers [4,5,6].

The study of impulsive differential equations have more attention in recent years due to its applications. Most of the research papers dealt with existence of solutions for equations with instantaneous impulses, see for more details [7,8,9,10,11]. In [12] E. Herenández et al. quoted that many authors used the concept of mild solutions inappropriately using the semigroup theory for fractional differential equations. To make the concept of mild solution more appropriate, he treated abstract differential equations with fractional derivatives based on the well developed theory of resolvent operators for integral equations.

Fractional functional differential equations with state-dependent delay viewed frequently in applications as model of equations and several authors have studied about these type of equations [13,14,15,16,17,18,19]. Also the theory of abstract neutral differential equations arise in many areas of applied mathematics and for this reason, the study of such equations have been treated in the literature recently, see[20,14,21,15,22,23,19].

These type of equations preferred to model the viscoelasticity and heat conduction equations. The neutral equations are depending on past and present scenarios along the derivative involved in delay as well as function itself. These features directs to study fractional neutral impulsive differential equations in many real life applications. The existence results and qualitative properties of fractional neutral delay differential equations was studied in [24,25,26,27,28,29].

Recently, Hernández and O'regan in [30] introduced a new class of non-instantaneous impulsive differential equations. In the model presented in [30], the impulsive action start abruptly at certain time and their process continue on a finite time interval. This non-instantaneous impulsive systems are more suitable to the study of dynamics of evolution processes in pharmacotheraphy. Existence results of solutions for non-instantaneous impulsive fractional/integer order differential equations have also been discussed in [31,32,33,34,35,36,28,37,38].

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In this contribution, we study the existence and uniqueness of fractional neutral differential equations with state-dependent delay subject to non-instantaneous impulsive conditions of the form

$$
\begin{aligned}
{}_{s_i}D_t^{\alpha}(y(t)-g(t,y_t)) &= A(y(t)+\int_0^t k(t-s)y(s)ds)+f(t,y_{\rho(t,y_t)}),\\
&\qquad t\in(s_i,t_{i+1}], i=0,1,...,N \qquad (1.1)\\
y(t) &= h_i(t,y_t), \quad t\in(t_i,s_i], i=1,..,N \qquad (1.2)\\
y_0 &= \phi, \qquad (1.3)
\end{aligned}
$$

where $\alpha\in(0,1)$, $\phi\in\mathscr{B}$, $A:\mathscr{D}(A)\subset\mathscr{Y}\to\mathscr{Y}$ is the infinitesimal generator of a C_0-semigroup of bounded linear operator $(\mathscr{S}(t))_{t\geq 0}$ defined on a Banach space $(\mathscr{Y},\|\cdot\|)$. The functions $y_s:(-\infty,0]\to\mathscr{Y},\ y_s(\theta)=y(s+\theta)$, belongs to some suitable abstract phase space $\mathscr{B}$ which is described axiomatically, $0=t_0=s_0<t_1\leq s_1\leq t_2<\cdots<t_N\leq s_N\leq t_{N+1}=a$ are prefixed numbers, $\rho:[0,a]\times\mathscr{B}\to(-\infty,a],\ g:[0,a]\times\mathscr{B}\to\mathscr{Y},\ f:[0,a]\times\mathscr{B}\to\mathscr{Y},\ h_i\in C((t_i,s_i]\times\mathscr{B};\mathscr{Y})$ and $k\in L^1_{loc}(\mathbb{R}^+)$ are appropriate functions.

We consider the Caputo fractional derivative and perturbed resolvent operator to make the concept of the mild solution more appropriate. We prove the existence and uniqueness of mild solutions of the above problem by utilizing fixed point theorem for condensing map and contraction mapping principle.

2 Preliminaries

Let the space $\mathscr{L}(\mathscr{Y},\mathscr{Z})$ to denote the bounded linear operators from the Banach space $\mathscr{Y}$ into $\mathscr{Z}$ endowed with the norm $\|\cdot\|_{\mathscr{L}(\mathscr{Y},\mathscr{Z})}$ and we write simply $\mathscr{L}(\mathscr{Y})$ when $\mathscr{Y}=\mathscr{Z}$. Let $\|y\|_{\mathscr{D}}=\|y\|+\|Ay\|$ where notation $\mathscr{D}$ represents the domain of the operator A. Consider $B_p(y,\mathscr{Y})$ to denote the closed ball with center at y and radius p in $\mathscr{Y}$. $C([0,a];\mathscr{Y})$ describes the space of all the continuous functions from $[0,a]$ into $\mathscr{Y}$ with the sup-norm $\|\cdot\|_{C([0,a];\mathscr{Y})}$, and $C^{\gamma}([0,a];\mathscr{Y}),\gamma\in(0,a)$ represents the space formed by all the $\mathscr{Y}$-valued γ-Hölder continuous functions from $[0,a]$ into $\mathscr{Y}$ with the norm $\|y\|_{C^{\gamma}([0,a];\mathscr{Y})}=\|y\|_{C([0,a];\mathscr{Y})}+[|y|]_{C^{\gamma}([0,a];\mathscr{Y})}$ where $[|y|]_{C^{\gamma}([0,a];\mathscr{Y})}=\sup_{\substack{t,s\in[0,a]\\ t\neq s}}\frac{\|y(t)-y(s)\|_{\mathscr{Y}}}{(t-s)^{\gamma}}$.

We introduce the space $\mathscr{PC}(\mathscr{Y})$ of all the functions $y:[0,a]\to\mathscr{Y}$ such that $y(\cdot)$ is continuous at $t\neq t_i, y(t_i^-)=y(t_i)$ and $y(t_i^+)$ exists for every $i=1,\cdots,N$, which is a Banach space with respect to the norm $\|y\|_{\mathscr{PC}(\mathscr{Y})}=\sup_{s\in[0,a]}\|y(s)\|$. For a function $y\in\mathscr{PC}(\mathscr{Y})$ and $i\in\{0,1,\cdots,N\}$, we introduce the function $\tilde{y}_i\in C([t_i,t_{i+1}];\mathscr{Y})$ by

$$
\tilde{y}_i(t)=\begin{cases} y(t), & \text{for } t\in(t_i,t_{i+1}],\\ y(t_i^+), & \text{for } t=t_i.\end{cases} \qquad (2.1)
$$

Moreover, for $E\subseteq\mathscr{PC}(\mathscr{Y})$ and $i\in\{0,1,\cdots,N\}$, we consider the notion $\tilde{E}_i$ for the set $\tilde{E}_i=\{\tilde{y}_i: y\in E\}$. We note the Ascoli-Arzela type criteria as below.

Lemma 1.*[16] A set $E\subseteq\mathscr{PC}(\mathscr{Y})$ is relatively compact in $\mathscr{PC}(\mathscr{Y})$ if and only if each set $\tilde{E}_i$ is relatively compact in $C([t_i,t_{i+1}];\mathscr{Y})$.*

We consider the phase space $(\mathscr{B},\|\cdot\|_{\mathscr{B}})$, is a linear space of function y_t mapping from $(-\infty,0]$ into $\mathscr{Y}$ with respect to the seminorm $\|\cdot\|_{\mathscr{B}}$, which is previously addressed in Hino et al., [39] to examine the infinite delay problem. We assume the space $\mathscr{B}$ meets the axioms given below:

(A)If $y:(-\infty,\kappa+b]\to\mathscr{Y},\ \kappa\in\mathbb{R},\ b>0$, is such that $y|_{[\kappa,\kappa+b]}\in\mathscr{PC}([\kappa,\kappa+b];\mathscr{Y})$ and $y_\kappa\in\mathscr{B}$, then for every $t\in[\kappa,\kappa+b]$ the subsequent conditions hold:
 (i)$y_t\in\mathscr{B}$,
 (ii)$\|y(t)\|\leq H\|y_t\|_{\mathscr{B}}$,
 (iii)$\|y_t\|_{\mathscr{B}}\leq K(t-\kappa)\{\sup\|y(s)\|:\kappa\leq s\leq t\}+M(t-s)\|x_\kappa\|_{\mathscr{B}}$, where $M,K:[0,\infty)\to[1,\infty)$, M is locally bounded, K is continuous; $H>0$ is a constant and K,M,H are independent of $y(\cdot)$.
(B)$\mathscr{B}$ is complete.

*Remark.*Since the domain of $\phi(\cdot)$ is $(-\infty,0]$, we observe that for every $t<0$, ϕ_t is well defined.

Next we find out the mild solution of the impulsive initial value problem (1.1)-(1.3). For, first we establish the equivalent integral equation of (1.1)-(1.3).

The Riemann-Liouville fractional integral of order $\alpha>0$ of a function u in the space $L^p(0,1)$, $p\in[1,\infty)$, is the integral

$$
{}_aI_t^{\alpha}u(t)=\frac{1}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}u(s)ds.
$$

The Caputo type fractional derivative of order $\alpha > 0$, $\alpha \in (n-1,n)$ is defined as

$$ {}_aD_t^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1}u^n(s)ds, $$

where the function $u(t)$ have absolutely continuous derivatives of order up to $n-1$.

Also we get the following results by the relation between Caputo fractional derivative and Riemann-Liouville fractional integral. If the fractional order derivative $D^{\alpha}u,\ \alpha \in (0,1)$, is integrable, then

$$ {}_aI_t^{\alpha}\,{}_aD_t^{\alpha}u(t) = u(t) - u(a), \tag{2.2} $$

and if u is integrable then

$$ {}_aD_t^{\alpha}\,{}_aI_t^{\alpha}u(t) = u(t), \tag{2.3} $$

holds on [1,2].

Suppose that $y \in \mathscr{PC}(\mathscr{Y})$ is a solution of the equations (1.1)-(1.3). It follows from equations (2.2) and (2.3), we have the corresponding fractional integral equations of (1.1)-(1.3) as

$$ y(t) = \phi(t),\ t \in (-\infty,0],\ \ y(t) = h(t,y_t),\ t \in (t_i,s_i],\ i = 1,\cdots,N, \tag{2.4} $$

and

$$ \begin{aligned} y(t) = {} & h_i(s_i,y_{s_i}) - g(s_i,y_{s_i}) + g(t,y_t) + \frac{1}{\Gamma(\alpha)}\int_{s_i}^t (t-s)^{\alpha-1}f(s,y_{\rho(s,y_s)})ds \\ & + \frac{1}{\Gamma(\alpha)}\int_{s_i}^t (t-s)^{\alpha-1}[A(y(s) + \int_0^s k(s-\tau)y(\tau)d\tau)]ds, \end{aligned} \tag{2.5} $$

for every $t \in (s_i,t_{i+1}]$ and $i = 0,\cdots,N$.

Now we make the concept of mild solution for (1.1)-(1.3).

Note that the following perturbed convolution equation

$$ y(t) = (a_{\alpha} + a_{\alpha} * k) * Ay(t) + f(t),\ t \in [0,a], \tag{2.6} $$

has a corresponding resolvent operator $(\mathscr{T}(t))_{t\geq 0}$ on $\mathscr{Y}$ and $f \in C([0,a];\mathscr{Y})$. Here $*$ denotes the convolution operator, $a_{\alpha} * y(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds$, where $a_{\alpha} = \frac{1}{\Gamma(\alpha)}(t)^{\alpha-1}$, see [12] and [40, Section 1.4].

Definition 1.[40, *Definition 1.3]* *A family $(\mathscr{T}(t))_{t\geq 0} \subset \mathscr{L}(\mathscr{Y})$ of bounded linear operators in $\mathscr{Y}$ is called a resolvent for (2.6)[or solution operator for (2.6)] if the subsequent conditions are satisfied.*

(S$_1$)*$\mathscr{T}(0) = I$ and $\mathscr{T}(t)$ is strongly continuous on $\mathbb{R}_+$;*
(S$_2$)*$\mathscr{T}(t)$ commutes with A, and $A\mathscr{T}(t)y = \mathscr{T}(t)Ay$ for every $y \in \mathscr{D}$ and $t \geq 0$;*
(S$_3$)*The resolvent equation holds*

$$ \mathscr{T}(t)y = y + a_{\alpha}A * \mathscr{T}(t)y + k * a_{\alpha}A * \mathscr{T}(t)y,\ \text{for every } y \in \mathscr{D},\ t \geq 0. $$

Definition 2.[40, *Definition 1.4]* *A resolvent operator $\mathscr{T}(t)$ for equation (2.6) is said to be differentiable, if $\mathscr{T}(\cdot)y \in \mathscr{W}^{1,1}([0,\infty);\mathscr{Y})$ for every $y \in \mathscr{D}$ and there is $\phi_A \in L^1_{loc}([0,\infty))$ with $\|\mathscr{T}'(t)y\| \leq \phi_A(t)\|y\|_{\mathscr{D}}$, a.e. on $[0,\infty)$, for every $u \in \mathscr{D}$.*

Definition 3.[40, *Definition 1.1]* *A function $y \in C([0,a];\mathscr{Y})$ is called a mild solution of (2.6) on $[0,a]$ if $(a_{\alpha} + a_{\alpha} * k) * y \in \mathscr{D}(A)$ for all $t \in [0,a]$ and*

$$ y(t) = A(a_{\alpha} + a_{\alpha} * k) * y(t) + f(t),\ t \in [0,a]. $$

The next lemma follows from [12, Lemma 1.1].

Lemma 2.*Suppose (2.6) admits a differentiable resolvent $\mathscr{T}(t)$.*

(i)If $y(\cdot)$ is a mild solution of (2.6) on $[0,a]$, then the function $t \to \int_0^t \mathscr{T}(t-s)f(s)ds$ is continuously differentiable on $[0,a]$, and

$$ y(t) = \frac{d}{dt}\int_0^t \mathscr{T}(t-s)f(s)ds,\ t \in [0,a], $$

(ii)If $f \in C([0,a];\mathscr{D})$ then the function $y : [0,a] \to \mathscr{Y}$ defined by

$$ y(t) = \int_0^t \mathscr{T}'(t-s)f(s)ds + f(t),\ t \in [0,a], $$

is a mild solution of (2.6) on $[0,a]$.

We consider the following fixed point theorem for the existence results.

Theorem 1.[41, Theorem 4.3.2] *Suppose that $\mathscr{Y}$ is a Banach space, D is a closed bounded convex subset of $\mathscr{Y}$, A is a continuous function from D into D with the property that there is a number ϑ such that $0 \leq \vartheta < 1$ and $\alpha[A(\Delta)] \leq \vartheta\alpha[\Delta]$ for all $\Delta \subset D$, where*

$\alpha[\Delta] = \inf\{\ \gamma > 0 :\ \Delta$ *can be covered by a finite number of sets having diameter to larger than* $\gamma\}$.

Then the set $F = \{z \in D : Az = z\}$ is nonempty and compact.

3 Existence and Uniqueness Results

In this section, we prove the existence and uniqueness of mild solution of the fractional differential equation (1.1)-(1.3).

Definition 4. *A function $y : (-\infty, a] \to \mathscr{Y}$ is said to be a mild solution of the equation (1.1)-(1.3) on $[0,a]$, if $y_0 = \phi,\ y_{\rho(s,y_s)} \in \mathscr{B}$ for all $s \in [0,a]$ and $\int_0^t (t-s)^{1-\alpha}(y(s) + \int_0^s k(s-\tau)y(\tau)d\tau)ds \in \mathscr{D}$ for all $t \in (0,a]$ and*

$$y(t) = \begin{cases} \phi(0) - g(0,\phi) + g(t,y_t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, y_{\rho(s,y_s)})ds \\ + \frac{A}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(y(s) + \int_0^s k(s-\tau)y(\tau)d\tau)ds,\ t \in (0,t_1], \\ h_i(s_i, y_{s_i}) - g(s_i, y_{s_i}) + g(t,y_t) + \frac{1}{\Gamma(\alpha)}\int_{s_i}^t (t-s)^{\alpha-1} f(s, y_{\rho(s,y_s)})ds \\ + \frac{A}{\Gamma(\alpha)}\int_{s_i}^t (t-s)^{\alpha-1}(y(s) + \int_0^s k(s-\tau)y(\tau)d\tau)ds,\ t \in (s_i, t_{i+1}], \\ h_i(t,y_t),\ t \in (t_i, s_i], \\ \phi,\ t \in (-\infty, 0] \end{cases} \tag{3.1}$$

for every $i = 1, \cdots, N$.

To prove the existence results we assume that ϕ is in $\mathscr{B}$ and that the function $\rho : [0,a] \times \mathscr{B} \to (-\infty, a]$ is continuous. Now we assume the necessary hypotheses.

(H$_1$) Consider the set $\mathscr{Q}(\rho^-) = \{\rho(s,\psi) : (s,\psi) \in [0,a] \times \mathscr{B},\ \rho(s,\psi) \leq 0\}$. The function $t \to \phi_t$ is well defined from $\mathscr{Q}(\rho^-)$ into $\mathscr{B}$ and there exists a bounded and continuous function $J^\phi : \mathscr{Q}(\rho^-) \to \mathbb{R}$ such that $\|\phi_t\|_{\mathscr{B}} \leq J^\phi(t)\|\phi\|_{\mathscr{B}},\ \forall t \in \mathscr{Q}(\rho^-)$.

(H$_2$) $f : [0,a] \times \mathscr{B} \to \mathscr{D}$ is continuous such that
  (i) Let $y : (-\infty, a] \to \mathscr{Y}$ be such that $y_0 = \phi$ and $y|_{[0,a]} \in \mathscr{PC}(\mathscr{Y})$. The map $t \to f(t, y_{\rho(t,y_t)})$ is measurable on $[0,a]$ and the map $t \to f(t,y_t)$ is continuous on $\mathscr{Q}(\rho^-) \cup [0,a]$ for all $s \in [0,a]$.
  (ii) For every $t \in [0,a]$, the function $f(t,\cdot) : \mathscr{B} \to \mathscr{D}$ is continuous.
  (iii) Let $m_f \in C([0,a];\mathbb{R}^+)$ and $W : [0,\infty) \to (0,\infty)$ is a non-decreasing function such that $\|f(t,\psi)\| \leq m(t)W(\|\psi\|_{\mathscr{B}})$, $(t,\psi) \in [0,a] \times \mathscr{B}$
  (iv) There exist a function $L_f \in C([0,a];\mathbb{R}^+)$ such that $\|f(t,y_t) - f(t,z_t)\|_{\mathscr{D}} \leq L_f(t)\|y_t - z_t\|_{\mathscr{B}}$.

(H$_3$) The function $g \in C([0,a] \times \mathscr{B}; \mathscr{D})$ and there exists a constant K_g and $L_g \in C([0,a];\mathbb{R}^+)$ such that $\|g(t,y_t)\|_{\mathscr{D}} \leq K_g(\|y_t\|_{\mathscr{B}} + 1)$, and $\|g(t,y_t) - g(t,z_t)\|_{\mathscr{D}} \leq L_g(t)\|y_t - z_t\|_{\mathscr{B}}$.

(H$_4$) The function $h_i \in C((t_i,s_i] \times \mathscr{B} \to \mathscr{D})$ and there exists a constant K_{h_i} and $L_{h_i} \in C((t_i,s_i];\mathbb{R}^+)$ such that $\|h_i(t,y_t)\|_{\mathscr{D}} \leq K_{h_i}(\|y_t\|_{\mathscr{B}} + 1)$, and $\|h_i(t,y_t) - h_i(t,z_t)\|_{\mathscr{D}} \leq L_{h_i}(t)\|y_t - z_t\|_{\mathscr{B}}$ for each $i = 1, \cdots, N$.

Lemma 3.[16] *Let $y : (-\infty, a] \to \mathscr{Y}$ be such that $y_0 = \phi$ and $y|_{[0,a]} \in \mathscr{PC}(\mathscr{Y})$. Then*
$\|y_s\|_{\mathscr{B}} \leq (M_a + J_0^\phi)\|\phi\|_{\mathscr{B}} + K_a \sup\{\|x(\theta)\|;\ \theta \in [0, max\{0,s\}]\},\ s \in \mathscr{Q}(\rho^-) \cup [0,a]$,
where $J_0^\phi = \sup_{t \in \mathscr{Q}(\rho^-)} J^\phi(t),\ K_a = \sup_{t \in [0,a]} K(t)$ *and* $M_a = \sup_{t \in [0,a]} M(t)$.

Theorem 1. *Assume that the hypotheses $(H_1) - (H_4)$ are satisfied and $\phi(0) \in \mathscr{D}$. Then the problem (1.1)-(1.3) has a unique mild solution if $\Omega < 1$, where*

$$\begin{aligned} \Omega = K_a \max_{i=\{1,\cdots,N\}} \{ & (|L_{h_i}|_{C((t_i,s_i];\mathbb{R}^+)} + 2|L_g|_{C((s_i,t_{i+1}];\mathbb{R}^+)} \\ & + \frac{t_{i+1}^\alpha}{\alpha\Gamma(\alpha)}|L_f|_{C((s_i,t_{i+1}];\mathbb{R}^+)}(1 + \|\phi_A\|_{L^1((s_i,t_{i+1}];\mathbb{R}^+)}), \\ & (|L_g|_{C([0,t_1];\mathbb{R}^+)} + \frac{t_1^\alpha}{\alpha\Gamma(\alpha)}|L_f|_{C([0,t_1];\mathbb{R}^+)})(1 + \|\phi_A\|_{L^1([0,t_1];\mathbb{R}^+)})\}. \end{aligned}$$

*Proof.*Set $Y = \{y \in \mathscr{PC}(\mathscr{Y}) : y(0) = \phi(0)\}$ endowed with uniform convergence topology. On the space Y and by considering Lemma 2(ii), we define the fixed point operator $\Gamma : Y \to Y$ by

$$\Gamma y(t) = \begin{cases} h_i(t,\overline{y}_t), & t \in (t_i,s_i], \\ \phi(0) - g(0,\phi) + g(t,\overline{y}_t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,\overline{y}_{\rho(s,\overline{y}_s)})ds & \\ +\int_0^t \mathscr{T}'(t-s)[\phi(0) - g(0,\phi) + g(s,\overline{y}_s) & \\ +\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1} f(\tau,\overline{y}_{\rho(\tau,\overline{y}_\tau)})d\tau]ds, & t \in [0,t_1], \\ h_i(s_i,\overline{y}_{s_i}) - g(s_i,\overline{y}_{s_i}) + g(t,\overline{y}_t) + \frac{1}{\Gamma(\alpha)}\int_{s_i}^t (t-s)^{\alpha-1} f(s,\overline{y}_{\rho(s,\overline{y}_s)})ds & \\ +\int_{s_i}^t \mathscr{T}'(t-s)[h_i(s_i,\overline{y}_{s_i}) - g(s_i,\overline{y}_{s_i}) + g(s,\overline{y}_s) & \\ +\frac{1}{\Gamma(\alpha)}\int_{s_i}^s (s-\tau)^{\alpha-1} f(\tau,\overline{y}_{\rho(\tau,\overline{y}_\tau)})d\tau]ds, & t \in (s_i,t_{i+1}], \end{cases}$$

where $\overline{y} : (-\infty,a] \to \mathscr{Y}$ is such that $\overline{y} = y$ and $\overline{y}_0 = \phi$ on $[0,a]$. Let $y \in Y$. From the assumptions on g, f, and h_i, we have

$$\begin{aligned} &\int_0^t \|\mathscr{T}'(t-s)[\phi(0) - g(0,\phi) + g(s,\overline{y}_s) + \frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1} f(\tau,\overline{y}_{\rho(\tau,\overline{y}_\tau)})d\tau]\|ds \\ &\le \int_0^t \|\mathscr{T}'(t-s)(\phi(0) - g(0,\phi) + g(s,\overline{y}_s))\|ds \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t \|\mathscr{T}'(t-s)\int_0^s (s-\tau)^{\alpha-1} f(\tau,\overline{y}_{\rho(\tau,\overline{y}_\tau)})d\tau\|ds \\ &\le \int_0^t \phi_A(t-s)\|\phi(0) - g(0,\phi) + g(s,\overline{y}_s)\|_{\mathscr{D}}ds \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t \phi_A(t-s)\int_0^s (s-\tau)^{\alpha-1}\|f(\tau,\overline{y}_{\rho(\tau,\overline{y}_\tau)})\|_{\mathscr{D}}d\tau ds \\ &\le \|\phi(0) - g(0,\phi) + g(s,\overline{y}_s)\|_{\mathscr{D}}\|\phi_A\|_{L^1([0,t_1];\mathbb{R}^+)} + \frac{\|f(\tau,\overline{y}_{\rho(\tau,\overline{y}_\tau)})\|_{\mathscr{D}}}{\Gamma(\alpha)}\int_0^t \phi_A(t-s)\frac{s^\alpha}{\alpha}ds \\ &\le (\|\phi(0) - g(0,\phi) + g(s,\overline{y}_s)\|_{\mathscr{D}} + \frac{t_1^\alpha}{\alpha\Gamma(\alpha)}\|f(\tau,\overline{y}_{\rho(\tau,\overline{y}_\tau)})\|_{\mathscr{D}})\|\phi_A\|_{L^1([0,t_1];\mathbb{R}^+)}. \end{aligned}$$

Then from the above inequality results the function
$s \to \mathscr{T}'(t-s)(\phi(0) - g(0,\phi) + g(t,\overline{y}_t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,\overline{y}_{\rho(s,\overline{y}_s)})ds)$
is integrable on $[0,t_1]$ for every $t \in [0,t_1]$. Likewise, we see

$$\begin{aligned} &\int_{s_i}^t \|\mathscr{T}'(t-s)[h_i(s_i,\overline{y}_{s_i}) - g(s_i,\overline{y}_{s_i}) + g(s,\overline{y}_s) + \int_{s_i}^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau,\overline{y}_{\rho(\tau,\overline{y}_\tau)})d\tau]\|ds \\ &\le \int_{s_i}^t \phi_A(t-s)\|h_i(s_i,\overline{y}_{s_i}) - g(s_i,\overline{y}_{s_i}) + g(s,\overline{y}_s)\|_{\mathscr{D}}ds \\ &\quad + \int_{s_i}^t \phi_A(t-s)\frac{1}{\Gamma(\alpha)}\int_{s_i}^s (s-\tau)^{\alpha-1}\|f(\tau,\overline{y}_{\rho(\tau,\overline{y}_\tau)})\|_{\mathscr{D}}d\tau ds \\ &\le (\|h_i(s_i,\overline{y}_{s_i}) - g(s_i,\overline{y}_{s_i}) + g(s,\overline{y}_s)\|_{\mathscr{D}} + \frac{t_{i+1}^\alpha}{\alpha\Gamma(\alpha)}\|f(\tau,\overline{y}_{\rho(\tau,\overline{y}_\tau)})\|_{\mathscr{D}})\|\phi_A\|_{L^1((s_i,t_{i+1}];\mathbb{R}^+)}. \end{aligned}$$

From this result we get that
$s \to \mathscr{T}'(t-s)(h_i(s_i,\overline{y}_{s_i}) - g(s_i,\overline{y}_{s_i}) + g(t,\overline{y}_t) + \frac{1}{\Gamma(\alpha)}\int_{s_i}^t (t-s)^{\alpha-1} f(s,\overline{y}_{\rho(s,\overline{y}_s)}))$
is also integrable on $[s_i,t]$ for all $t \in (s_i,t_{i+1}]$. This implies that $\Gamma y \in PC$ and Γ is well defined.

For $y,z \in Y$ and $t \in (s_i,t_{i+1}],\ i=\{1,\cdots,N\}$, we obtain
$\|\Gamma y(t)-\Gamma z(t)\|_{C((s_i,t_{i+1}];\mathscr{Y})}$

$$\begin{aligned}
&\le \|h_i(s_i,\overline{y}_{s_i})-h_i(s_i,\overline{z}_{s_i})\|_{\mathscr{D}}+\|g(s_i,\overline{y}_{s_i})-g(s_i,\overline{z}_{s_i})\|_{\mathscr{D}}\\
&\quad +\|g(t,\overline{y}_t)-g(t,\overline{z}_t)\|_{\mathscr{D}}+\frac{1}{\Gamma(\alpha)}\int_{s_i}^{t}\frac{\|f(s,\overline{y}_{\rho(s,\overline{y}_s)})-f(s,\overline{z}_{\rho(s,\overline{z}_s)})\|_{\mathscr{D}}}{(t-s)^{1-\alpha}}ds\\
&\quad +\int_{s_i}^{t}\phi_A(t-s)\left(\|h_i(s_i,\overline{y}_{s_i})-h_i(s_i,\overline{z}_{s_i})\|_{\mathscr{D}}\right.\\
&\quad \left.+\|g(s_i,\overline{y}_{s_i})-g(s_i,\overline{z}_{s_i})\|_{\mathscr{D}}+\|g(s,\overline{y}_s)-g(s,\overline{z}_s)\|_{\mathscr{D}}\right)ds\\
&\quad +\frac{1}{\Gamma(\alpha)}\int_{s_i}^{t}\phi_A(t-s)\int_{s_i}^{s}\frac{\|f(\tau,\overline{y}_{\rho(\tau,\overline{y}_\tau)})-f(\tau,\overline{z}_{\rho(\tau,\overline{z}_\tau)})\|_{\mathscr{D}}}{(s-\tau)^{1-\alpha}}d\tau ds\\
&\le \left(|L_{h_i}|_{C((t_i,s_i];\mathbb{R}^+)}+|L_g|_{C((s_i,t_{i+1}];\mathbb{R}^+)}\right)\|\overline{y}_{s_i}-\overline{z}_{s_i}\|_{\mathscr{B}}+|L_g|_{C((s_i,t_{i+1}];\mathbb{R}^+)}\|\overline{y}_t-\overline{z}_t\|_{\mathscr{B}}\\
&\quad +\frac{1}{\Gamma(\alpha)}\int_{s_i}^{t}\frac{L_f(s)}{(t-s)^{1-\alpha}}ds\|\overline{y}_{\rho(s,\overline{y}_s)}-\overline{z}_{\rho(s,\overline{z}_s)}\|_{\mathscr{B}}+\left(|L_{h_i}|_{C((t_i,s_i];\mathbb{R}^+)}\|\overline{y}_{s_i}-\overline{z}_{s_i}\|_{\mathscr{B}}\right.\\
&\quad \left.+|L_g|_{C((s_i,t_{i+1}];\mathbb{R}^+)}\|\overline{y}_{s_i}-\overline{z}_{s_i}\|_{\mathscr{B}}+|L_g|_{C((s_i,t_{i+1}];\mathbb{R}^+)}\|\overline{y}_t-\overline{z}_t\|_{\mathscr{B}}\right)\|\phi_A\|_{L^1((s_i,t_{i+1}];\mathbb{R}^+)}\\
&\quad +\frac{1}{\Gamma(\alpha)}\int_{s_i}^{t}\phi_A(t-s)\int_{s_i}^{s}\frac{L_f(\tau)}{(s-\tau)^{1-\alpha}}\|\overline{y}_{\rho(\tau,\overline{y}_\tau)}-\overline{z}_{\rho(\tau,\overline{z}_\tau)}\|_{\mathscr{B}}d\tau ds\\
&\le K_a(|L_{h_i}|_{C((t_i,s_i];\mathbb{R}^+)}+2|L_g|_{C((s_i,t_{i+1}];\mathbb{R}^+)}+\frac{t_{i+1}^{\alpha}}{\alpha\Gamma(\alpha)}|L_f|_{C((s_i,t_{i+1}];\mathbb{R}^+)})\\
&\quad \times(1+\|\phi_A\|_{L^1((s_i,t_{i+1}];\mathbb{R}^+)})\|y-z\|_Y.
\end{aligned}$$

Proceeding as above, we obtain that

$$\begin{aligned}
\|\Gamma y(t)-\Gamma z(t)\|_{C([0,t_1];\mathscr{Y})} &\le K_a(|L_g|_{C([0,t_1];\mathbb{R}^+)}+\frac{t_1^{\alpha}}{\alpha\Gamma(\alpha)}|L_f|_{C([0,t_1];\mathbb{R}^+)})\\
&\quad \times(1+\|\phi_A\|_{L^1([0,t_1];\mathbb{R}^+)})\|y-z\|_Y,
\end{aligned}$$

and

$$\|\Gamma y(t)-\Gamma z(t)\|_{C((t_i,s_i];\mathscr{Y})}\le K_a|L_{h_i}|_{C((t_i,s_i];\mathbb{R}^+)}\|y-z\|_Y.$$

From the above results we get

$$\|\Gamma y(t)-\Gamma z(t)\|_Y\le \Omega\|y-z\|_Y.$$

Since $\Omega<1$, this shows that $\Gamma(\cdot)$ is a contraction map and from the contraction mapping principle there exists a unique mild solution for the equations (1.1)-(1.3).

In the accompanying theorem we prove the existence results of a mild solution via fixed point technique for condensing operators. For that we require the following lemma.

Lemma 4.[42] *If $f\in C([0,a];\mathscr{D})$ and $F:[0,a]\to\mathscr{Y}$ is the function defined by $F(t)=\frac{1}{\Gamma(\alpha)}\int_0^t\frac{f(s)}{(t-s)^{1-\alpha}}ds$, then $F\in C^{\alpha}([0,a];\mathscr{D})$ and*

$$[|F|]_{C^{\alpha}([0,a];\mathscr{D})}\le\frac{2}{\alpha\Gamma(\alpha)}\parallel f\parallel_{C([0,a];\mathscr{D})}.$$

Theorem 2.*Assume that $(H_1)-(H_4)$ are satisfied, $\mathscr{T}(t)$ is compact for all $t>0$, $\phi(0)\in\mathscr{D}$, and there exist a constant K_ϕ such that $\|\phi(0)\|_{\mathscr{D}}\le K_\phi\|\phi\|_{\mathscr{B}}$. If*
$[(K_{h_i}+2K_g)K_a+\frac{t_{i+1}^{\alpha}\|m_f\|}{\alpha\Gamma(\alpha)}\lim_{r\to\infty}\sup\frac{1}{r}W(K_ar)](1+\|\phi_A\|_{L^1((s_i,t_{i+1}];\mathbb{R}^+)})<1,$
$[K_gK_a+\frac{t_1^{\alpha}}{\alpha\Gamma(\alpha)}\|m_f\|\lim_{r\to\infty}\sup\frac{1}{r}W(K_ar)](1+\|\phi_A\|_{L^1([0,t_1];\mathbb{R}^+)})<1,$
then there exists a mild solution of (1.1) - (1.3).

Proof. Choose $r > 0$ such that

$$((K_{h_i} + 2K_g)(K_a s + M_a\|\phi\|_{\mathscr{B}}) + \frac{t_{i+1}^{\alpha}}{\alpha\Gamma(\alpha)}\|m_f\|_{C((s_i,t_{i+1}];\mathbb{R}^+)}W(K_a s$$
$$+(M_a + J_0^{\phi})\|\phi\|_{\mathscr{B}}))(1 + \|\phi_A\|_{L^1((s_i,t_{i+1}];\mathbb{R}^+)}) \leq s,$$

and

$$((K_\phi + K_g)\|\phi\|_{\mathscr{B}} + K_g(K_a s + M_a\|\phi\|_{\mathbb{B}}) + \frac{t_1^{\alpha}}{\alpha\Gamma(\alpha)}\|m_f\|_{C([0,t_1];\mathbb{R}^+)}W((M_a$$
$$+J_0^{\phi})\|\phi\|_{\mathscr{B}} + K_a s))(1 + \|\phi_A\|_{L^1([0,t_1];\mathbb{R}^+)}) \leq s,$$

for all $s \geq r$.

On the space $Y = \{y \in \mathscr{PC}([0,a];\mathscr{Y}) : y(0) = \phi(0)\}$ endowed with a uniform convergence norm. Let $\Gamma : B_r(0,Y) \to B_r(0,Y)$ be the operator considered as in the proof of Theorem 1. Arguing as in the proof of Theorem 1, we can show that Γ is well defined. Next, we prove that Γ is a condensing map from $B_r(0,Y)$ into $B_r(0,Y)$.

First, we show that there exists $r > 0$ such that $\Gamma(B_r(0,Y)) \subset B_r(0,Y)$.

Let $y \in B_r(0,Y)$. For $i = 1,\cdots,N,\ t \in (s_i,t_{i+1}]$ we get

$$\begin{aligned}
\|\Gamma y(t)\| &\leq \|h_i(s_i,\bar{y}_{s_i})\|_{\mathscr{D}} + \|g(s_i,\bar{y}_{s_i})\|_{\mathscr{D}} + \|g(t,\bar{y}_t)\|_{\mathscr{D}} + \frac{1}{\Gamma(\alpha)}\int_{s_i}^{t}\frac{\|f(s,\bar{y}_{\rho(s,\bar{y}_s)})\|_{\mathscr{D}}}{(t-s)^{1-\alpha}}ds \\
&\quad + \int_{s_i}^{t}\phi_A(t-s)\left(\|h_i(s_i,\bar{y}_{s_i})\|_{\mathscr{D}} + \|g(s_i,\bar{y}_{s_i})\|_{\mathscr{D}} + \|g(s,\bar{y}_s)\|_{\mathscr{D}}\right)ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_{s_i}^{t}\phi_A(t-s)\int_{s_i}^{s}\frac{\|f(\tau,\bar{y}_{\rho(\tau,\bar{y}_\tau)})\|_{\mathscr{D}}}{(s-\tau)^{1-\alpha}}d\tau ds \\
&\leq ((K_{h_i} + K_g)(\|\bar{y}_{s_i}\|_{\mathscr{B}} + 1) + K_g(\|\bar{y}_t\|_{\mathscr{B}} + 1))(1 + \|\phi_A\|_{L^1((s_i,t_{i+1}];\mathbb{R}^+)}) \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_{s_i}^{t}\frac{m_f(s)W(\|\bar{y}_{\rho(s,\bar{y}_s)}\|_{\mathscr{B}})}{(t-s)^{1-\alpha}}ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_{s_i}^{t}\phi_A(t-s)\int_{s_i}^{s}\frac{m_f(\tau)W(\|\bar{y}_{\rho(\tau,\bar{y}_\tau)}\|_{\mathscr{B}})}{(s-\tau)^{1-\alpha}}d\tau ds \\
&\leq [(K_{h_i} + 2K_g)(K_a r + M_a\|\phi\|_{\mathscr{B}}) + \frac{t_{i+1}^{\alpha}}{\alpha\Gamma(\alpha)}\|m_f\|_{C((s_i,t_{i+1}];\mathbb{R}^+)} \\
&\quad \times W(K_a r + (M_a + J_0^{\phi})\|\phi\|_{\mathscr{B}})](1 + \|\phi_A\|_{L^1((s_i,t_{i+1}];\mathbb{R}^+)}),
\end{aligned}$$

which implies that $\|\Gamma y(t)\|_{C((s_i,t_{i+1}])} \leq r$ for all $i = 1,\cdots,N$. In the similar way we can proceed as for $t \in [0,t_1]$,

$$\begin{aligned}
\|\Gamma y(t)\| &\leq \|\phi(0)\|_{\mathscr{D}} + \|g(0,\phi)\|_{\mathscr{D}} + \|g(t,\bar{y}_t)\|_{\mathscr{D}} + \frac{1}{\Gamma(\alpha)}\int_{0}^{t}\frac{\|f(s,\bar{y}_{\rho(s,\bar{y}_s)})\|_{\mathscr{D}}}{(t-s)^{1-\alpha}}ds \\
&\quad + \int_{0}^{t}\phi_A(t-s)(\|\phi(0)\|_{\mathscr{D}} + \|g(0,\phi)\|_{\mathscr{D}} + \|g(s,\bar{y}_s)\|_{\mathscr{D}})ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_{0}^{t}\phi_A(t-s)\int_{0}^{s}\frac{\|f(\tau,\bar{y}_{\rho(\tau,\bar{y}_\tau)})\|_{\mathscr{D}}}{(s-\tau)^{1-\alpha}}d\tau ds \\
&\leq (K_\phi + K_g)\|\phi\|_{\mathscr{B}} + K_g\|\bar{y}_t\|_{\mathscr{B}} + \frac{t_1^{\alpha}}{\alpha\Gamma(\alpha)}\|m_f\|_{C([0,t_1];\mathbb{R}^+)}W(\|\bar{y}_{\rho(s,\bar{y}_s)})\|_{\mathscr{B}}) \\
&\quad + ((K_\phi + K_g)\|\phi\|_{\mathscr{B}} + K_g\|\bar{y}_t\|_{\mathscr{B}})\|\phi_A\|_{L^1([0,t_1];\mathbb{R}^+)} \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_{0}^{t}\phi_A(t-s)\int_{0}^{s}\frac{\|m_f\|_{C([0,t_1];\mathbb{R}^+)}W(\|\bar{y}_{\rho(s,\bar{y}_s)})\|_{\mathscr{B}}}{(s-\tau)^{1-\alpha}}d\tau ds \\
&\leq \Big((K_\phi + K_g)\|\phi\|_{\mathscr{B}} + K_g(K_a r + M_a\|\phi\|_{\mathbb{B}}) + \frac{t_1^{\alpha}}{\alpha\Gamma(\alpha)}\|m_f\|_{C([0,t_1];\mathbb{R}^+)} \\
&\quad \times W((M_a + J_\phi^0)\|\phi\|_{\mathscr{B}} + K_a r)\Big)(1 + \|\phi_A\|_{L^1([0,t_1];\mathbb{R}^+)}) \\
&\leq r.
\end{aligned}$$

On the other hand, from the properties of function $h_i(\cdot)$ we easily see that, for $t \in (s_i, t_i]$

$\|\Gamma y(t)\|_{C((t_i,s_i];\mathbb{R}^+)} \le K_{h_i}(K_a r + M_a\|\phi\|_{\mathscr{B}}) \le r.$

From the above inequalities, we infer that $\|\Gamma y(t)\|_Y \le r$, and Γ has values in $B_r(0,Y)$.
Next, we consider the decomposition $\Gamma = \sum_{j=1}^{3}\Gamma^j$ where

$$\Gamma^1 y(t) = \begin{cases} h_i(t,\bar{y}_t), & t \in (t_i,s_i],\ i \ge 1,\\ h_i(s_i,\bar{y}_{s_i}) - g(s_i,\bar{y}_{s_i}) + g(t,\bar{y}_t) & \\ + \int_{s_i}^t \mathscr{T}'(t-s)(h_i(s_i,\bar{y}_{s_i}) - g(s_i,\bar{y}_{s_i}) + g(s,\bar{y}_s))ds, & t \in (s_i,t_{i+1}],\ i \ge 1,\\ \phi(0) - g(0,\phi) + g(t,\bar{y}_t) + \int_0^t \mathscr{T}'(t-s)(\phi(0) - g(0,\phi) + g(s,\bar{y}_s))ds, & t \in [0,t_1],\\ 0, & \text{otherwise}, \end{cases}$$

$$\Gamma^2 y(t) = \begin{cases} \frac{1}{\Gamma(\alpha)}\int_{s_i}^t (t-s)^{\alpha-1} f(s,\bar{y}_{\rho(s,\bar{y}_s)})ds, & t \in (s_i,t_{i+1}],\ i \ge 0,\\ 0, & \text{otherwise}, \end{cases}$$

$$\Gamma^3 y(t) = \begin{cases} \frac{1}{\Gamma(\alpha)}\int_{s_i}^t \mathscr{T}'(t-s)\int_{s_i}^s (s-\tau)^{\alpha-1} f(\tau,\bar{y}_{\rho(\tau,\bar{y}_\tau)})d\tau ds, & t \in (s_i,t_{i+1}],\ i \ge 0,\\ 0, & \text{otherwise}. \end{cases}$$

We consider the subsequent proof into the following steps.

Step 1. The map Γ^1 is a contraction on $B_r(0,Y)$.
Let $y,z \in B_r(0,Y)$. For $t \in (s_i,t_{i+1}],\ i = 1,\cdots,N$, it is easy to see that

$\|\Gamma^1 y - \Gamma^1 z\|_{C((s_i,t_{i+1}];\mathbb{R}^+)} \le K_a(K_{h_i} + 2K_g)\|y-z\|_Y(1 + \|\phi_A\|_{L^1((s_i,t_{i+1}];\mathbb{R}^+)}),$

For $t \in (t_i,s_i],\ i = 1,\cdots,N$, $\|\Gamma^1 y - \Gamma^1 z\|_{C((t_i,s_i];\mathbb{R}^+)} \le K_a K_{h_i}\|y-z\|_Y$.
For $t \in [0,t_1]$, $\|\Gamma^1 y - \Gamma^1 z\|_{C([0,t_1];\mathbb{R}^+)} \le K_a K_g\|y-z\|_Y(1 + \|\phi_A\|_{L^1([0,t_1];\mathbb{R}^+)})$
which implies that

$$\begin{aligned}\|\Gamma^1 y - \Gamma^1 z\|_Y \le \max_{\{i=1,\cdots,N\}}\{&K_a(K_{h_i} + 2K_g)(1 + \|\phi_A\|_{L^1((s_i,t_{i+1}];\mathbb{R}^+)}),\\ &K_a K_g(1 + \|\phi_A\|_{L^1([0,t_1];\mathbb{R}^+)})\}\|y-z\|_Y,\end{aligned}$$

since

$$\max_{\{i=1,\cdots,N\}}\{K_a(K_{h_i} + 2K_g)(1 + \|\phi_A\|_{L^1((s_i,t_{i+1}];\mathbb{R}^+)}),\ K_a K_g(1 + \|\phi_A\|_{L^1([0,t_1];\mathbb{R}^+)})\} < 1.$$

Hence Γ^1 is a contraction on $B_r(0,Y)$.

Next, we show that the maps Γ^2 and Γ^3 are completely continuous on $B_r(0,Y)$. Consider the constant r_1 defined by $r_1 = \|m_f\|_{C((s_i,t_{i+1}];\mathbb{R}^+)}W((M_a + J_0^\phi) + K_a r)$. We note that $\|f(s,\bar{y}_{\rho(s,\bar{y}_s)})\| \le r_1$ for $y \in B_r(0,Y)$ and all $s \in (s_i,t_{i+1}]$.

Step 2. The map Γ^2 is completely continuous on $B_r(0,Y)$.
From the properties of the function $f(\cdot)$ it is easy to see that Γ^2 is continuous. Next, we show that Γ^2 is a compact operator on $B_r(0,Y)$.
From Lemma 4 we have

$$\|\Gamma^2 y(t+h) - \Gamma^2 y(t)\| \le \frac{2r_1}{\alpha\Gamma(\alpha)}h^\alpha$$

for all $y \in B_r(0,Y)$, which shows that $\Gamma^2 B_r(0,Y)$ is equicontinuous on $(s_i,t_{i+1}]$, and this result is obvious when $t \in (t_i,s_i],\ i = 1,\cdots,N$. Hence $[\Gamma^2 \widetilde{B_r(0,Y)}]_i$ is an equicontinuous subset of $C([t_i,t_{i+1}];\mathscr{Y})$.

On the other hand, we show that the set $\{\Gamma^2 y(s) : y \in B_r(0,Y), s \in [s_i,t_{i+1}]\}$ is relatively compact in $\mathscr{Y}$.
Let $s_i < \varepsilon < t_{i+1},\ i = 1,\cdots,N$. Consider [41, Lemma II.1.3] the mean value theorem for the Bochner integral, for $y \in B_r(0,Y)$, and $\varepsilon \le t \le t_{i+1}$, we see that

$$\begin{aligned}\Gamma^2 y(t) &= \frac{1}{\Gamma(\alpha)}\int_t^{t-\varepsilon}\frac{f(s,\bar{y}_{\rho(s,\bar{y}_s)})}{(t-s)^{1-\alpha}}ds + \frac{1}{\Gamma(\alpha)}\int_{s_i}^{t-\varepsilon}\frac{f(s,\bar{y}_{\rho(s,\bar{y}_s)})}{(t-s)^{1-\alpha}}ds\\ &\in B_{\frac{r_1\varepsilon^\alpha}{\alpha\Gamma(\alpha)}}(0,\mathscr{Y}) + \frac{(t-\varepsilon-s_i)}{\Gamma(\alpha)}\overline{co((t-\varepsilon)^{1-\alpha}f(s,\bar{y}_{\rho(s,\bar{y}_s)}) : s \in [s_i,t_{i+1}])}.\end{aligned}$$

Since the map i_c is compact (see [12, Lemma 2.2]) and $f \in C([0,a] \times \mathscr{B}; \mathscr{D})$, from the above we assert that

$$\{\Gamma^2 y(s) : y \in B_r(0,Y), s \in [\varepsilon, t_{i+1}]\} \subset B_{\frac{r_1 \varepsilon^\alpha}{\alpha \Gamma(\alpha)}}(0, \mathscr{Y}) + K_\varepsilon,$$

where K_ε is a compact subset of $\mathscr{Y}$. Moreover,

$$\{\Gamma^2 y(s) : y \in B_r(0,Y), s \in [s_i, \varepsilon]\} \subset B_{\frac{r_1 \varepsilon^\alpha}{\alpha \Gamma(\alpha)}}(0, \mathscr{Y}),$$

Hence we find that

$$\{\Gamma^2 y(s) : y \in B_r(0,Y), s \in [s_i, t_{i+1}]\} \subset B_{\frac{r_1 \varepsilon^\alpha}{\alpha \Gamma(\alpha)}}(0, \mathscr{Y}) \cup (B_{\frac{r_1 \varepsilon^\alpha}{\alpha \Gamma(\alpha)}}(0, \mathscr{Y}) + K_\varepsilon),$$

is relatively compact in $\mathscr{Y}$, since $\frac{r_1 \varepsilon^\alpha}{\alpha \Gamma(\alpha)} \to 0$ as $\varepsilon \to 0$. It follows that Γ^2 is relatively compact.

Step 3. The map Γ^3 is completely continuous. Now we prove that $\{\Gamma^3 y(t) : y \in B_r(0,Y)\}$ is relatively compact in $\mathscr{Y}$ for all $t \in [s_i, t_{i+1}],\ i \geq 0$.
Let $s_i \leq t \leq t_{i+1}$ and $r_2 = \|m_f\|_{C((s_i,t_{i+1}];\mathbb{R}^+)} W((M_a + J_0^\phi) + K_a r)\|\phi_A\|_{L^1((s_i,t_{i+1}];\mathbb{R}^+)}$. From step 2, we know that the set $\bar{F_y} = \{F_y(s) : s \in (s_i, t_{i+1}], y \in B_r(0,Y)\}$ is relatively compact in $\mathscr{Y}$, where $F_y(s) = \frac{1}{\Gamma(\alpha)} \int_{s_i}^t (t-s)^{\alpha-1} f(s, \bar{y}_{\rho(s,\bar{y}_s)}) ds$.
If $y \in B_r(0,Y)$, using mean value theorem presented in [41, Lemma II.1.3] for the Bochner integral we get

$$\begin{aligned}\Gamma^3 y(t) &= \int_{s_i}^{t-\varepsilon} \mathscr{T}'(t-s) F_y(s) ds + \int_{t-\varepsilon}^{t} \mathscr{T}'(t-s) F_y(s) ds \\ &\in B_{r_2} + (t-\varepsilon)\overline{co(\{\mathscr{T}'(s)x : s \in [\varepsilon, t], x \in F_y\})},\end{aligned}$$

and hence $\{\Gamma^3 y(t) : y \in B_r(0,Y)\} \subset B_{r_2} + K_\varepsilon$, where K_ε is compact and $r_2 \to 0$ as $\varepsilon \to 0$.
Thus the set is relatively compact in $\mathscr{Y}$.
Further we show that the map $[\Gamma^3 B_r(0,Y)]_i,\ i = 1, \cdots, N$ is an equicontinuous subset of $C([t_i, t_{i+1}]; \mathscr{Y})$.
The proof is trivial on $(t_i, s_i]$. Assume $t \in (s_i, t_{i+1})$. From Lemma 4, for $y \in B_r(0,Y)$ and $0 < l < \varepsilon$ such that $t + l \leq t_{i+1}$, we have

$$\begin{aligned}&\|\Gamma^3 y(t+l) - \Gamma^3 y(t)\| = \| \int_{s_i}^{t+l} \mathscr{T}'(t+l-s) F_y(s) ds - \int_{s_i}^{t} \mathscr{T}'(t-s) F_y(s) ds\| \\ \leq &\int_{s_i}^{l} \phi_A(t+l-s) \|F_y(s)\|_{\mathscr{D}} ds + \int_0^t \phi_A(t) \|F_y(t+l-s) - F_y(t-s)\|_{\mathscr{D}} ds \\ &+ \int_0^{s_i} \phi_A(t-s) \|F_y(s)\|_{\mathscr{D}} ds \\ \leq &\frac{r_1 (t-s_i)^\alpha}{\alpha \Gamma \alpha} \int_{s_i}^{l} \phi_A(t+l-s) ds + \int_0^t \phi_A(t) [\|F_y\|]_{C^\alpha((s_i,t_{i+1}];\mathscr{D})} l^\alpha ds \\ &+ \frac{r_1 (t-s)^\alpha}{\alpha \Gamma(\alpha)} \int_0^{s_i} \phi_A(t-s) ds \\ \leq &\frac{r_1 t_{i+1}^\alpha}{\alpha \Gamma(\alpha)} \left(\int_{s_i}^{l} \phi_A(t+l-s) ds + \int_0^{s_i} \phi_A(t-s) ds \right) + \frac{2 r_1 l^\alpha}{\alpha \Gamma(\alpha)} \|\phi_A\|_{L^1((s_i,t_{i+1}])},\end{aligned}$$

then it follows that $\Gamma^3 B_r(0,Y)$ is right equicontinuous at $t \in (s_i, t_{i+1})$. From the similar argument we can say that $\Gamma^3 B_r(0,Y)$ is left continuous at $t \in (s_i, t_{i+1}]$ and right equicontinuous at s_i. Hence, $\Gamma^3 B_r(0,Y)$ is a equicontinuous map.
Conclude from the above steps and Lemma 1 the operators Γ^2 and Γ^3 are completely continuous and Γ^1 is contraction.
Hence it shows that Γ is a condensing operator and followed from Theorem 1 we have that there exists a at least one mild solution of the problem (1.1)-(1.3).

4 Application

Consider the space $\mathscr{Y} := L^2([0,\pi])$ and let A be defined by $Ax = x''$ with domain $\mathscr{D}$ consist of set of all y and $y'' \in \mathscr{Y}$, such that $y(0) = y(\pi) = 0$. $(\mathscr{S}(t))_{t \geq 0}$ is the analytic semigroup on $\mathscr{Y}$ which is generated by the operator A.

The operator A has discrete spectrum with eigenvalues of the form $-n^2,\ n \in \mathbb{N}$ and the corresponding normalized eigenfunctions given by $y_n(\eta) = (\frac{2}{\pi})^{1/2} \sin(n\eta)$. Further, for all $y \in \mathscr{Y}$ and every $t > 0$, $\mathscr{S}(t)y = \sum_{n=1}^\infty e^{-n^{2t}} < y, y_n > y_n$, also the set $\{y_n : n \in \mathbb{N}\}$ is an orthonormal basis for $\mathscr{Y}$. The semigroup $(\mathscr{S}(t))_{t\geq 0}$ is a uniformly bounded and compact. $R(\lambda, A) = (\lambda - A)^{-1}$ is a compact operator for every $\lambda \in \rho(A)$.

We know that the following integral equation

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}(1+(1*k))(t-s)Ay(s)ds,\ t \geq 0,$$

has a corresponding analytic resolvent operator $(\mathscr{T}(t))_{t\geq 0}$ on $\mathscr{Y}$ which is given by

$$\mathscr{T}(t) = \begin{cases} I, & t = 0, \\ \frac{1}{2\pi i}\int_{\Gamma_{\delta,\theta}} e^{\lambda t}(\frac{1}{\lambda}(1-(1+\hat{k}(\lambda))\lambda^{-\alpha}A))^{-1}d\lambda,\ t \geq 0, \end{cases}$$

where $\Gamma_{\delta,\theta}$ represents a contour consisting of the rays $\{\delta e^{i\nu} : \delta \geq 0\}$ and $\{\delta e^{-i\nu} : \delta \geq 0\}$ for some $\nu \in (\pi, \pi/2)$, see [40].

Next, we consider the fractional neutral differential equations with state delay and the non-instantaneous impulsive conditions are given by

$$\frac{\partial^\alpha}{\partial t^\alpha}\left(y(t,\eta) - \int_{-\infty}^t a_1(s-t)y(s,\eta)ds\right) = \frac{\partial^2}{\partial t^2}\left(y(t,\eta) - \int_0^t k(t-s)y(s,\eta)ds\right)$$
$$+\int_{-\infty}^t a_2(s-t)x(s-\varsigma_1(t)\varsigma_2(\|u(t)\|),\eta)ds,\ t\in[0,a],\ \eta\in[0,\pi] \tag{4.1}$$
$$y(t,0) = y(t,\pi) = 0,\ \ t\in[0,a], \tag{4.2}$$
$$y(\tau,\eta) = \phi(\tau,\eta),\ \tau \leq 0, \phi \in \mathscr{B} = PC_0 \times L^2([0,a];\mathscr{Y}), \tag{4.3}$$
$$y(t,\eta) = \int_{-\infty}^t a_3(s-t)y(s,\eta)ds,\ t\in(t_i,s_i],\ i = 1,\cdots,N, \tag{4.4}$$

for $(t,\eta) \in [0,a]\times[0,\pi]$, $0 = t_0 = s_0 < t_1 \leq s_1 < \cdots < t_N \leq s_N < t_{N+1} = a$ are prefixed real numbers, $k \in L^1(\mathbb{R})$.

To study this system, we suppose that $\varsigma_i : [0,\infty) \to [0,\infty),\ i = 1,2$ are continuous functions and we impose the subsequent conditions

–The function $a_j \in C(\mathbb{R};\mathbb{R}),\ j = 1,2,3,$
$L_g = \left(\int_{-\infty}^0 a_1^2(s)/g(s)\right)^{1/2} < \infty$, $L_f = \left(\int_{-\infty}^0 a_2^2(s)/g(s)\right)^{1/2} < \infty$, and $L_{h_i} = \left(\int_{-\infty}^0 a_3^2(s)/g(s)\right)^{1/2} < \infty$.

Under these conditions, we can define the map $h_i : (t_i,s_i]\times\mathscr{B} \to \mathscr{Y},\ g,f : [0,a]\times\mathscr{B}\to\mathscr{Y},\ \rho : I\times\mathscr{B}\to\mathbb{R},$, by

$$\rho(s,\psi) = s - \rho_1(s)\rho_2(\|\psi(0)\|),$$
$$f(t,\psi)(\eta) = \int_{-\infty}^0 a_2(s)\psi(s,\eta)ds,$$
$$g(t,\psi)(\eta) = \int_{-\infty}^0 a_1(s)\psi(s,\eta)ds,$$
$$h_i(t,\psi)(\eta) = \int_{-\infty}^0 a_3(s)\psi(s,\eta)ds,\ i = 1,\cdots,N.$$

From these conditions we can represent the equations (4.1)-(4.4) by the abstract fractional impulsive problem (1.1)-(1.3). Furthermore, the functions f,g,h_i, are bounded linear operators with $\|f\|_{\mathscr{L}(\mathscr{B},\mathscr{Y})} \leq L_f$, $\|g\|_{\mathscr{L}(\mathscr{B},\mathscr{Y})} \leq L_g$ and $\|h_i\|_{\mathscr{L}(\mathscr{B},\mathscr{Y})} \leq L_{h_i}, i = 1,\cdots,N$.

We can prove that $y \in \mathscr{PC}(\mathscr{Y})$ is a mild solution of (4.1)-(4.4) if $y(\cdot)$ is a mild solution of the corresponding abstract fractional impulsive problem (1.1)-(1.3). Under the above assumptions, the equations (4.1)-(4.4) has at least one mild solution in the view of Theorem 2 and the uniqueness of the solution verified from the Theorem 1.

5 Conclusion

In this study, we make the concept of mild solution for fractional impulsive differential equations more appropriate by using the perturbed resolvent operator technique. By utilizing the fixed point theorem we established the existence results of non-instantaneous impulsive differential equations with sate-dependent delay. The non-instantaneous impulsive systems are more suitable to the study of dynamics of evolution processes in pharmacotheraphy. The natural characteristics of fractional derivatives and neutral type equations considered with non-instantaneous impulses are more effective to describe the real world phenomena.

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