Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.18576/pfda/020303

Exponential Stability of Solutions of a Second Order System of Integrodifferential Equations with the Caputo-Fabrizio Fractional Derivatives

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Received: 3 Jan. 2016, Revised: 21 Apr. 2016, Accepted: 23 Apr. 2016
Published online: 1 Jul. 2016

Abstract: The paper deals with the stability problem for a nonlinear fractional differential equation depending on the Caputo-Fabrizio fractional derivatives without singular kernels of different orders and on power nonlinearities of different orders. We give conditions under which the equilibrium of the equation is exponentially stable. The proof of this result is based on the Pinto's integral inequality.

Keywords: Riemann-Liouville integral, Caputo-Fabrizio derivative, fractional differential equation, exponential stability.

1 Introduction

In the paper [1] a sufficient condition for the exponential stability of solutions of a fractionally perturbed ODEs is proved, where the fractional parts of their right-hand sides depend several Riemann-Liouville integrals of different orders. It is well-known that fractional differential equations with the Caputo or Riemann-Lilouville derivatives on their left-hand sides do not have exponentially stable solutions (see [2], [3], [4], [5], [6]). They can have asymptotically stable solutions only (see [3], [4]). We study the same problem for the case of fractionally perturbed ODEs, where instead of the Liouville integrals there are integrals from the definition of the Caputo-Fabrizio fractional derivative defined below. The problem of the existence of global solutions for a functional-differential equation depending on several Riemann-Liouville integrals of different orders is studied in the paper [7] and the problem of asymptotic integration of this type of equations is studied in [8].

Let us consider the following fractionally perturbed pendulum equation:

$$u''(t)+\eta u'(t)+\omega^2 u(t)=g\big(t,u(t),u'(t),{}^{CF}D^{\alpha_1}u(t),\dots,{}^{CF}D^{\alpha_p}u(t)\big),\qquad t\in[0,\infty),\tag{1}$$

where $\omega\neq 0$, $\eta>0$,

$${}^{CF}D^{\alpha_i}u(t):=\frac{M(\alpha_i)}{1-\alpha_i}\int_0^t \exp\left(-\frac{\alpha_i}{1-\alpha_i}(t-s)\right)u'(s)ds\tag{2}$$

is the Caputo-Fabrizio fractional derivative without singular kernel of the function $u(t)$ of the order $\alpha_i\in(0,1)$, $i\in\{1,2,\dots,p\}$, defined recently in the paper [9]. The classical Caputo fractional derivative, defined by M. Caputo in the paper [10] and the corresponding fractional differential equations (see [6]) are frequently used in applications. This new fractional derivative can also be very useful tool for modeling of real world problems. In the paper [11] the Duffing-like oscillator

$$m\frac{d^2x(t)}{dt^2}+c\int_0^t \mu e^{-\mu(t-\tau)}x'(\tau)d\tau+kx(t)+\alpha kx^3(t)=A\cos(\Omega t)\tag{3}$$

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is studied, where x represents the displacement of the oscillator mass m, the linear stiffness is given by k, the coefficient α represents the form of the cubic stiffness nonlinearity, c is the viscous damping coefficient and the non-viscous damping effects are represented by the parameter μ. The forcing amplitude is $A = x_0 k$, where x_0 is the equivalent static displacement. The damping term $c\int_0^t \mu e^{-\mu(t-\tau)}x'(\tau)d\tau$ has the form (2).

M. Caputo and M. Fabrizio present some applications related to their new definition of fractional derivative in the paper [12] and some applications of this derivative are recently presented also in the papers [13], [14], [15], [16]. Fractional differential equations with the Caputo-Fabrizio derivatives are studied in [17], where an existence and uniqueness theorem for this type of equations is proved.

The equation (1) can be written as the system

$$\begin{aligned} x_1'(t) &= x_2(t), \\ x_2'(t) &= -\omega^2 x_1(t) - \eta x_2(t) + g\big(t, x_1(t), x_2(t), {}^{CF}I^{\alpha_1}x_2(t), \dots, {}^{CF}I^{\alpha_p}x_2(t)\big), \quad t \in [0,\infty), \end{aligned} \tag{4}$$

where $x_1 = u$, $x_2 = u'$,

$${}^{CF}I^{\alpha_i}x_2(t) = \frac{M(\alpha_i)}{1-\alpha_i}\int_0^t \exp\Big(-\frac{\alpha_i}{1-\alpha_i}(t-s)\Big)x_2(s)ds, \qquad i = 1,2,\dots,p. \tag{5}$$

In the paper [18] an abstract second order differential equation of the form

$$u''(t) + Cu(t) = g\big(t, u(t), {}^{C}D^{\alpha_1}u(t), \dots, {}^{C}D^{\alpha_p}u(t)\big), \qquad t \in [0,\infty),\ u(t) \in X, \tag{6}$$

where X is a Banach space, C is a strongly continuous cosine family of linear operators in X, f is a nonlinear mapping and

$${}^{C}D^{\alpha_i}u(t) = \frac{1}{\Gamma(1-\alpha_i)}\int_0^t (t-s)^{-\alpha_i}u'(s)ds$$

is the Caputo fractional derivative of the mapping $u(t)$ at $t \in \mathbb{R}$ of order $\alpha \in (0,1)$, $i \in \{1,2,\dots,p\}$.

Some existence results for this equation are proved there. Some generalizations of these results to an abstract integrodifferential equation are proved in the paper [15].

We consider the following finite dimensional second order integrodifferential equation:

$$u''(t) + Du'(t) + Cu = g\big(t, u(t), u'(t), {}^{CF}D^{\alpha_1}u(t), \dots, {}^{CF}D^{\alpha_p}u(t)\big), \qquad t \in [0,\infty),\ u \in \mathbb{R}^n, \tag{7}$$

where A, B are constant matrices, $g\colon \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^{np} \to \mathbb{R}^n$ is a continuous mapping. If we denote $x_1(t) = u(t)$, $x_2(t) = u'(t)$, then we obtain the following system for $x(t) = (x_1(t), x_2(t))$:

$$\begin{aligned} x_1'(t) &= x_2(t), \\ x_2'(t) &= -Cx_1(t) - Dx_2(t) + g\big(t, x_1(t), x_2(t), {}^{CF}D^{\alpha_1}x_2(t), \dots, {}^{CF}D^{\alpha_p}x_2(t)\big), \qquad t \in [0,\infty),\ x_i(t) \in \mathbb{R}^n,\ i = 1,2. \end{aligned} \tag{8}$$

Motivated by this example we will consider a more general system of nonlinear fractional differential equations in the next section with the aim to prove a sufficient condition for the exponential stability of its solutions. A similar problem for fractional differential equations with several Riemann-Liouville integrals is studied in the paper [1]. In the paper [7] a sufficient conditions for the non- existence of blowing-up solutions to some functional-differential equations with the same type of nonlinearities as in [18], [19] and [7] are proved.

2 Exponential Stability Result

In this section we study the problem of exponential stability of solutions of the system

$$x'(t) = Ax(t) + f\big(t, x(t), {}^{CF}I^{\alpha_1}x(t), \dots, {}^{CF}I^{\alpha_p}x(t)\big), \qquad t \in [0,\infty), \tag{9}$$

where

$${}^{CF}I^{\alpha_i}x(t) = \big({}^{CF}I^{\alpha_i}x_1(t), {}^{CF}I^{\alpha_i}x_2(t), \dots, {}^{CF}I^{\alpha_i}x_n(t)\big), \qquad i = 1,2,\dots,p, \tag{10}$$

A is a constant matrix, $f\colon \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^{pn}$ is a continuous mapping. Before we need to present a lemma due to D. Bainov and P. Semionov (see [20, Theorem 10.3]), originally proved by M. Pinto in [21]. First we define an ordering $\propto$ of continuous functions $\omega_1, \omega_2\colon [a,b) \to \mathbb{R}$, where ω_1 is positive on $(0,\infty)$. We write $\omega_2 \propto \omega_1$ if $\frac{\omega_2}{\omega_1}$ is nondecreasing $(0,b)$.

Lemma 1. *Let $c>0$ be a constant. Assume that $\Psi_j(t)$ are nonnegative continuous functions on $[a,b)$, $\omega_j(u)$, $j=1,2,\dots,k$ are nondecreasing continuous functions on $[0,\infty)$, positive for $u>0$, $\omega_1 \propto \omega_2 \propto \cdots \propto \omega_k$ and $u(t)$ is a nonnegative continuous function on $[0,\infty)$ such that*

$$u(t) \le c + \sum_{j=1}^{k} \int_0^t \Psi_j(s)\omega_j\big(u(s)\big)ds, \qquad t \in [a,b). \tag{11}$$

Then

$$u(t) \le \eta_k(t), \qquad t \in [a,b_1), \tag{12}$$

where

$$\eta_k(t) = W_k^{-1}\Big[W_k(\eta_{k-1}(t)) + \int_a^t \Psi_k(s)ds\Big], \qquad t \in [a,b_1)$$

for some $b_1 \in (a,b)$, where

$$\eta_1(t) = W_1^{-1}[W_1(c) + \int_a^t \Psi_1(s)ds]$$
$$W_j(u) = \int_{u_j}^{u} \frac{dz}{\omega_j(z)}, \qquad z \ge u_j > 0,\ j=1,2,\dots,k,$$

W_j^{-1} is the inverse of W_j and

$$\eta_j(t) = W_j^{-1}\Big[W_j(\eta_{j-1}(t)) + \int_0^t \psi_j(s)ds\Big], \qquad j=1,2,3,\dots,k.$$

Corollary 1. *Let $\omega_j(u) = u^{m_j}$, $j=1,2,\dots,k$, where $1 \le m_1 < m_2 < \cdots < m_k$, $[a,b)=[0,\infty)$ and let the following conditions be satisfied:*

$$\int_0^\infty \Psi_j(s)ds < \infty, \qquad j=1,2,\dots,k;$$
$$(m_j-1)\big(cD_j\big)^{m_j-1}\int_0^\infty \Psi_j(s)ds < 1, \qquad j=1,2,\dots,k,$$

where

$$D_1 = \begin{cases} e^{\int_0^\infty \psi_1(s)ds}, & \text{if } m_1 = 1,\\ \left(1-(m_1-1)c^{m_1-1}\int_0^\infty \Psi_1(s)ds\right)^{-\frac{1}{m_1-1}}, & \text{if } m_1 > 1 \end{cases}$$

$$D_j = \left(1-(m_j-1)(D_j)c^{m_j-1}\int_0^\infty \Psi_j(s)ds\right)^{-\frac{1}{m_j-1}}, \qquad j=2,\dots,k.$$

Then

$$u(t) \le cD,$$

where $D = D_k$.

Proof. For $m_1 = 1$ then $u(t) \le ce^{\int_0^t} \le ce^{\int_0^\infty \Psi_1(s)ds} \le cD_1$. If $m_1 > 1$ then

$$\eta_1(t) \le \frac{c}{\left(1-(m_1-1)c^{m_1-1}\int_0^\infty \Psi_1(s)ds\right)^{\frac{1}{m_1-1}}} \le cD_1.$$

One can show by induction that

$$\eta_j(t) \le cD_j, \quad j=2,\dots,n$$

and from Lemma 1 we obtain that $u(t) \le \eta_k(t) \le cD_k = cD$.

We assume that all solutions of the equation (1) exist on the interval $[0,\infty)$ and that the following conditions hold:

(C1)

$$\|f(t,x,v_1,v_2,\ldots,v_p)\| \le \sum_{j=1}^{k} \lambda_j(t)\|x\|^{m_j} + \sum_{i=1}^{p}\sum_{j=1}^{k} \mu_{ij}(t)\|v_i\|^{m_j}, \qquad t\in\mathbb{R},\ x\in\mathbb{R}^n,\ v_i\in\mathbb{R}^n,\ i=1,2,\ldots,p, \tag{13}$$

where $1\le m_1 < m_2 < \cdots < m_k$ and $\lambda_i(t), \mu_{ij}(t),\ i=1,2,\ldots,p;\ j=1,2,\ldots,k$ are nonnegative continuous functions on $[0,\infty)$, $\|x\| = \max\{|x_1|,|x_2|,\ldots,|x_n|\}$.

(C2) There exist constants $K>0$, $a>0$ such that

$$\|e^{\mathbf{A}t}x\| \le Ke^{-at}\|x\| \qquad \text{for all} \qquad t\in[0,\infty),\ x\in\mathbb{R}^n. \tag{14}$$

Theorem 1. *Let the conditions* **(C1)**, **(C2)** *and the following conditions be satisfied:*

(C3)

$$a > \kappa := \max\{\sigma_i : 1\le i\le p\}, \quad \textit{where} \quad \sigma_i = \frac{\alpha_i}{1-\alpha_i}, \qquad i=1,2,\ldots,p; \tag{15}$$

(C4)

$$L_{ij} = \int_0^\infty \mu_{ij}(s)e^{(a-m_j\sigma_i)s}s^{m_j-1}ds < \infty, \qquad i=1,2,\ldots,p;\ j=1,2,\ldots,k; \tag{16}$$

(C5)

$$R_j = \int_0^\infty \lambda_j(s)e^{(1-m_j)as}ds < \infty, \qquad j=1,2,\ldots,k. \tag{17}$$

Then there exist constants $\gamma>0$, $\rho>0$ such that

$$\|x(t)\| \le \gamma e^{-at}\|x_0\| \qquad \textit{for all } t\in[0,\infty)$$

and for any solutions $x(t)$ of the equation (1) satisfying the initial condition $x(0)=x_0$ with $\|x_0\|<\rho$.

Proof. Let $x(t)$ be a solutions of the equation (1) with $x(0)=x_0$. Then

$$x(t) = e^{\mathbf{A}t}x_0 + \int_0^t e^{A(t-s)} f\left(s, x(s), N_1e^{-\sigma_1 s}\int_0^s e^{\sigma_1\tau}x(\tau)d\tau,\ldots,N_pe^{-\sigma_p s}\int_0^s e^{\sigma_p\tau}x(\tau)d\tau\right)ds, \tag{18}$$

where

$$N_i = \frac{M(\alpha_i)}{1-\alpha_i}, \qquad \sigma_i = \frac{\alpha_i}{1-\alpha_i}, \qquad i=1,2,\ldots,p.$$

Using the condition **(C1)** and the inequality (14) we can estimate $\|x(t)\|$ as follows:

$$\begin{aligned}\|x(t)\| \le Ke^{-at}\|x_0\| &+ Ke^{-at}\sum_{j=1}^{k}\int_0^t e^{as}\lambda_j(s)\|x(s)\|^{m_j}d\tau \\ &+ Ke^{-at}\sum_{j=1}^{k}\sum_{i=1}^{p}\int_0^t e^{as}\mu_{ij}(s)N_i^{m_j}e^{-\sigma_i m_j s}\left(\int_0^s e^{\sigma_j\tau}\|x(\tau)\|d\tau\right)^{m_j}ds.\end{aligned} \tag{19}$$

If we denote $v(t)=e^{at}\|x(t)\|$, i.e., $\|x(t)\| = e^{-at}v(t)$, then $\|x(t)\|^{m_j} = e^{-m_jat}v(t)^{m_j}$ and using the Hölder inequality we obtain

$$\begin{aligned}v(t) \le K\|x_0\| &+ K\sum_{j=1}^{k}\int_0^t e^{(1-m_j)as}\lambda_j(s)v(s)^{m_j}ds \\ &+ K\sum_{j=1}^{k}\sum_{i=1}^{p}\int_0^t \mu_{ij}(s)e^{(a-m_j\sigma_i)s}N_i^{m_j}\left(\int_0^s e^{-[a-\sigma_i]\tau}v(\tau)d\tau\right)^{m_j}ds.\end{aligned} \tag{20}$$

If $m_1=1$ then

$$\int_0^t \mu_{i1}(s)e^{(a-m_j\sigma_i)s}N_i\left(\int_0^s e^{-[a-\sigma_i]\tau}v(\tau)d\tau\right)ds \le N_i\left(\int_0^t \mu_{i1}(s)e^{(a-\sigma_i)s}ds\right)\left(\int_0^t e^{-[a-\sigma_i]\tau}v(\tau)d\tau\right).$$

If $m_j > 1$ then using the Hölder inequality we obtain

$$\int_0^t \mu_{ij}(s) N_i^{m_j} e^{(a-\sigma_i m_j)s} \left(\int_0^s e^{-[a-\sigma_i]\tau} v(\tau) d\tau \right)^{m_j} ds \leq N_i^{m_j} \int_0^t \mu_{ij}(s) e^{(a-\sigma_i m_j)s} s^{m_j-1} \left(\int_0^s e^{-[a-\sigma_i]m_j\tau} v(\tau)^{m_j} d\tau \right) ds$$
$$\leq N_i^{m_j} L_{ij} \int_0^t e^{-[a-\sigma_i]m_j\tau} v(\tau)^{m_j} d\tau.$$

From the both above inequalities we obtain the following inequality for $v(t)$:

$$v(t) \leq K\|x_0\| + \sum_{j=1}^{k} \int_0^t F_j(s) v(s)^{m_j} ds, \qquad t \geq 0, \tag{21}$$

where

$$F_j(s) = K\lambda_j(s) e^{(1-m_j)as} + KNLe^{-[a-\kappa]m_j s}$$

with $N = \max\{N_i^{m_j} : 1 \leq i \leq p, 1 \leq j \leq k\}$, $L = \max\{L_{ij} : 1 \leq i \leq p, 1 \leq j \leq k\}$ and κ defined by (15).

From the conditions **(C4)**, **(C5)** it follows that $\int_0^t F_j(s)ds < \int_0^\infty F_j(s)ds < \infty$, $j = 1, 2, \ldots, k$. Therefore by the Corollary 1 there is a constant $D > 0$ such that $v(t) = e^{at}\|x(t)\| \leq KD\|x_0\|, \quad t \geq 0$, i.e., $\|x(t)\| \leq \gamma e^{-at}\|x_0\|$ for all $t \in [0, \infty)$, where $\gamma = KD$.

Acknowledgement

The both authors were supported by the Slovak Grant Agency VEGA No. 1/0071/14 and the second author was supported also by the Slovak Research and Development Agency under the contract No. APVV-14-0378.

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