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Bayesian Estimation and Two-sample Prediction Based on Unified Hybrid Censored Sample

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Abstract: In this paper, a general inverse exponential form of the underlying distribution and a general conjugate prior are used to discuss the maximum likelihood and Bayesian estimation based on unified hybrid censored sample. A general procedure for deriving two-sample Bayesian prediction is developed using unified hybrid censoring scheme. Special cases of the inverse Weibull model such as the inverse exponential and the inverse Rayleigh distributions are then used as illustrative examples. Finally, numerical examples are presented for illustrating all the inferential procedures developed here.

Keywords: Bayesian estimation; Bayesian prediction; the inverse Weibull distribution; the inverse Rayleigh distribution; Inverse exponential distribution; Unified hybrid censored sample.

1 Introduction

In life-testing experiments, the experimenter may stop the experiment before all the units on the test be have failed due to some considerations such as time and cost. In such cases, the obtained data is called censored data. The most two common forms of censoring are Type-I and Type-II censoring schemes. Type-I hybrid censoring scheme is introduced by Epstein in [1] as a mixture of Type-I and Type-II censoring schemes. Type-II hybrid censoring scheme (Type-II HCS) is proposed by Childs et al. in [2] to fix the disadvantages inherent in Type-I hybrid censoring scheme. Chandrasekar et al. in [3] introduced generalized Type-I hybrid and generalized Type-II HCS as mixtures of Type-I hybrid and Type-II HCS. For more details about HCS, one may refer to [4].

Recently, Balakrishnan et al. in [5] proposed the unified HCS to fix the disadvantages inherent in the generalized Type-I hybrid and generalized Type-II HCS, suggested by Chandrasekar et al. in [3]. This censoring scheme can be described as follows. Consider a life-testing experiment in which n identical units are placed on a life-test. Fix integers $k,r \in \{0,\ldots,n\}$ and $T_1,T_2 \in (0,\infty)$ such that $k<r$ and $T_1<T_2$. If the k^{th} failure occurs before time T_1, the experiment is terminated at $\min\{\max(X_{r:n},T_1),T_2\}$. If the k^{th} failure occurs between T_1 and T_2, the experiment is terminated at $\min(X_{r:n},T_2)$ and if the k^{th} failure occurs after time T_2, the experiment is terminated at $X_{k:n}$. Under this censoring scheme, we can guarantee that the experiment would be completed at most in time T_2 with at least k failure and if not, we can guarantee exactly k failures. The described unified HCS and inferential methods based on such a scheme have been discussed earlier in the literature; see, for example; [4], [6], [7], [8], and [9].

We consider here the inverse exponential form for the underlying distribution, suggested by Mohie El-Din et al. in [10], that is described as follows; Motivated by the fact that the survival function (SF) $\bar{F}(x|\theta)=1-F(x|\theta)$ corresponding to any cumulative distribution function (CDF) $F(x|\theta)$ can be written in the form

$$\bar{F}(x|\theta)=1-\exp[-\psi(x;\theta)], \tag{1}$$

where $\psi(x;\theta)=-\ln F(x|\theta)$. Of course, some conditions need to be imposed so that $\bar{F}(x|\theta)$ is a valid SF. These conditions are: $\psi(x;\theta)$ is continuous, monotone decreasing and differentiable function, with $\psi(x;\theta)\to 0$ as $x\to\infty$ and $\psi(x;\theta)\to\infty$

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as $x \to 0^+$. The probability density function (*PDF*) corresponding to (1) is given by

$$f(x|\theta) = -\psi'(x;\theta)\exp[-\psi(x;\theta)], \tag{2}$$

where $\psi'(x;\theta)$ is the first derivative of $\psi(x;\theta)$ with respect to x. With an appropriate choice of $\psi(x;\theta)$, several distributions that are used in reliability studies can be obtained as special cases such as inverse exponential, inverse Weibull and inverse Rayleigh distributions. Inverse exponential distribution has been considered by Killer and Kamath in [11], Duran and Lewis in [12], and Abdel-Aty et al. in [13].

The rest of this paper is organized as follows. In Section 2, the maximum likelihood (ML) and Bayesian estimators of the unknown parameters under squared error loss function are developed. The problem of predicting the order statistics from a future sample is then discussed in Section 3. The inverse Weibull, the inverse exponential, and the inverse Rayleigh distributions are presented in Section 4 as special cases from the general inverse exponential form (1). Finally, in Section 5, some computational results for the inverse exponential distribution are presented for illustrating all the inferential methods developed here.

2 The ML and Bayesian Estimations

In this section, we use the general inverse exponential form for the underline distribution in (1) to develop general procedure for deriving the ML and Bayesian estimators of the unknown parameters based on an observed unified HCS.

Let $X_{1:n} < X_{2:n} < \ldots < X_{n:n}$ be the order statistics from a random sample of size n from an absolutely continuous CDF $F(x) \equiv F(x|\theta)$ with PDF $f(x) \equiv f(x|\theta)$, where the parameter $\theta \in \Theta$ may be a real vector. Let D_j denote the number of $X_{i:n}$'s that are at most T_j, $j = 1,2$. Then, D_j is a discrete random variable has the binomial distribution $B(n, F(T_j))$, $j = 1,2$, with support $\{0,1,\ldots,n\}$. Therefore, under the unified HCS described above, we have one of the following six types of observations:

1. Case I: $0 < X_{k:n} < X_{r:n} < T_1 < T_2$ the experiment terminate at T_1 and we will observe $X_{1:n} < \ldots < X_{k:n} < \ldots < X_{r:n} < \ldots < X_{D_1:n}$.
2. Case II: $0 < X_{k:n} < T_1 < X_{r:n} < T_2$ the experiment terminate at $X_{r:n}$ and we will observe $X_{1:n} < \ldots < X_{k:n} < \ldots < X_{D_1:n} < \ldots < X_{r:n}$.
3. Case III: $0 < X_{k:n} < T_1 < T_2 < X_{r:n}$ the experiment terminate at T_2 and we will observe $X_{1:n} < \ldots < X_{k:n} < \ldots < X_{D_1:n} < \ldots < X_{D_2:n}$.
4. Case IV: $0 < T_1 < X_{k:n} < X_{r:n} < T_2$ the experiment terminate at $X_{r:n}$ and we will observe $X_{1:n} < \ldots < X_{D_1:n} < \ldots < X_{k:n} < \ldots < X_{r:n}$.
5. Case V: $0 < T_1 < X_{k:n} < T_2 < X_{r:n}$ the experiment terminate at T_2 and we will observe $X_{1:n} < \ldots < X_{D_1:n} < \ldots < X_{k:n} < \ldots < X_{D_2:n}$.
6. Case VI: $0 < T_1 < T_2 < X_{k:n} < X_{r:n}$ the experiment terminate at $X_{k:n}$ and we will observe $X_{1:n} < \ldots < X_{D_1:n} < \ldots < X_{D_2:n} < \ldots < X_{k:n}$.

Thus, the joint density function of the unified hybrid censored sample $\underline{\mathbf{X}} = (X_{1:n}, X_{2:n}, \ldots, X_{D^*:n})$ is as follows:

$$\begin{aligned} f_{\underline{\mathbf{X}}}(\underline{\mathbf{x}}) &= \frac{n!}{(n-D^*)!}\prod_{j=1}^{D^*} f(x_j)\{1-F(T^*)\}^{n-D^*} \\ &= n!\sum_{i=0}^{n-D^*} C_i\,[F(T^*)]^{n-D^*-i}\prod_{j=1}^{D^*} f(x_j), \end{aligned} \tag{3}$$

where $C_i = \frac{(-1)^{n-D^*-i}}{(n-D^*-i)!i!}$,

$$(D^*, T^*) = \begin{cases} (D_1, T_1), \text{ if } 0 < X_{k:n} < X_{r:n} < T_1 < T_2, & \text{Case I}, \\ (r, X_{r:n}), \text{ if } 0 < X_{k:n} < T_1 < X_{r:n} < T_2, & \text{Case II}, \\ \qquad\quad \text{ if } 0 < T_1 < X_{k:n} < X_{r:n} < T_2, & \text{Case IV}, \\ (D_2, T_2), \text{if } 0 < X_{k:n} < T_1 < T_2 < X_{r:n}, & \text{Case III}, \\ \qquad\quad \text{ if } 0 < T_1 < X_{k:n} < T_2 < X_{r:n} & \text{Case V}, \\ (k, X_{k:n}), \text{ if } 0 < T_1 < T_2 < X_{k:n} < X_{r:n}, & \text{Case VI}, \end{cases} \tag{4}$$

and $\underline{\mathbf{x}} = (x_1, x_2, \ldots, x_{D^*})$ is a vector of realizations.

Upon using (1) and (2) in (3), we obtain the likelihood function of θ based on unified HCS as

$$L(\theta;\underline{\mathbf{x}}) = n!\left(\prod_{j=1}^{D^*}\left(-\psi'(x_j;\theta)\right)\right)\sum_{i=0}^{n-D^*} C_i \exp\left\{-\left[\sum_{j=1}^{D^*}\psi(x_j;\theta)+(n-D^*-i)\,\psi(T^*;\theta)\right]\right\}, \tag{5}$$

the log-likelihood function of θ is given by

$$\begin{aligned}\log L(\theta;\underline{\mathbf{x}}) &= \log n! + \sum_{j=1}^{D^*}\log\left(-\psi'(x_j;\theta)\right)\\ &\quad+\log\left\{\sum_{i=0}^{n-D^*} C_i \exp\left\{-\left[\sum_{j=1}^{D^*}\psi(x_j;\theta)+(n-D^*-i)\,\psi(T^*;\theta)\right]\right\}\right\}.\end{aligned} \tag{6}$$

Thus, we can calculate the ML estimate of θ by solving the equation

$$\frac{d\log L(\theta|\underline{\mathbf{x}})}{d\theta} = 0.$$

This equation is appropriate for a single value θ, but for a vector θ of course, the partial derivatives produce a system of equations that are solved simultaneously.

For the Bayesian approach, the unknown parameter is regarded as a realization of a random variable, which has some prior distribution. We consider here a general conjugate prior, suggested by AL-Hussaini in [14], that is given by

$$\pi(\theta;\delta) \propto A(\theta;\delta)\exp[-B(\theta;\delta)], \tag{7}$$

where $\theta \in \Theta$ is the vector of parameters of the distribution in (1) and δ is the vector of prior parameters. The prior family in (7) includes several priors used in the literature as special cases.

Upon combining (3) and (7), the posterior density function of θ, given unified HCS, is obtained as

$$\begin{aligned}\pi^*(\theta|\underline{\mathbf{x}}) &= L(\theta;\underline{\mathbf{x}})\pi(\theta;\delta)\Big/\int_{\theta\in\Theta} L(\theta;\underline{\mathbf{x}})\pi(\theta;\delta)d\theta\\ &= I^{-1}\sum_{i=0}^{n-D^*} C_i\eta_j(\theta;\underline{\mathbf{x}})\exp[-\zeta_i(\theta;\underline{\mathbf{x}})],\end{aligned} \tag{8}$$

where

$$\eta_j(\theta;\underline{\mathbf{x}}) = \left(\prod_{j=1}^{D^*}\left(-\psi'(x_j;\theta)\right)\right)[A(\theta;\delta)],$$

$$\zeta_i(\theta;\underline{\mathbf{x}}) = (n-D^*-i)\,\psi(T^*;\theta)+B(\theta;\delta)+\sum_{j=1}^{D^*}\psi(x_j;\theta),$$

and

$$I = \sum_{i=0}^{n-D^*} C_i \int_{\theta\in\Theta}\eta_j(\theta;\underline{\mathbf{x}})\exp[-\zeta_i(\theta;\underline{\mathbf{x}})].$$

The Bayesian estimator of θ under the squared error loss function is the mean of the posterior density function, given by

$$\widehat{\theta} = I^{-1}\sum_{i=0}^{n-D^*} C_i \int_{\theta\in\Theta}\theta\eta_j(\theta;\underline{\mathbf{x}})\exp[-\zeta_i(\theta;\underline{\mathbf{x}})]d\theta. \tag{9}$$

3 Two-Sample Bayesian Prediction

Let $Y_{1:m} \le Y_{2:m} \le \ldots \le Y_{m:m}$ be the order statistics from a future random sample of size m from the same population. We develop here a general procedure for deriving the point and interval predictions for $Y_{s:m}$, $1 \le s \le m$, based on the observed unified HCS. It is well known that the marginal density function of the s^{th} order statistic from a sample of size m from a continuous distribution with CDF $F(x)$ and PDF $f(x)$ is given, see; [15], by

$$\begin{aligned} f_{Y_{s:m}}(y_s|\theta) &= \frac{m!}{(s-1)!(m-s)!}[F(y_s)]^{s-1}[1-F(y_s)]^{m-s}f(y_s), \;\; y_s>0 \\ &= \frac{m!}{(s-1)!}\sum_{w=0}^{m-s} C_w [F(y_s)]^{m-w-1} f(y_s), \end{aligned} \tag{10}$$

where $C_w = \frac{(-1)^{m-s-w}}{w!(m-s-w)!}$.

Upon substituting (1) and (2) in (10), the marginal density function of $Y_{s:m}$ becomes

$$f_{Y_{s:m}}(y_s|\theta) = \frac{m!}{(s-1)!}\sum_{w=0}^{m-s} C_w \left(-\psi'(y_s,\theta)\right) \exp[-(m-w)\psi(y_s,\theta)], \;\; y_s>0. \tag{11}$$

Upon combining (8) and (11), the Bayesian predictive density function of $Y_{s:m}$, given unified HCS, is obtained as

$$\begin{aligned} f^*_{Y_{s:m}}(y_s|\underline{\mathbf{x}}) &= \int_{\theta\in\Theta} f_{Y_{s:m}}(y_s|\theta)\pi^*(\theta|\underline{\mathbf{x}})d\theta \\ &= \frac{m!I^{-1}}{(s-1)!}\sum_{i=0}^{n-D^*} C_i \int_{\theta\in\Theta} \eta_j(\theta;\underline{\mathbf{x}})\exp[-\zeta_i(\theta;\underline{\mathbf{x}})] \\ &\quad \times \sum_{w=0}^{m-s} C_w \left(-\psi'(y_s,\theta)\right)\exp[-(m-w)\psi(y_s,\theta)]d\theta. \end{aligned} \tag{12}$$

From (12), we simply obtain the cumulative distribution function $F^*_{Y_{s:m}}(t|\underline{\mathbf{x}})$, for $t\geq 0$, as

$$\begin{aligned} F^*_{Y_{s:m}}(t|\underline{\mathbf{x}}) &= \int_0^t f^*_{Y_{s:m}}(y_s|\underline{\mathbf{x}})dy_s \\ &= \frac{m!I^{-1}}{(s-1)!}\sum_{i=0}^{n-D^*} C_i \int_{\theta\in\Theta} \eta_j(\theta;\underline{\mathbf{x}})\exp[-\zeta_i(\theta;\underline{\mathbf{x}})] \\ &\quad \times \sum_{w=0}^{m-s} \frac{C_w}{(m-w)}\exp[-(m-w)\psi(t,\theta)]d\theta. \end{aligned} \tag{13}$$

The Bayesian point predictor of $Y_{s:m}$, $1\leq s\leq m$, under the squared error loss function is the mean of the predictive density, given by

$$\widehat{Y}_{s:m} = \int_0^\infty y_s f^*_{Y_{s:m}}(y_s|\underline{\mathbf{x}})dy_s, \tag{14}$$

where $f^*_{Y_{s:m}}(y_s|\underline{\mathbf{x}})$ is given as in (12).

The Bayesian predictive bounds of $100(1-\gamma)\%$ two-sided equi-tailed (ET) interval for $Y_{s:m}$, $1\leq s\leq m$, can be obtained by solving the following two equations:

$$F^*_{Y_{s:m}}(U_{ET}|\underline{\mathbf{x}}) = \frac{\gamma}{2} \quad \text{and} \quad F^*_{Y_{s:m}}(L_{ET}|\underline{\mathbf{x}}) = 1-\frac{\gamma}{2}, \tag{15}$$

where $F^*_{Y_{s:m}}(t|\underline{\mathbf{x}})$ is given as in (13), and L_{ET} and U_{ET} denote the lower and upper bounds, respectively. For the highest posterior density (HPD) method, the following two equations need to be solved:

$$F^*_{Y_{s:m}}(U_{HPD}|\underline{\mathbf{x}}) - F^*_{Y_{s:m}}(L_{HPD}|\underline{\mathbf{x}}) = 1-\gamma,$$

and

$$f^*_{Y_{s:m}}(U_{HPD}|\underline{\mathbf{x}}) - f^*_{Y_{s:m}}(L_{HPD}|\underline{\mathbf{x}}) = 0,$$

where $f^*_{Y_{s:m}}(y_s|\underline{\mathbf{x}})$ is as in (12), and L_{HPD} and U_{HPD} denote the HPD lower and upper bounds, respectively.

4 Illustrative Examples

Several distributions that are used in reliability studies can be obtained as special cases from the general inverse exponential form given in (1). In this section, we apply the general procedure derived in the preceding sections for the the inverse Weibull (IW) distribution, inverse Rayleigh and inverse exponential distributions as illustrative examples.

4.1 The Inverse Weibull Distribution (α,β)

In this section we study ML estimates and Bayesian estimates for unknown parameters based on unified hybrid from the inverse Weibull distribution. Also we study two sample Bayesian prediction intervals for order statistics (OS) based on the inverse Weibull distribution which is one of the most important distributions in the inverse exponential-type class of distributions. For example the inverse Weibull (IW) distribution has been used to distribution the degradation of mechanical components ([16]) such as the dynamic components (pistons, crankshaft,etc.) of diesel engines. Properties of IW distribution have been obtained by, for example; [17], [18], and [19].

The distribution function of the inverse Weibull distribution is given by

$$F(x|\theta)=\exp[-(\alpha x)^{-\beta}],\quad x>0, \tag{16}$$

where $\theta=(\alpha,\beta)$, $\alpha>0$, and $\beta>0$ so we have

$$\psi(x;\theta)=\frac{\alpha^{-\beta}}{x^{\beta}}\quad\text{and}\quad\psi'(x;\theta)=-\frac{\beta\alpha^{-\beta}}{x^{\beta+1}}. \tag{17}$$

Suppose that α is an unknown and β is known.Therefore, the likelihood function of α and β based on unified HCS, is given by

$$L(\alpha,\beta;\underline{\mathbf{x}})=n!\left(\prod_{j=1}^{D^*}\frac{1}{x_j^{\beta+1}}\right)\beta^{D^*}\sum_{i=0}^{n-D^*}C_i\alpha^{-\beta D^*}\exp\left\{-\alpha^{-\beta}\left[\sum_{j=1}^{D^*}\frac{1}{x_j^{\beta}}+\frac{n-D^*-i}{T^{*\beta}}\right]\right\}. \tag{18}$$

Thus, the log-likelihood function of α and β is given by

$$\begin{aligned}\log L(\alpha,\beta;\underline{\mathbf{x}})=&\log n!+\sum_{j=1}^{D^*}\log\left(\frac{1}{x_j^{\beta+1}}\right)+D^*\log(\beta)\\&+\log\left\{\sum_{i=0}^{n-D^*}C_i\alpha^{-\beta D^*}\exp\left\{-\alpha^{-\beta}\left[\sum_{j=1}^{D^*}\frac{1}{x_j^{\beta}}+\frac{n-D^*-i}{T^{*\beta}}\right]\right\}\right\},\end{aligned} \tag{19}$$

and so the ML estimator $\widehat{\alpha}_{ML}$ of α is readily obtained by solving the following equation

$$\sum_{i=0}^{n-D^*}\left\{C_i\beta\left(\alpha^{-\beta}-D^*\right)\alpha^{-\beta D^*-1}\exp\left[-\alpha^{-\beta}\left(\sum_{j=1}^{D^*}\frac{1}{x_j^{\beta}}+\frac{n-D^*-i}{T^{*\beta}}\right)\right]\right\}=0. \tag{20}$$

For the case when α is an unknown and β is known, we use the prior density function which was suggested by Calabria and Pulcini in [18] (when β is known) as

$$\pi(\lambda;\delta)=\alpha^{-c\beta-1}\exp[-d\alpha^{-\beta}], \tag{21}$$

where $\alpha>0, \delta=(c,d)$ and $c,d>0$.

Hence $A(\theta;\delta)=\alpha^{-c\beta-1}$ and $B(\theta;\delta)=d\alpha^{-\beta}$, from (8) the posterior density function is then given by,

$$\pi^*(\theta|\underline{\mathbf{x}})=I^{-1}\sum_{i=0}^{n-D^*}C_i\,\alpha^{-\beta(D^*+c)-1}\exp\left[-\alpha^{-\beta}\left(\sum_{j=1}^{D^*}\frac{1}{x_j^{\beta}}+\frac{n-D^*-i}{T^{*\beta}}+d\right)\right], \tag{22}$$

where

$$I=\Gamma(D^*+c)\sum_{i=0}^{n-D^*}C_i\left(\sum_{j=1}^{D^*}\frac{1}{x_j^{\beta}}+\frac{n-D^*-i}{T^{*\beta}}+d\right)^{-(D^*+c)}.$$

Hence, the Bayesian estimator of α under the squared error loss function is obtained as

$$\widehat{\alpha}_B = \frac{\Gamma\left(D^*+c+1\right)}{I}\sum_{i=0}^{n-D^*} C_i \left(\sum_{j=1}^{D^*}\frac{1}{x_j^{\beta}} + \frac{n-D^*-i}{T^{*\beta}} + d\right)^{-(D^*+c+1)}. \tag{23}$$

From (12), the Bayesian predictive density function of $Y_{s:m}$, given unified HCS, is obtained as

$$\begin{aligned} f^*_{Y_{s:m}}(y_s|\underline{\mathbf{x}}) &= \frac{m!}{I(s-1)!}\sum_{i=0}^{n-D^*}\sum_{w=0}^{m-s} C_i C_w \int_{\theta\in\Theta} \frac{\beta}{y_s^{\beta+1}}\alpha^{-\beta(D^*+c+1)-1} \\ &\quad \times \exp[-\alpha^{-\beta}\left(\sum_{j=1}^{D^*}\frac{1}{x_j^{\beta}} + \frac{n-D^*-i}{T^{*\beta}} + \frac{m-w}{y_s^{\beta}} + d\right) d\alpha \\ &= \frac{m!\Gamma\left(D^*+c+1\right)}{I(s-1)!}\sum_{i=0}^{n-D^*}\sum_{w=0}^{m-s}\frac{C_i C_w \beta}{y_s^{\beta+1}}\left(\sum_{j=1}^{D^*}\frac{1}{x_j^{\beta}}\right. \\ &\quad \left. + \frac{n-D^*-i}{T^{*\beta}} + \frac{m-w}{y_s^{\beta}} + d\right)^{-(D^*+c+1)}. \end{aligned} \tag{24}$$

Using (13), (17), and (22) then the predictive cumulative distribution function of $Y_{s:m}$ is given by

$$\begin{aligned} F^*_{Y_{s:m}}(t|\underline{\mathbf{x}}) &= \frac{m!\Gamma\left(D^*+c+1\right)}{I(s-1)!}\sum_{i=0}^{n-D^*}\sum_{w=0}^{m-s}\int_0^t \frac{C_i C_w \beta}{y_s^{\beta+1}}\left(\sum_{j=1}^{D^*}\frac{1}{x_j^{\beta}}\right. \\ &\quad \left. + \frac{n-D^*-i}{T^{*\beta}} + \frac{m-w}{y_s^{\beta}} + d\right)^{-(D^*+c+1)} dy_s \\ &= \frac{m!\Gamma\left(D^*+c\right)}{I(s-1)!}\sum_{i=0}^{n-D^*}\sum_{w=0}^{m-s}\frac{C_i C_w}{(m-w)}\left(\sum_{j=1}^{D^*}\frac{1}{x_j^{\beta}}\right. \\ &\quad \left. + \frac{n-D^*-i}{T^{*\beta}} + \frac{m-w}{t^{\beta}} + d\right)^{-(D^*+c)}. \end{aligned} \tag{25}$$

4.1.1 The inverse exponential distribution.

We can obtain the inverse exponential distribution as special case of the inverse Weibull distribution by setting $\beta = 1$. Hence the distribution function of the inverse exponential distribution is given by

$$F(x|\theta) = \exp[-\frac{1}{\alpha x}], \quad x > 0, \tag{26}$$

where $\alpha > 0$, and we have

$$\psi(x;\theta) = \frac{1}{\alpha x} \quad \text{and} \quad \psi'(x;\theta) = -\frac{1}{\alpha x^2}. \tag{27}$$

and so the ML estimator $\widehat{\alpha}_{ML}$ of α is readily obtained by solving the following equation

$$\sum_{i=0}^{n-D^*}\left\{C_i\left(\alpha^{-1}-D^*\right)\alpha^{-D^*-1}\exp\left[-\alpha^{-1}\left(\sum_{j=1}^{D^*}\frac{1}{x_j} + \frac{(n-D^*-i)}{T^*}\right)\right]\right\} = 0. \tag{28}$$

Also, the Bayesian estimator of α under the squared error loss function is obtained as

$$\widehat{\alpha}_B = \frac{\Gamma\left(D^*+c+1\right)}{I}\sum_{i=0}^{n-D^*} C_i \left(\sum_{j=1}^{D^*}\frac{1}{x_j} + \frac{n-D^*-i}{T^*} + d\right)^{-(D^*+c+1)}. \tag{29}$$

Putting $\beta = 1$ in (24), then the predictive density function of $Y_{s:m}$ is given by

$$f^*_{Y_{s:m}}(y_s|\mathbf{x}) = \frac{m!\Gamma(D^*+c+1)}{I(s-1)!}\sum_{i=0}^{n-D^*}\sum_{w=0}^{m-s}\frac{C_iC_w}{y_s^2}\left(\sum_{j=1}^{D^*}\frac{1}{x_j}+\frac{n-D^*-i}{T^*}+\frac{m-w}{y_s}+d\right)^{-(D^*+c+1)}, \tag{30}$$

and putting $\beta = 1$ in (25), then the predictive cumulative distribution function of $Y_{s:m}$ is given by

$$F^*_{Y_{s:m}}(t|\mathbf{x}) = \frac{m!\Gamma(D^*+c)}{I(s-1)!}\sum_{i=0}^{n-D^*}\sum_{w=0}^{m-s}\frac{C_iC_w}{(m-w)}\left(\sum_{j=1}^{D^*}\frac{1}{x_j}+\frac{n-D^*-i}{T^*}+\frac{m-w}{t}+d\right)^{-(D^*+c)}, \tag{31}$$

where

$$I = \Gamma(D^*+c)\sum_{i=0}^{n-D^*} C_i\left(\sum_{j=1}^{D^*}\frac{1}{x_j}+\frac{n-D^*-i}{T^*}+d\right)^{-(D^*+c)}. \tag{32}$$

4.1.2 The inverse Rayleigh distribution.

We can obtain the inverse Rayleigh distribution as special case of the inverse Weibull distribution by setting $\beta = 2$. Hence the distribution function of the inverse exponential distribution is given by

$$F(x|\theta) = \exp\left[-\frac{1}{(\alpha x)^2}\right], \quad x > 0, \tag{33}$$

where $\alpha > 0$, and we have

$$\psi(x;\theta) = \frac{1}{(\alpha x)^2} \quad \text{and} \quad \psi'(x;\theta) = -\frac{2}{\alpha^2 x^3}, \tag{34}$$

and so the ML estimator $\widehat{\alpha}_{ML}$ of α is readily obtained by solving the following equation

$$\sum_{i=0}^{n-D^*}\left\{C_i\left(\alpha^{-2}-D^*\right)\alpha^{-2D^*-1}\exp\left[-\alpha^{-2}\left(\sum_{j=1}^{D^*}\frac{1}{x_j^2}+\frac{(n-D^*-i)}{T^{*2}}\right)\right]\right\} = 0. \tag{35}$$

Also, the Bayesian estimator of α under the squared error loss function is obtained as

$$\widehat{\alpha}_B = \frac{\Gamma(D^*+c+1)}{I}\sum_{i=0}^{n-D^*} C_i\left(\sum_{j=1}^{D^*}\frac{1}{x_j^2}+\frac{n-D^*-i}{T^{*2}}+d\right)^{-(D^*+c+1)}. \tag{36}$$

Putting $\beta = 2$ in (24), then the predictive denstiy function of $Y_{s:m}$ is given by

$$f^*_{Y_{s:m}}(y_s|\mathbf{x}) = \frac{m!\Gamma(D^*+c+1)}{I(s-1)!}\sum_{i=0}^{n-D^*}\sum_{w=0}^{m-s}\frac{C_iC_w}{y_s^3}\left(\sum_{j=1}^{D^*}\frac{1}{x_j^2}+\frac{n-D^*-i}{T^{*2}}+\frac{m-w}{y_s^2}+d\right)^{-(D^*+c+1)}, \tag{37}$$

and putting $\beta = 2$ in (25), then the predictive cumulative distribution function of $Y_{s:m}$ is given by

$$F^*_{Y_{s:m}}(t|\mathbf{x}) = \frac{m!\Gamma(D^*+c)}{I(s-1)!}\sum_{i=0}^{n-D^*}\sum_{w=0}^{m-s}\frac{C_iC_w}{(m-w)}\left(\sum_{j=1}^{D^*}\frac{1}{x_j^2}+\frac{n-D^*-i}{T^{*2}}+\frac{m-w}{t^2}+d\right)^{-(D^*+c)}, \tag{38}$$

where

$$I = \Gamma(D^*+c)\sum_{i=0}^{n-D^*}\left[C_i\left(\sum_{j=1}^{D^*}\frac{1}{x_j^2}+\frac{n-D^*-i}{T^{*2}}+d\right)^{-(D^*+c)}\right]. \tag{39}$$

Table 1: The ML and Bayesian estimates of α.

Scheme	$\widehat{\alpha}_{ML}$	$\widehat{\alpha}_B$	
		IP	*NIP*
1	0.099	0.083	0.080
2	0.079	0.075	0.063
3	0.073	0.066	0.057
4	0.067	0.059	0.051

5 Numerical example for the inverse exponential distribution

In order to illustrate all the inferential results established for the inverse exponential distribution, we generated order statistics from a sample of size $n = 20$ from the inverse exponential distribution with $\alpha = 0.1$. The generated order statistics as follows: 0.026, 0.045, 0.061, 0.064, 0.090, 0.105, 0.107, 0.108, 0.113, 0.118, 0.127, 0.274, 0.319, 0.327, 0.348, 0.388, 0.449, 0.584, 1.765, and 27.861. We will apply the following four unified HCS:

1. Scheme 1: Suppose $k = 4$, $r = 6$, $T_1 = 0.110$ and $T_2 = 0.200$, then $x_{4:20} < x_{6:20} < T_1$ and the experiment would have terminated at $T_1 = 0.110$. Therefore, we would have the following data: 0.026, 0.045, 0.061, 0.064, 0.090, 0.105, 0.107, and 0.108;
2. Scheme 2: Suppose $k = 6$, $r = 10$, $T_1 = 0.110$ and $T_2 = 0.200$, then $x_{6:20} < T_1 < x_{10:20} < T_2$ and the experiment would have terminated at $x_{10:20} = 0.118$. Therefore, we would have the following data: 0.026, 0.045, 0.061, 0.064, 0.090, 0.105, 0.107, 0.108, 0.113, and 0.118;
3. Scheme 3: Suppose $k = 9$, $r = 13$, $T_1 = 0.110$ and $T_2 = 0.200$, then $T_1 < x_{9:20} < T_2 < x_{13:20}$ and the experiment would have terminated at $T_2 = 0.200$. Therefore, we would have the following data: 0.026, 0.045, 0.061, 0.064, 0.090, 0.105, 0.107, 0.108, 0.113, 0.118, and 0.127;
4. Scheme 4: Suppose $k = 13$, $r = 16$, $T_1 = 0.110$ and $T_2 = 0.200$, then $T_1 < T_2 < x_{13:20} < x_{16:20}$ and the experiment would have terminated at $x_{13:20} = 0.319$. Therefore, we would have the following data: 0.026, 0.045, 0.061, 0.064, 0.090, 0.105, 0.107, 0.108, 0.113, 0.118, 0.127, 0.274, and 0.319.

Based on the above four unified HCS, we used the results presented in Subsection 4.2 to calculate the ML and Bayesian estimates of the unknown parameter α. Also, we calculate the point predictor and 95% ET and HPD prediction intervals for the future order statistics $Y_{s:10}$, where $1 \le s \le 10$, from a future unobserved sample with size $m = 10$. All obtained results for the Bayesian estimation and prediction, presented in Tables 1,2, are computed based on two different choices of the hyperparameters (c, d), namely,

1. $(0.1, 10)$: informative prior (IP).
2. $(0, 0)$: noninformative prior (NIP).

6 Conclusions and discussion

In this paper, based on the general inverse exponential form (1) for the underline distribution, a general procedure for calculating the ML and Bayesian estimators of the unknown parameters has been discussed when the observed sample is unified hybrid censored sample. Both Bayesian point and interval predictions of the future order statistics from an unobserved future sample have been developed. We can apply this general procedure for several important distributions that are used in reliability studies such as inverse Weibull, inverse Rayleigh, and inverse exponential distributions. We applied in this paper the general procedures for the inverse exponential distribution as illustrative example.

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From the results in Table 2, we notice that, the point predictor of mean is between the upper and lower bounds of the prediction intervals. Also, a comparison of the results for the informative priors with the corresponding ones for non-informative priors reveals that the former produce more precise results. Moreover, the HPD prediction intervals seem to be more precise than the ET prediction intervals.

Table 2: Bayesian point predictor and 95% ET and HPD prediction intervals $Y_{s:\ell:N}$, for $s = 1,\ldots,\ell$, from the exponential distribution.

		IP			*NIP*		
Scheme	s	$\widehat{Y}_{s:N}$	ET interval	HPD interval	$\widehat{Y}_{s:N}$	ET interval	HPD interval
1	1	0.463	(0.011,2.754)	(0.007,2.121)	0.509	(0.011,3.009)	(0.005,2.293)
	2	0.847	(0.131,4.529)	(0.009,3.630)	0.986	(0.135,4.998)	(0.008,3.957)
	3	1.321	(0.359,6.747)	(0.109,5.526)	1.537	(0.367,7.488)	(0.104,6.049)
	4	1.788	(0.656,9.006)	(0.291,7.479)	2.085	(0.669,10.037)	(0.278,8.212)
	5	2.271	(1.009,11.385)	(0.529,9.542)	2.654	(1.024,12.727)	(0.508,10.500)
	6	3.021	(1.489,15.017)	(0.851,12.638)	3.533	(1.509,16.816)	(0.819,13.926)
	7	3.835	(2.057,19.006)	(1.245,16.057)	4.492	(2.080,21.316)	(1.198,17.716)
	8	4.751	(2.723,23.516)	(1.716,19.931)	5.571	(2.749,26.412)	(1.651,22.015)
	9	6.563	(3.739,32.142)	(2.378,27.169)	7.678	(3.773,36.071)	(2.291,29.998)
	10	8.956	(5.089,43.544)	(3.259,36.773)	10.464	(5.133,48.854)	(3.142,40.597)
2	1	0.491	(0.012,2.828)	(0.008,2.194)	0.535	(0.012,3.056)	(0.007,2.352)
	2	0.914	(0.141,4.620)	(0.011,3.734)	1.015	(0.145,5.030)	(0.010,4.028)
	3	1.409	(0.386,6.855)	(0.124,5.667)	1.565	(0.396,7.496)	(0.120,6.133)
	4	1.894	(0.707,9.124)	(0.327,7.654)	2.109	(0.725,10.009)	(0.320,8.303)
	5	2.392	(1.089,11.507)	(0.594,9.748)	2.670	(1.113,12.656)	(0.582,10.594)
	6	3.171	(1.611,15.161)	(0.955,12.899)	3.543	(1.643,16.696)	(0.936,14.033)
	7	4.014	(2.227,19.165)	(1.395,16.374)	4.491	(2.269,21.133)	(1.368,17.832)
	8	4.959	(2.951,23.689)	(1.922,20.308)	5.556	(3.003,26.152)	(1.885,22.136)
	9	6.851	(4.055,32.398)	(2.661,27.690)	7.662	(4.124,35.742)	(2.613,30.174)
	10	9.348	(5.519,43.900)	(3.645,37.482)	10.445	(5.611,48.419)	(3.581,40.839)
3	1	1.305	(0.030,6.088)	(0.009,4.748)	1.470	(0.034,6.843)	(0.011,5.305)
	2	2.079	(0.316,9.889)	(0.030,8.044)	2.208	(0.344,11.182)	(0.030,9.031)
	3	3.076	(0.863,14.623)	(0.293,12.178)	3.280	(0.934,16.594)	(0.302,13.708)
	4	4.046	(1.583,19.414)	(0.761,16.419)	4.329	(1.708,22.090)	(0.790,18.517)
	5	5.038	(2.440,24.440)	(1.375,20.881)	5.409	(2.626,27.866)	(1.431,23.585)
	6	6.599	(3.611,32.167)	(2.204,27.606)	7.097	(3.881,36.716)	(2.295,31.209)
	7	8.282	(4.997,40.620)	(3.218,35.014)	8.923	(5.364,46.416)	(3.352,39.618)
	8	10.166	(6.625,50.165)	(4.431,43.395)	10.972	(7.105,57.377)	(4.616,49.139)
	9	13.976	(9.106,68.644)	(6.129,59.181)	15.068	(9.760,78.464)	(6.390,66.997)
	10	18.998	(12.398,93.030)	(8.392,80.115)	20.474	(13.284,106.317)	(8.751,90.687)
4	1	0.910	(0.022,4.417)	(0.012,3.471)	0.990	(0.024,4.776)	(0.015,3.739)
	2	1.515	(0.239,7.118)	(0.024,5.843)	1.623	(0.253,7.728)	(0.024,6.315)
	3	2.247	(0.658,10.475)	(0.236,8.817)	2.411	(0.694,11.400)	(0.242,9.544)
	4	2.952	(1.212,13.856)	(0.612,11.856)	3.174	(1.275,15.108)	(0.628,12.851)
	5	3.668	(1.873,17.393)	(1.105,15.046)	3.952	(1.969,18.992)	(1.135,16.327)
	6	4.804	(2.779,22.857)	(1.769,19.867)	5.181	(2.917,24.978)	(1.818,21.572)
	7	6.023	(3.853,28.819)	(2.582,25.167)	6.502	(4.041,31.518)	(2.654,27.345)
	8	7.384	(5.116,35.542)	(3.556,31.158)	7.980	(5.362,38.898)	(3.655,33.872)
	9	10.179	(7.038,48.680)	(4.913,42.509)	10.990	(7.372,53.250)	(5.053,46.203)
	10	13.860	(9.586,65.996)	(6.725,57.555)	14.956	(10.039,72.179)	(6.917,62.551)

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