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Fractional Differential Geometry of Curves & Surfaces

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Abstract: Following the concepts of fractional differential and Leibnitz's L-Fractional Derivatives, proposed by the author [1], the L-fractional chain rule is introduced. Furthermore, the theory of curves and surfaces is revisited, into the context of Fractional Calculus. The fractional tangents, normals, curvature vectors and radii of curvature of curves are defined. Moreover, the Serret-Frenet equations are revisited, into the context of fractional calculus. The proposed theory is implemented into a parabola and the curve configured by the Weierstrass function as well. The fractional bending problem of an inhomogeneous beam is also presented, as implementation of the proposed theory. Further, the theory is extended on manifolds, defining the fractional first differential (tangent) spaces, along with the revisiting first and second fundamental forms for the surfaces. In addition revisited operators like fractional gradient, divergence and rotation are introduced, outlining revision of the vector field theorems..

Keywords: Fractional derivatives, Caputo derivatives, fractional differentials, L- Fractional derivatives, fractional tangent space, fractional normal, fractional curvature, fractional surfaces.

1 Introduction

Fractional calculus, originated by Leibnitz [2], Liouville [3], and Riemann [4], has recently applied to modern advances in physics and engineering. Fractional derivative models account for long-range (non-local) dependence of phenomena, resulting in better description of their behavior. Various material models, based upon Fractional time derivatives, have been presented, describing their viscoelastic interaction, Refs [5,6]. Lazopoulos [7] has proposed an elastic uniaxial model, based upon fractional derivatives for lifting Noll's axiom of local-action. Carpinteri et al. [8] have also proposed a fractional approach to non-local mechanics. Applications in various physical areas may also be found in various books Refs. [9,10,11,12]. Since the need for Fractional Differential Geometry has extensively been discussed in various places, researchers have presented different aspects, concerning Fractional Geometry of Manifolds [13,14] with applications in fields of mechanics, quantum mechanics, relativity, finance, probabilities etc. Nevertheless, researchers are raising doubtfulness about the existence of Fractional Differential Geometry and their argument is not easily rejected. Basically, the classical differential $df(x) = f'(x)dx$ has been substituted by the fractional one introduced by Adda [16,17] in the form

$$d^a f = g(x)(dx)^a$$

.

Nevertheless that definition of the differential is valid in the case of positive increments dx, whereas in the case of negative increments, the differential $d^a f(x)$ may be complex. That is exactly the reason why many researchers reasonably reject the existence of Fractional Differential Geometry. However the variable x accepts its own fractional differential

$$d^a x = \sigma(x)(dx)^a$$

with $\sigma(x) \neq 1$, differently of the conventional case when $a = 1$, where $\sigma(x)$ is always one. Relating both equations, it appears that

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$$d^a f = \frac{g(x)}{\sigma(x)} d^a x$$

. In this case $d^\sigma x$ is always a real quantity accepting positive or negative incremental real values alike. On those bases, the development of Fractional Differential Geometry may be established. Further, fractal functions exhibiting self-similarity are non-differentiable functions, but they exhibit fractional differentiability of order $0 < \alpha < 1$. See Ref. [18,19,20,21]. Golmankhaneh and Baleanu [15] introduced the generalized fractional Riemann-Liouville and Caputo like derivatives for functions defined on fractal sets. Fractional Calculus in mechanics has been suggested by many researchers, Tarasov [13,22], Drapaca & Sivaloganathan [23], Sumelka [24], Lazopoulos & Lazopoulos [25], in problems of continuum mechanics with microstructure where non-local elasticity is necessary. Fractional Continuum mechanics has been applied to various problems in hydrodynamics Ref [13,26]. Recently Fractional Calculus has been introduced by the author [27] for the description of peridynamic theory [28,29]. Yet, fractional calculus has been considered as the best frame for describing viscoelastic problems [5,6]. In addition Fractional Differential Geometry affects rigid body dynamics, in holonomic and non-holonomic systems [30,31,32]. Recent applications in Quantum Mechanics, Physics and relativity demand differential geometry revisited by Fractional Calculus [33,34]. In the present work, the fractional differential established in Lazopoulos [1] will be recalled along with the introduced Leibnitz L-fractional derivatives. Those differentials are always real and proper for establishing the Fractional Differential Geometry. Correcting the picture of fractional differential of a function, the fractional tangent space of a manifold was defined, introducing also Leibnitz L-fractional derivative that is the only one having physical meaning. Moreover, the present work reviews the theory of Fractal Geometry of curves, describing their tangent spaces, their normals, the curvature vectors and the corresponding radii of curvature. In addition the Serret-Frenet equations will be revisited into the fractional calculus context. The theory is implemented to a parabola, to the Weierstrass function and the beam bending [35], considered as applications of the curves theory to the solid mechanics. Yet, the theory is extended on manifolds, just to describe the fractional differential geometry of surfaces. Finally outline of fractional vector field theory is included, along with the revisited fractional vector field theorems.

2 Basic Properties of Fractional Calculus

Fractional Calculus has recently become a branch of pure mathematics, with many applications in Physics and Engineering, Tarasov [13,22]. Many definitions of fractional derivatives exist. In fact, Fractional Calculus originated by Leibniz, is looking for the possibility of defining the derivative $\frac{d^n g}{dx^n}$ when $n = \frac{1}{2}$. The various types of the fractional derivatives exhibit some advantages over the others. Nevertheless they are almost all non local, contrary to the conventional ones. The detailed properties of fractional derivatives may be found in Kilbas et al. [9], Podlubny[10], Samko et al.[11]. Starting from Cauchy formula for the n-fold integral of a primitive function $f(x)$.

$$I^n f(x) = \int_0^x f(s)(ds)^n = \int_a^x dx_n \int_a^{x_n} dx_{n-1} \int_a^{x_{n-1}} dx_{n-2} ..... \int_a^{x_2} f(x_1) dx_1 \tag{1}$$

expressed by

$$ {}_aI_x{}^n f(x) = \frac{1}{(n-1)!} \int_a^x (x-s)^{n-1} f(s) ds, \quad x > 0, n \in N \tag{2}$$

and

$$ {}_xI_b{}^n f(x) = \frac{1}{(n-1)!} \int_x^b (s-x)^{n-1} f(s) ds, \quad x > 0, n \in N \tag{3}$$

the left and right fractional integral of f are defined as

$$ {}_aI_x{}^a f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(s)}{(x-s)^{1-a}} ds, \tag{4}$$

$$ {}_xI_b{}^a f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(s)}{(s-x)^{1-a}} ds. \tag{5}$$

In Eqs.(4,5) we assume that is the order of fractional integrals with $0 < a \leqslant 1$, considering $\Gamma(x) = (x-1)!$ with $\Gamma(\alpha)$ Euler's Gamma function. Thus the left and right Riemann-Liouville (R-L) derivatives are defined by

$$ {}_{\mathrm{a}}D_x{}^a f(x) = \frac{d}{dx}({}_{\mathrm{a}}I_x{}^{1-a}\, f(x)) \tag{6}$$

and

$$ {}_xD_b{}^a f(x) = -\frac{d}{dx}({}_bI_x{}^{1-a}\, f(x)). \tag{7}$$

Pointing out that the R-L derivatives of a constant c are non zero, Caputos derivative has been introduced, yielding zero for any constant. Thus, it is considered as more suitable in the description of physical systems.
In fact Caputos derivative is defined by

$$ {}_{\mathrm{a}}{}^cD_x{}^a f(x) = \frac{1}{\Gamma(1-\alpha)}\int_{\mathrm{a}}^x \frac{f'(s)}{(x-s)^a}ds \tag{8}$$

and

$$ {}_x{}^cD_b{}^a f(x) = -\frac{1}{\Gamma(1-\alpha)}\int_x^b \frac{f'(s)}{(s-x)^a}ds. \tag{9}$$

Evaluating Caputos derivatives for functions of the type
$f(x) = (x-a)^n or\, f(x) = (b-x)^n$ we get

$$ {}_a{}^cD_x{}^a(x-\mathrm{a})^\nu = \frac{\Gamma(\nu+1)}{\Gamma(-\alpha+\nu+1)}(x-\mathrm{a})^{\nu-\alpha} \tag{10}$$

and for the corresponding right Caputos derivative

$$ {}_xD_b{}^a(b-x)^\nu = \frac{\Gamma(\nu+1)}{\Gamma(-\alpha+\nu+1)}(b-x)^{\nu-\alpha}$$

.

Likewise, Caputos derivatives are zero for constant functions

$$ f(x) = c. \tag{11}$$

3 The Geometry of Fractional Differential

It is reminded, the n-fold integral of the primitive function $f(x)$, Eq.(1) is

$$ I^n f(x) = \int_{\mathrm{a}}^x f(s)\,(ds)^n \tag{12}$$

which is real for any positive or negative increment ds. Passing to the fractional integral

$$ I^\alpha(f(x)) = \int_{\mathrm{a}}^x f(s)\,(ds)^\alpha \tag{13}$$

the integer n is simply substituted by the fractional number . Nevertheless, that substitution is not at all straightforward. The major difference between passing from Eq.(11) to Eq.(12) is that although $(ds)^n$ is real for negative values of ds, $(ds)^\alpha$ is complex. Therefore, the fractional integral, Eq.(13), is not compact for any increment ds. Hence the integral of Eq.(13) is misleading. In other words, the differential, necessary for the existence of the fractional integral, Eq.(13), is wrong. Hence, a new fractional differential, real and valid for positive and negative values of the increment ds, should be established. It is reminded that the a-Fractional differential of a function f(x) is defined by, [17]

$$ d^a f(x) = {}_{\mathrm{a}}{}^cD_x{}^a f(x)(dx)^a. \tag{14}$$

It is evident that the fractional differential, defined by Eq.(14), is valid for positive incremental dx, whereas for negative ones, that differential might be complex. Hence considering for the moment that the increment dx is positive, and recalling that ${}_{a}^{c}D_{x}^{a}x \neq 1$, the a-fractional differential of the variable x is

$$d^{a}x = {}_{a}^{c}D_{x}^{a}x(dx)^{a}. \quad (15)$$

Hence

$$d^{a}f(x) = \frac{{}_{a}^{c}D_{x}^{a}f(x)}{{}_{a}^{c}D_{x}^{a}x}d^{a}x. \quad (16)$$

It is evident that $d^{a}f(x)$ is a non-linear function of dx, although it is a linear function of $d^{a}x$. That fact suggests the consideration of the fractional tangent space that we propose. Now the definition of fractional differential, Eq.(16), is imposed either for positive or negative variable differentials $d^{\alpha}x$. In addition the proposed L-fractional (in honour of Leibnitz) derivative ${}_{0}^{L}D_{x}^{a}f(x)$ is defined by,

$$d^{a}f(x) = {}_{0}^{L}D_{x}^{a}f(x)d^{a}x \quad (17)$$

with the Leibnitz L-fractional derivative,

$${}_{0}^{L}D_{x}^{a}f(x) = \frac{{}_{a}^{c}D_{x}^{a}f(x)}{{}_{a}^{c}D_{x}^{a}x}. \quad (18)$$

Hence only Leibnizs derivative has any geometrical of physical meaning. In addition, Eq.(3), is deceiving and the correct form of Eq.(3), should be substituted by,

$$f(x) - f(\mathrm{a}) = {}_{\mathrm{a}}^{L}I_{x}^{a}\left({}_{\mathrm{a}}^{L}D_{x}^{\alpha}f(x)\right) = \frac{1}{\Gamma(\alpha)\Gamma(2-a)}\int_{a}^{x}\frac{(s-\mathrm{a})^{1-\alpha}}{(x-s)^{1-a}}{}_{0}^{L}D_{x}^{\alpha}f(s)ds. \quad (19)$$

It should be pointed out that the correct forms are defined for the fractional differential by Eq.(17), the Leibniz derivative, Eq.(18), and the fractional integral by Eq.(19). All the other forms are misleading. Configurating the fractional differential, along with the first fractional differential space (fractional tangent space), the function y=f(x) has been drawn in Fig.1, with the corresponding first differential space at a point x, according to Adda's definition, Eq.(14).

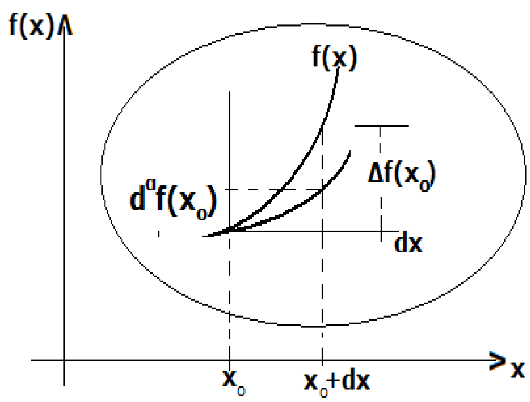

Fig. 1: he non-linear differential of f(x).

The tangent space , according to Adda's [17] definition, Eq. (14), is configurated by the nonlinear curve $d^{\alpha}f(x)$ versus dx. Nevertheless, there are some questions concerning the correct picture of the configuration, Fig. 1, concerning the fractional differential presented by Adda [17]. Indeed,

(a)The tangent space should be linear. There is not conceivable reason for the nonlinear tangent spaces.

(b)The differential should be configured for positive and negative increments dx. However, the tangent spaces, in the present case, do not exist for negative increments dx.
(c)The axis $d^{\alpha} f(x)$, in Fig.1, presents the fractional differential of the function f(x), however dx denotes the conventional differential of the variable x. It is evident that both axes along x and f(x) should correspond to differentials of the same order.

Therefore, the tangent space (first differential space), should be configured in the coordinate system with axes $(d^{\alpha}x,\ d^{\alpha} f(x))$. Hence, the fractional differential, defined by Eq. (17), is configured in the plane $(d^{\alpha}x,\ d^{\alpha} f(x))$ by a line, as it is shown in Fig.2.

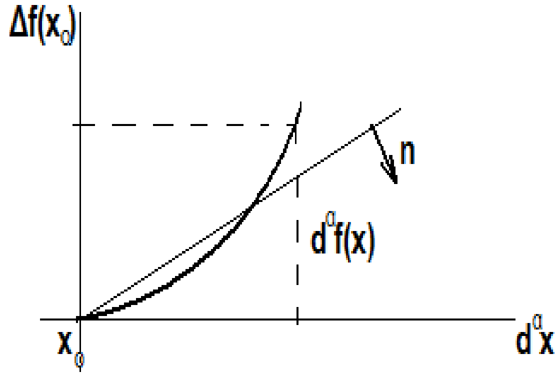

Fig. 2: The virtual tangent space of the f(x) at the point $x = x_0$.

It is evident that the differential space is not tangent (in the conventional sense) to the function at x_0, but intersects the figure $y = f(x)$ at least at one point x_0. This space, we introduce, is the tangent space. Likewise, the normal is perpendicular to the line of the fractional tangent. Hence we are able to establish Fractional Differential Geometry of curves and surfaces with the Fractional Field Theory. Consequently when $\alpha = 1$, the tangent spaces, we propose, coincide with the conventional tangent spaces.

4 The L-Fractional Chain Rule and Fractional Differential

It is evident the fractional differential may be valid when the variable changes. Nevertheless the chain rule is not valid in Fractional Calculus, [12], p.80. Therefore the idea of fractional differential may seem useless. However variation of chain rule may be considered for L-Fractional derivatives. Let us consider the function

$$f(x) = x^{\beta}, \tag{20}$$

where β may be a rational number. Then the L-fractional derivative is defined by

$$ {}_{a}^{L}D_{x}^{a} f(x) = \frac{\Gamma(\beta+1)\Gamma(2-\alpha)}{\Gamma(\beta+1-\alpha)} x^{\beta-1}. \tag{21a}$$

In case that $x = t^{\gamma}$, the L-fractional derivative is

$$ {}^{L}D_{t}^{a} f(x) = {}^{L}D_{t}^{a} t^{\gamma} = \frac{\Gamma(\gamma+1)\Gamma(2-\alpha)}{\Gamma(\gamma+1-\alpha)} x^{\gamma-1}. \tag{21b}$$

Hence the conventional chain rule applied in the present case yields

$$ {}^{L}D_{t}^{a} x(t) = \frac{\Gamma(\gamma+1)\Gamma(\beta+1)(\Gamma(2-\alpha))^{2}}{\Gamma(\gamma+1-\alpha)\Gamma(\beta+1-\alpha)} x^{\beta-1}\cdot t^{\gamma-1}. \tag{22a}$$

Nevertheless:

$$ {}^{L}D_t{}^{a} f(x(t)) = {}^{c}D_t{}^{a} = \frac{\Gamma(\beta\gamma+1)\Gamma(2-\alpha)}{\Gamma(\beta\gamma+1-\alpha)} t^{\beta\gamma-1}. \tag{22b} $$

Hence a fractional chain rule may be established for any monomial $f(x) = x^{\beta}$ where $x = t^{\gamma}$ with

$$ {}^{L}D_t{}^{a} f(x(t)) = \kappa_{\gamma}{}^{\beta\, L}D_x{}^{a} f(x)\, {}^{L}D_t{}^{a} x(t) \tag{23} $$

with

$$ \kappa_{\gamma}{}^{\beta} = \frac{\Gamma(\beta\gamma+1)\Gamma(\beta+1-\alpha)\Gamma(\gamma+1-\alpha)}{\Gamma(\beta\gamma+1-\alpha)\Gamma(\beta+1)\Gamma(\gamma+1)\Gamma(2-\alpha)}. \tag{24} $$

Example:
Let us consider $\beta = 2.5$ and $\gamma = \frac{1}{3}$
then

$$ \kappa_{1/3}{}^{2.5} = \frac{\Gamma(2.5/3+1)\Gamma(2.5-\alpha)\Gamma(1/3+1-\alpha)}{\Gamma(2.5/3+1-a)\Gamma(2.5+1)\Gamma(1/3+1)\Gamma(2-\alpha)}. \tag{25} $$

Furthermore

$$ {}_{a}{}^{L}D_x{}^{a} f(x) = {}_{a}{}^{L}D_x{}^{a} x^{2.5} = \frac{\Gamma(2.5+1)\Gamma(2-\alpha)}{\Gamma(2.5+1-a)} x^{1.5} \tag{26a} $$

and

$$ {}^{L}D_t{}^{a} x(t) = {}^{L}D_t{}^{a} t^{1/3} = \frac{\Gamma(1/3+1)\Gamma(2-\alpha)}{\Gamma(1/3+1-a)} t^{-2/3}. \tag{26b} $$

Hence,

$$ {}^{L}D_t{}^{a} f(x(t)) = k_{1/3}{}^{2.5}\, {}_{\mathrm{a}}{}^{L}D_x{}^{\alpha} f(x)\, {}^{L}D_t{}^{a} x(t) = \frac{\Gamma(2.5/3+1)\Gamma(2-\alpha)}{\Gamma(2.5/3+1-a)} t^{-0.5/3}. \tag{27} $$

Although the procedure of L-fractional chain rule is valid up to now for rational monomials, it may be valid for any rational polynomial

$$ f(x) = \sum{}^{\beta_i} y^{\beta_t} \text{ with } x = \sum_{\gamma_j} t^{\gamma_i}. \tag{28} $$

Likewise the Fractional Chain rule for rational polynomials may be expressed by:

$$ {}^{L}D_t{}^{a} x(t) = \sum{}^{\beta_i} \sum_{\gamma_j} \kappa_{\gamma_j}{}^{\beta_i\, L}D_x{}^{a} f(x)\, {}^{L}D_t{}^{a} x(t). \tag{29} $$

However, another view, maybe more physical, may reject the chain rule, since the influence of both the β and γ should be present and the path from β to γ should be expressed. That quite physical idea cancels the independence of the L-chain rule from the powers β and γ.

5 The Fractional Arc Length

Let y=f(x) be a function, which may be non-differentiable but has a fractional derivative of order , $0 < \alpha < 1$. The fractional differential of y=f(x) in the differential space is defined by

$$ d^{a}y = \frac{{}_{0}D_x{}^{a} f(x)}{{}_{0}D_x{}^{a} x} d^{a}x = {}_{0}{}^{L}D_x{}^{a} f(x) d^{a}x. \tag{30} $$

Therefore, the arc length is defined by

$$ s_1(x,a) = {}_{0}I_x{}^{a}[(d^{a}y)^2 + (d^{a}x)^2]^{1/2} = {}_{0}I_x{}^{a}[(\frac{{}_{0}D_x{}^{a} f(x)}{{}_{0}D_x{}^{a} x})^2 + 1]^{\frac{1}{2}} d^{a}x. \tag{31} $$

Furthermore, for parametric curves of the type

$$y = f(t), \quad x = g(t). \tag{32}$$

The fractional -differentials are defined by:

$$\begin{aligned} d^a x &= \frac{{}_0D_t{}^a g(t)}{{}_0D_t{}^a t} d^a t, \\ d^a y &= \frac{{}_0D_t{}^a f(t)}{{}_0D_t{}^a t} d^a t \end{aligned} \tag{33}$$

and the fractional differential of the arc-length is expressed by

$$d^a s = \sqrt{(d^a y)^2 + (d^a x)^2} = [(\frac{{}_0D_t{}^a f(t)}{{}_0D_t{}^a t})^2 + (\frac{{}_0D_t{}^a g(t)}{{}_0D_t{}^a t})^2]^{\frac{1}{2}} d^a t \tag{34}$$

and

$$\begin{aligned} s &= {}_0{}^L I_x{}^a d^a s = {}_0{}^L I_t{}^a [(\frac{{}_0D_t{}^a f(t)}{{}_0D_x{}^a t})^2 + (\frac{{}_0D_t{}^a g(t)}{{}_0D_x{}^a t})^2]^{\frac{1}{2}} d^\alpha t \\ &= {}_0{}^L I_t{}^a [({}_0{}^L D_t{}^a f(t))^2 + ({}_0{}^L D_t{}^a g(t))^2]^{\frac{1}{2}} d^\alpha t. \end{aligned} \tag{35}$$

6 The Fractional Tangent Space

Let $\mathbf{r} = \mathbf{r}(s)$ be a natural representation of a curve C, where s is the -fractional length of the curve. Since the velocity of a moving material point on the curve r(s) defines the tangent space, the fractional tangent space of the curve $\mathbf{r} = \mathbf{r}(s)$ is defined by the first derivative

$$\mathbf{r}_1 = \frac{d^a \mathbf{r}}{d^a s} = \frac{{}_0D_s{}^a \mathbf{r}}{{}_0D_s{}^a s} = {}_0{}^L D_s{}^a \mathbf{r}. \tag{36}$$

Recalling

$$d^a |\mathbf{r}| = d^a s \tag{37}$$

the length $|\mathbf{r}_1|$ of the fractional tangent vector is unity.

The tangent space line of the curve $\mathbf{r} = \mathbf{r}(s)$ at the point $r_0 = r(s_0)$ is defined by

$$\mathbf{r} = \mathbf{r}_0 + k\mathbf{t}_0 \qquad 0 < k < \infty, \tag{38}$$

where $\mathbf{t}_0 = \mathbf{t}(s_0)$ is the unit tangent vector at $\mathbf{r}$.

The plane through $\mathbf{r}_0$, orthogonal to the tangent line at $\mathbf{r}_0$, is called the normal plane to the curve C at s_0. The points of that orthogonal plane are defined by

$$(\mathbf{y} - \mathbf{r}_0) \cdot \mathbf{t}(s_0) = (\mathbf{y} - \mathbf{r}_0) \cdot \mathbf{r}_1(s_0) = 0. \tag{39}$$

7 Fractional Curvature of Curves

Considering the fractional tangent vector

$$\mathbf{t} = \mathbf{r}_1(s) = \frac{{}_0D_s{}^a \mathbf{r}}{{}_0D_s{}^a s} = {}_0{}^L D_s{}^a \mathbf{r} \tag{40}$$

its fractional derivative may be considered

$$\mathbf{r}_2(s) = \frac{d^a \mathbf{t}}{d^a s} = \frac{{}_0D_s{}^a \mathbf{t}}{{}_0D_s{}^a s} = {}_0{}^L D_s{}^a \mathbf{t} = \mathbf{t}_1(s). \tag{41}$$

The vector $t_1(s)$ is called the fractional curvature vector on C at the point r(s) and is denoted by $\kappa = \kappa(s) = \mathbf{t_1}(s)$.

Since is a unit vector

$$\mathbf{t}\cdot\mathbf{t}=1. \tag{42}$$

Restricted to fractional derivatives that yield zero for a constant function, such as Caputos derivatives, the curvature vector $\mathbf{t}_1(s)$ on C is orthogonal to $\mathbf{t}$ and parallel to the normal plane. The magnitude of the fractional curvature vector:

$$\kappa=|\kappa(s)| \tag{43}$$

is called the fractional curvature of C at r(s).The reciprocal of the curvature k is the fractional radius of curvature at r(s)

$$\rho=\frac{1}{\kappa}=\frac{1}{|\kappa(s)|}. \tag{44}$$

8 The Fractional Radius of Curvature of a Curve

Following Porteous [37], for the fractional curvature of a plane curve $\mathbf{r}$, we study at each point $\mathbf{r}(t)$ of the curve, how closely the curve approximates there to a parameterized circle. Now in the tangent or first differential space at a point $\mathbf{r}(t_0)$, the circle, with centre c and radius ρ, consists of all **r(t)** in the differential space such that

$$(\mathbf{r}-\mathbf{c})\cdot(\mathbf{r}-\mathbf{c})=\rho^2. \tag{45}$$

Further Eq.(35) yields

$$\mathbf{c}\cdot\mathbf{r}-\frac{1}{2}\mathbf{r}\cdot\mathbf{r}=\frac{1}{2}(\mathbf{c}\cdot\mathbf{c}-\rho^2) \tag{46}$$

with the right hand side been constant. Therefore, the derivation of the function

$$V(c):t\to\mathbf{c}\cdot\mathbf{r}(t)-\frac{1}{2}\mathbf{r}(t)\cdot\mathbf{r}(t). \tag{47}$$

Hence

$$V(c)_1=(\mathbf{c}-\mathbf{r}(t))\cdot\mathbf{r_1}(t)=0, \tag{48}$$

$$V(c)_2=(\mathbf{c}-\mathbf{r}(t))\cdot\mathbf{r_2}(t)-\mathbf{r_1}(t)\cdot\mathbf{r_1}(t)=0. \tag{49}$$

Suppose that r is a parametric curve with $\mathbf{r}(t)$ in the virtual tangent space. Then $V(c)_1(t)=0$ when the vector $\mathbf{c}-\mathbf{r}(t)$ in the tangent space is orthogonal to the tangent vector $\mathbf{r}_1(t)$. Indeed when the point c, in the tangent space, lies on the normal to $\mathbf{r}_1(t)$ at t, the line through is orthogonal to the tangent line. When $\mathbf{r_2}(t)$ is not linearly dependent upon $\mathbf{r_1}(t)$, there will be a unique point $\mathbf{c}\neq\mathbf{r}(t)$,on the normal line, such that also $V(\mathbf{c})_2(t)=0$.

9 The Serret-Frenet Equations

Let r be a curve with unit speed, where the fractional velocity vector [36]

$$\mathbf{t}(s)=\mathbf{r}_1(s)=\frac{{}_0{}^cD_s{}^a\mathbf{r}(s)}{{}_0{}^cD_s{}^a s}={}_0{}^LD_s{}^a\mathbf{r}(s) \tag{50}$$

is of unit length. Let **r(s)** be such a curve. The vector

$$\mathbf{t}_1(s)=\mathbf{r}_2(s)=\frac{{}_0{}^cD_s{}^a\mathbf{r}_1(s)}{{}_0{}^cD_s{}^a s}=\frac{{}_0{}^cD_s{}^a}{{}_0{}^cD_s{}^a s}\left(\frac{{}_0{}^cD_s{}^a\mathbf{r}(s)}{{}_0{}^cD_s{}^a s}\right)={}_0{}^LD_s{}^a\left({}_0{}^LD_s{}^a\mathbf{r}(s)\right) \tag{51}$$

is normal to the curve **r= r(s)** since $\mathbf{t}(s)\cdot\mathbf{t}(s)=1$ and

$$\mathbf{t_1}(s)\cdot\mathbf{t}(s)=0 \tag{52}$$

since for Caputos derivative ${}_0D_s{}^a c = 0$ for any constant c.

Consider $\mathbf{t}_1(s) = \kappa(s)\mathbf{n}(s)$, where $\mathbf{n}(s)$ is the unit principal normal to r at s, provided that $\kappa(s) \neq 0$ where $\kappa(s)$ is the curvature of r at s.

Hence the equations for the focal line are defined by:

$$\begin{aligned}(\mathbf{c}-\mathbf{r}(s))\cdot\mathbf{r}_1(s) &= 0 \quad ,\\ (\mathbf{c}-\mathbf{r}(s))\cdot\kappa(s)\mathbf{n}(s) &= 1.\end{aligned} \tag{53}$$

Thus, the principal centre of curvature c at s is the point $\mathbf{r}(s)+\rho(s)\mathbf{n}(s)$, where $\rho(s) = \frac{1}{\kappa(s)}$.

Furthermore, the principal normal vector n(s) orthogonal to the tangent line is pointing towards the focal line (locus of the curvature centers). Likewise, the (unit) binormal b(s) is defined to be the vector $\mathbf{t}(s)\times\mathbf{n}(s)$, the triad of unit vectors $\mathbf{t}$(s), $\mathbf{n}$(s), $\mathbf{b}$(s) forming a right-handed orthonormal basis for the tangent vector space to the curvature $\mathbf{r}$(s).
Each of the derivative vectors $\mathbf{t_1}(s),\ \mathbf{n_1}(s),\ \mathbf{b_1}(s)$ linearly depends on $\mathbf{t}(s),\ n(s),\ \mathbf{b}(s)$. Considering the equations:

$\mathbf{t_1}\cdot\mathbf{t} = 0$ and $\mathbf{t_1}\cdot\mathbf{n} = 0$ with $\mathbf{t_1}\cdot\mathbf{n}+\mathbf{n_1}\cdot\mathbf{t} = 0$, we get the fractional Serret-Frenet equations:

$$\begin{aligned}\mathbf{t_1} &= \kappa\mathbf{n},\\ \mathbf{n_1} &= -\kappa\mathbf{t}+\tau\mathbf{b},\\ \mathbf{b_1} &= -\tau\mathbf{n}.\end{aligned} \tag{54}$$

The coefficient is defined to be the torsion of the curve r. These equations are the Fractional Equations for the fractional Serret Frenet system. Considering plane curves,

$$\mathbf{r}(x) = x\mathbf{i}+y(x)\mathbf{j}. \tag{55}$$

Eqs.(38,39), defining the fractional centres of curvature $\mathbf{c} = c_x\mathbf{i}+c_y\mathbf{j}$ become,

$$\begin{aligned}&(c_x-x)+(c_y-y(x))\ {}_0{}^LD_x{}^\alpha y(x) = 0\\ &(c_y-y(x))\,{}_0{}^LD_x{}^\alpha\left({}_0{}^LD_x{}^\alpha y(x)\right)-\left(1+{}_0{}^LD_x{}^\alpha y(x)^2\right) = 0.\end{aligned} \tag{56}$$

Since, the fractional radius of curvature is defined by

$$\rho^\alpha = \rho_x{}^\alpha\mathbf{i}+\rho_y{}^\alpha\mathbf{j} = (c_x-x)\,\mathbf{i}+(c_y-y(x))\,\mathbf{j} \tag{57}$$

the components of the fractional curvature are given by

$$\begin{aligned}\rho_x{}^\alpha &= -\frac{1+{}_0{}^LD_s{}^\alpha y(x)^2}{{}_0{}^LD_s{}^\alpha\left({}_0{}^LD_s{}^\alpha y(x)\right)}\ {}_0{}^LD_s{}^\alpha y(x),\\ \rho_y{}^\alpha &= \frac{1+{}_0{}^LD_s{}^\alpha y(x)^2}{{}_0{}^LD_s{}^\alpha\left({}_0{}^LD_s{}^\alpha y(x)\right)}.\end{aligned} \tag{58}$$

Further, for the case of $|y(x)| << 1$, that we consider in linear bending, Eq.(21) yields

$$d^a s = d^a x+o(d^a x)^2 \tag{59}$$

with

$${}_0{}^LD_s{}^\alpha\,() = {}_0{}^LD_x{}^\alpha\,() \tag{60}$$

and

$$\rho^\alpha = |\mathbf{r}^\alpha| \approx \frac{1}{{}_0{}^LD_x{}^\alpha\left({}_0{}^LD_x{}^\alpha y(x)\right)}. \tag{61}$$

Let us consider a fractional beam with its source point (x,y,z)=(0,0,0). That means, the fractional u Caputos derivatives of any function, concerning the beam, are defined by,

$${}_0{}^cD_u{}^a f(u) = \frac{1}{\Gamma(1-\alpha)}\int_0^u \frac{f'(s)}{(u-s)^a}ds,$$

where, u might be one of the variables (x,y,z).

10 Applications

a. The Fractional Geometry of a parabola. Let r be a parabola $t \to (t, t^2)$. Then, we have

$$\mathbf{r}(t) = t\mathbf{e_1} + t^2\mathbf{e_2}. \tag{62}$$

Hence

$$\mathbf{r}_1(t) = \mathbf{e_1} + \frac{{}_0{}^cD_t{}^a(t^2)}{{}_0{}^cD_t{}^a t}\mathbf{e_2} = \mathbf{e_1} + \frac{2t}{2-a}\mathbf{e_2} \tag{63}$$

and

$$\mathbf{r}_2(t) = \frac{2}{2-a}\mathbf{e_2}.$$

Then, the centers of curvature of the parabola describe a curve:

$$\mathbf{c}(t) = c_1(t)\mathbf{e_1} + c_2(t)\mathbf{e_2}. \tag{64}$$

Satisfying Eqs.(48 and 49) with

$$c_1 + \frac{2t}{2-a}c_2 = t + \frac{2t^3}{2-a}, \tag{65}$$

$$\frac{2}{2-a}c_2 = \frac{2t^2}{2-a} + 1 + \frac{4t^2}{(2-a)^2}. \tag{66}$$

Solving the system of Eqs.(65,66) we get

$$c_1 = -\frac{4t^3}{(-2+a)^2}, \tag{67}$$

$$c_2 = -\frac{4+a^2+8t^2-2a(2+t^2)}{(4-2a)}. \tag{68}$$

Fig.3 shows the tangent space of the parabola at the point t=1.5 for various values of the fractional dimension $\alpha = (1, 0.7, 0.3)$.

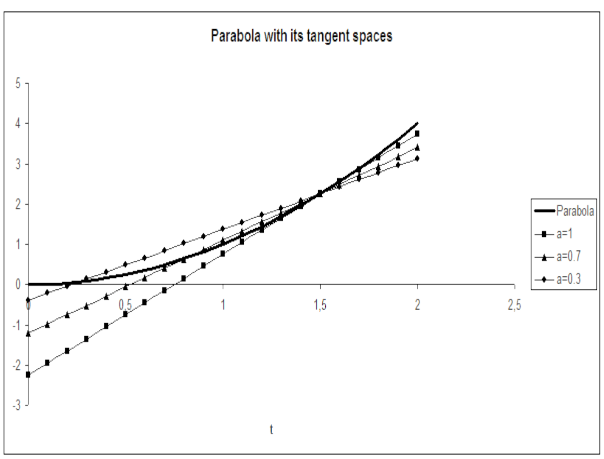

Fig. 3: The parabola with its tangent spaces at $t = 1.5 for \alpha = 1, \alpha = 0.7, \alpha = 0.3$.

It is clear that the tangent spaces for $\alpha = 0.7$ and $\alpha = 0.3$ intersect the parabola at the point t=1.5, although the conventional tangent space with fractional dimension $\alpha = 1.0$ touches the parabola at t=1.5. Furthermore the centers of curvature for various values of the fractional dimension α the point t=1,5 are (for the conventional case)

$\alpha = 1.0$	c_1= -13.5	and c_2=7.25
$\alpha = 0.7$	c_1= -7.98	and c_2=6.36
$\alpha = 0.3$	c_1= -4.67	and c_2=5.75

b. The tangent and curvature center of the Weierstrass function. Let us consider the function

$$W(t) = \sum_{n=1}^{\infty} \lambda^{-\alpha n} \left\{ \sin\left(\frac{\lambda^n t}{2}\right) - \sin\left(2\lambda^n t\right) \right\} \tag{69}$$

the well known Weierstrass function, continuous with discontinuous conventional derivatives at any point, [21]. The parameter has been proved to be related to the fractional dimension of the function W(t). Restricting the function to w(t) with

$$w(t) = \sum_{n=1}^{6} \lambda^{-\alpha n} \left\{ \sin\left(\frac{\lambda^n t}{2}\right) - \sin\left(2\lambda^n t\right) \right\} \tag{70}$$

and for $\alpha = 0.5$ and $\lambda = 2$, the fractional tangent to the curve at the point t=1.0 has been drawn, Fig.4, with the help of the Mathematica computerized pack.

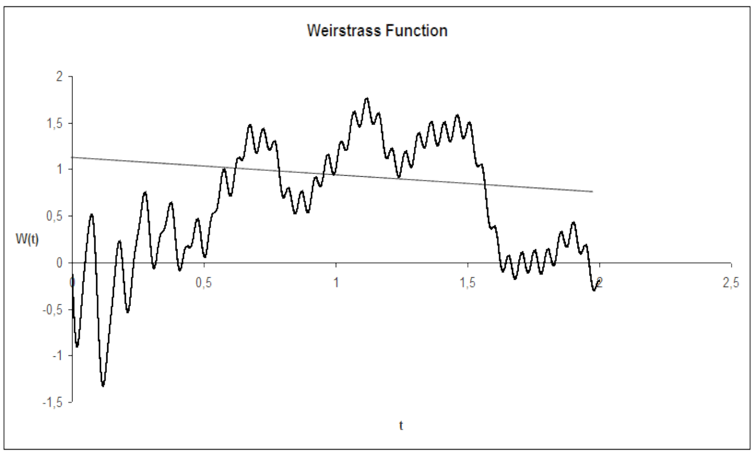

Fig. 4: The function w(t) with its fractional $(\alpha = 0.5)$ tangent at t=1.0.

c. Bending of fractional beams
Considering the pure fractional bending problem of a beam with microcracks, microvoids, various other defects, we get the fractional strain,

$$\varepsilon_{xx}{}^{\alpha} = -\frac{y}{\rho^{\alpha}}, \tag{71}$$

where the fractional curvature is defined by

$$\frac{1}{\rho^{\alpha}} = D_2 w(x) = \frac{1}{{}_0D_x{}^{\alpha} x} {}_0D_x{}^{\alpha} \left(\frac{{}_0D_x{}^{\alpha} w(x)}{{}_0D_x{}^{\alpha} x} \right) \tag{72}$$

with w(x) denoting the elastic line of the beam. Likewise, the fractional bending moment is expressed by:

$$M = -2\int_0^{h/2} \sigma_{xx}{}^{\alpha} y\, d^{\alpha} y = \frac{EI^{\alpha}}{\rho^{\alpha}} \tag{73}$$

with the fractional stress, see Lazopoulos and Lazopoulos [35],

$$\sigma_{xx}{}^{\alpha} = -\frac{M}{I^{\alpha}} y. \tag{74}$$

Hence the fractional bending of beams formula is revisited and expressed as

$$M = EI^{\alpha} D_2 w(x) = \frac{EI^{\alpha}}{{}_0^c D_x{}^{\alpha} x}\, {}_0^c D_x{}^{\alpha} \left(\frac{{}_0^c D_x{}^{\alpha} w(x)}{{}_0^c D_x{}^{\alpha} x} \right), \tag{75}$$

Therefore, the deflection curve w(x) is defined by

$$w(x) = \int_0^x \left(\int_0^s \frac{M(t)}{EI^{\alpha}}\, d^{\alpha} t \right) d^{\alpha} s + c_1 x + c_2. \tag{76}$$

In conventional integration, the deflection curve is defined by

$$w(x) = \int_0^x \frac{s^{1-\alpha}}{\Gamma(2-\alpha)} \left(\int_0^s \frac{M(t)}{EI^{\alpha}} \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \frac{1}{\Gamma(\alpha)(s-t)^{1-\alpha}} dt \right) \frac{ds}{(x-s)^{1-\alpha}} + c_1 x + c_2. \tag{77}$$

11 The Fractional Tangent Plane of a Surface

Let us consider a manifold, with points M(u,v), defined by the vectors

$$M(u,v) = \mathbf{x}(u,v) \tag{78}$$

with

$$x_i = x_i(u,v), \qquad u_1 \leqslant u \leqslant u_2, \quad v_1 \leqslant v \leqslant v_2, \qquad i = 1,2,3. \tag{79}$$

The infinitesimal distance between two points P and Q on the manifold M is defined by

$$d^{\alpha}\mathbf{x} = \frac{{}_0^c D_u{}^{\alpha} \mathbf{x}}{{}_0^c D_u{}^{\alpha} u} d^{\alpha} u + \frac{{}_0^c D_u{}^{\alpha} \mathbf{x}}{{}_0^c D_u{}^{\alpha} v} d^{\alpha} v. \tag{80}$$

In fact for the surface

$$z = u^2 v^2 \tag{81}$$

see, Fig. 2, the tangent space according to Eq.(16) is expressed by

$$d^{\alpha}\mathbf{r} = d^{\alpha} x\,\mathbf{i} + d^{\alpha} y\,\mathbf{j} + \frac{2xy}{(2-\alpha)} \left(y d^{\alpha} x + x d^{\alpha} y \right) \mathbf{k}. \tag{82}$$

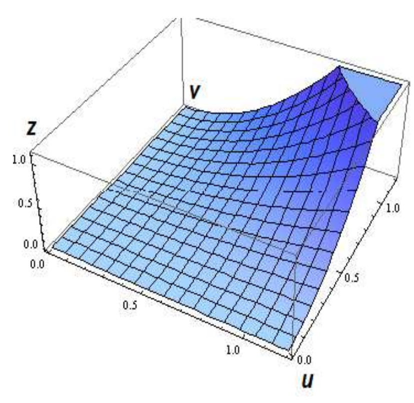

Fig. 5: The surface $z = u^2 v^2$.

Fig.6. shows the surface defined by Eq.(72) with its fractional tangent plane (space) at the point (u,v)=(0.5, 0.5) for two fractional dimensions, $\alpha = 1$ (the conventional case) and $\alpha = 0.3$. It is clear that the fractional tangent plane is different from the conventional one $(\alpha = 1)$.

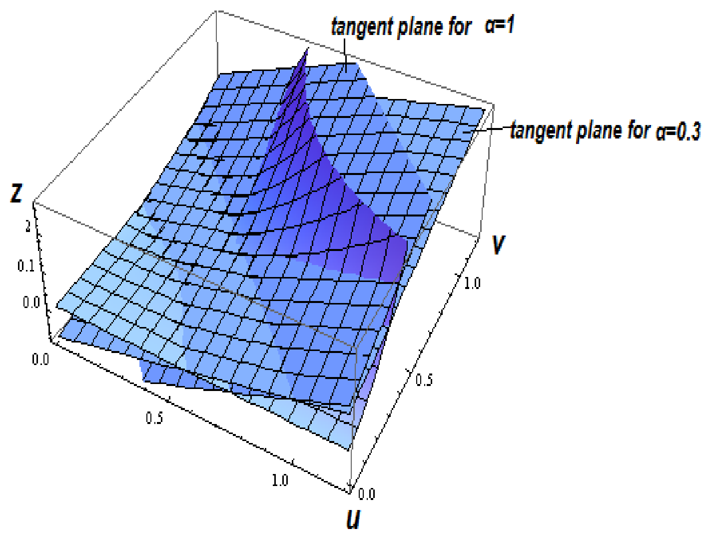

Fig. 6: The tangent planes for various values of the fractional dimension α.

12 Fundamental Differential Forms on Fractional Differential Manifolds

a. The First Fractional Fundamental Form Following formal procedure [36], the quantity

$$\begin{aligned} I^{\alpha} &= d^{\alpha}\mathbf{x}\cdot d^{\alpha}\mathbf{x} = \left(\frac{{}_0^cD_u{}^{\alpha}\mathbf{x}}{{}_0^cD_u{}^{\alpha}u}d^{\alpha}u+\frac{{}_0^cD_v{}^{\alpha}\mathbf{x}}{{}_0^cD_v{}^{\alpha}v}d^{\alpha}v\right)\cdot\left(\frac{{}_0^cD_u{}^{\alpha}\mathbf{x}}{{}_oD_u{}^{\alpha}u}d^{\alpha}u+\frac{{}_0^cD_v{}^{\alpha}\mathbf{x}}{{}_oD_v{}^{\alpha}v}d^{\alpha}v\right) \\ &= E\,d^{\alpha}u^2+2Fd^{\alpha}u\,d^{\alpha}v+G\,d^{\alpha}v^2 \end{aligned} \tag{83}$$

defined upon the tangent space of the manifold, as it has been clarified earlier, the I^{α} stands for the first fractional differential form, with the dot meaning the inner product.

$$\begin{aligned} E &= \frac{{}_0^cD_u{}^{\alpha}\mathbf{x}}{{}_0^cD_u{}^{\alpha}u}\cdot\frac{{}_0^cD_u{}^{\alpha}\mathbf{x}}{{}_0^cD_u{}^{\alpha}u}, \\ F &= \frac{{}_0^cD_u{}^{\alpha}\mathbf{x}}{{}_0^cD_u{}^{\alpha}u}\cdot\frac{{}_0^cD_v{}^{\alpha}\mathbf{x}}{{}_0^cD_v{}^{\alpha}v}, \\ G &= \frac{{}_0^cD_v{}^{\alpha}\mathbf{x}}{{}_0^cD_v{}^{\alpha}v}\cdot\frac{{}_0^cD_v{}^{\alpha}\mathbf{x}}{{}_0^cD_v{}^{\alpha}v}, \end{aligned} \tag{84}$$

corresponding to

$$I^{\alpha} = E\,du^2+2F\,du\,dv+G\,dv^2.$$

Furthermore the first fundamental form is positive definite i.e., $0\leqslant I^{\alpha}$ with $I^{\alpha}=0$ if and only if $d^{\alpha}u$ and $d^{\alpha}v$ are equal to zero. Hence,

$$EG-F^2>0.$$

b. The Second Fractional Fundamental Form. Consider the manifold $M(u,v)=\mathbf{x}(u,v)$. Then, at each point of the manifold, there is a fractional unit normal N to the fractional tangent plane,

$$\mathbf{N}=\frac{\frac{{}_oD_u{}^{\alpha}\mathbf{x}}{{}_oD_u{}^{\alpha}u}\times\frac{{}_oD_v{}^{\alpha}\mathbf{x}}{{}_oD_v{}^{\alpha}v}}{\left|\frac{{}_oD_u{}^{\alpha}\mathbf{x}}{{}_oD_u{}^{\alpha}u}\times\frac{{}_oD_v{}^{\alpha}\mathbf{x}}{{}_oD_v{}^{\alpha}v}\right|} \tag{85}$$

that is a function of u and v with the fractional differential

$$d^{\alpha}\mathbf{N} = \frac{{}_oD_u{}^{\alpha}\mathbf{N}}{{}_oD_u{}^{\alpha}u}d^{\alpha}u + \frac{{}_oD_u{}^{\alpha}\mathbf{N}}{{}_oD_u{}^{\alpha}v}d^{\alpha}v. \quad (86)$$

Restricting only to Caputo fractional derivatives with the property of zero fractional derivative of any constant, and taking into consideration that $\mathbf{N}\cdot\mathbf{N} = 1$, we get,

$$d^{\alpha}\mathbf{N}\cdot\mathbf{N} = 0, \quad (87)$$

where the vector d^{α}N is parallel to the fractional tangent space. The second fractional fundamental form is defined by [36]

$$\begin{aligned} II^{\alpha} &= -d^{\alpha}\mathbf{x}\cdot d^{\alpha}\mathbf{N} = -\left(\frac{{}_o{}^cD_u{}^{\alpha}\mathbf{x}}{{}_oD_u{}^{\alpha}u}d^{\alpha}u + \frac{{}_o{}^cD_u{}^{\alpha}\mathbf{x}}{{}_oD_u{}^{\alpha}v}d^{\alpha}v\right)\cdot\left(\frac{{}_o{}^cD_u{}^{\alpha}\mathbf{N}}{{}_oD_u{}^{\alpha}u}d^{\alpha}u + \frac{{}_o{}^cD_u{}^{\alpha}\mathbf{N}}{{}_oD_u{}^{\alpha}v}d^{\alpha}v\right) \\ &= L\,d^{\alpha}u^2 + 2Md^{\alpha}u\,d^{\alpha}v + N\,d^{\alpha}v^2 \end{aligned} \quad (88)$$

with

$$\begin{aligned} L &= -\frac{{}_o{}^cD_u{}^{\alpha}\mathbf{x}}{{}_oD_u{}^{\alpha}u}\cdot\frac{{}_o{}^cD_u{}^{\alpha}\mathbf{N}}{{}_oD_u{}^{\alpha}u}, \\ M &= -\frac{1}{2}\left(\frac{{}_o{}^cD_u{}^{\alpha}\mathbf{x}}{{}_o{}^cD_u{}^{\alpha}u}\cdot\frac{{}_o{}^cD_v{}^{\alpha}\mathbf{N}}{{}_o{}^cD_v{}^{\alpha}v} + \frac{{}_o{}^cD_u{}^{\alpha}\mathbf{N}}{{}_o{}^cD_u{}^{\alpha}u}\cdot\frac{{}_o{}^cD_v{}^{\alpha}\mathbf{x}}{{}_o{}^cD_v{}^{\alpha}v}\right), \\ N &= -\frac{{}_o{}^cD_v{}^{\alpha}\mathbf{x}}{{}_o{}^cD_v{}^{\alpha}v}\cdot\frac{{}_o{}^cD_u{}^{\alpha}\mathbf{N}}{{}_o{}^cD_v{}^{\alpha}v}. \end{aligned} \quad (89)$$

It is pointed again that the geometric procedures, that use quantities not defined upon the correct tangent spaces, are questionable. Even if analytically may yield the same results, geometrically are confusing.

13 The Fractional Normal Curvature

Let P be a point on a surface $\mathbf{x} = \mathbf{x}(u,v)$ and $\mathbf{x}(t) = \mathbf{x}(u(t), v(t))$ a regular curve C at P. The fractional curvature of curves has been discussed in chapter 6. The normal curvature $\mathbf{k}_n{}^{\alpha}$ vector of C at P is the vector projection of the curvature vector $\mathbf{k}^{\alpha}$ onto the normal vector N at P. The component of $\mathbf{k}^{\alpha}$ in the direction of the normal N is called the normal fractional curvature of C at P and is denoted by $k_n{}^{\alpha}$. Therefore,

$$k_n{}^{\alpha} = \mathbf{k}^{\alpha}\cdot\mathbf{N}. \quad (90)$$

Since the unit tangent to C at P is the vector,

$$\mathbf{t} = \frac{d^{\alpha}\mathbf{x}}{d^{\alpha}s} = \frac{d^{\alpha}\mathbf{x}}{d^{\alpha}t}\Big/\left|\frac{d^{\alpha}\mathbf{x}}{d^{\alpha}t}\right|, \quad (91)$$

where s denotes the fractional arc length of the curve and t is the unit perpendicular to the normal N along the curve, we get,

$$0 = \frac{d^{\alpha}(\mathbf{t}\cdot\mathbf{N})}{d^{\alpha}t} = \frac{d^{\alpha}\mathbf{t}}{d^{\alpha}t}\cdot\mathbf{N} + \mathbf{t}\cdot\frac{d^{\alpha}\mathbf{N}}{d^{\alpha}t}. \quad (92)$$

Therefore, the normal curvature of a curve is equal to

$$\begin{aligned} k_n{}^{\alpha} &= \mathbf{k}\cdot\mathbf{N} = \frac{d^{\alpha}\mathbf{t}}{d^{\alpha}t}\cdot\mathbf{N}\Big/\left|\frac{d^{\alpha}\mathbf{x}}{d^{\alpha}t}\right| = -\mathbf{t}\cdot\frac{d^{\alpha}\mathbf{N}}{d^{\alpha}t}\Big/\left|\frac{d^{\alpha}\mathbf{x}}{d^{\alpha}t}\right| \\ &= -\frac{d^{\alpha}\mathbf{x}}{d^{\alpha}t}\cdot\frac{d^{\alpha}\mathbf{N}}{d^{\alpha}t}\Big/\left|\frac{d^{\alpha}\mathbf{x}}{d^{\alpha}t}\right|^2 \\ &= -\frac{\left(\frac{{}_o{}^cD_u{}^{\alpha}\mathbf{x}}{{}_o{}^cD_u{}^{\alpha}u}\frac{d^{\alpha}u}{d^{\alpha}t} + \frac{{}_o{}^cD_u{}^{\alpha}\mathbf{x}}{{}_o{}^cD_u{}^{\alpha}v}\frac{d^{\alpha}v}{d^{\alpha}t}\right)\cdot\left(\frac{{}_o{}^cD_u{}^{\alpha}\mathbf{N}}{{}_o{}^cD_u{}^{\alpha}u}\frac{d^{\alpha}u}{d^{\alpha}t} + \frac{{}_o{}^cD_u{}^{\alpha}\mathbf{N}}{{}_o{}^cD_u{}^{\alpha}v}\frac{d^{\alpha}v}{d^{\alpha}t}\right)}{\left(\frac{{}_o{}^cD_u{}^{\alpha}\mathbf{x}}{{}_o{}^cD_u{}^{\alpha}u}\frac{d^{\alpha}u}{d^{\alpha}t} + \frac{{}_o{}^cD_u{}^{\alpha}\mathbf{x}}{{}_o{}^cD_u{}^{\alpha}v}\frac{d^{\alpha}v}{d^{\alpha}t}\right)\cdot\left(\frac{{}_o{}^cD_u{}^{\alpha}\mathbf{x}}{{}_o{}^cD_u{}^{\alpha}u}\frac{d^{\alpha}u}{d^{\alpha}t} + \frac{{}_o{}^cD_u{}^{\alpha}\mathbf{x}}{{}_o{}^cD_u{}^{\alpha}v}\frac{d^{\alpha}v}{d^{\alpha}t}\right)} \\ &= \frac{L(d^{\alpha}u/d^{\alpha}t)^2 + 2M(d^{\alpha}u/d^{\alpha}t)(d^{\alpha}v/d^{\alpha}t) + N(d^{\alpha}v/d^{\alpha}t)^2}{E(d^{\alpha}u/d^{\alpha}t)^2 + 2F(d^{\alpha}u/d^{\alpha}t)(d^{\alpha}v/d^{\alpha}t) + G(d^{\alpha}v/d^{\alpha}t)^2}. \end{aligned} \quad (93)$$

Recalling Eqs.(83, 88), the normal curvature is defined by

$$k_n{}^{\alpha} = \frac{II^{\alpha}}{I^{\alpha}}. \tag{94}$$

14 Fractional Vector Operators

In the present section the fractional tangent spaces along with their fractional normal vectors should be reminded, as they were defined in the preceding sections 4 and 5. For Cartesian coordinates, fractional generalizations of the divergence or gradient operators are defined by

$$\nabla^{(a)} f(\mathbf{x}) = grad^{(a)} f(\mathbf{x}) = \nabla_i{}^{(\alpha)} f(\mathbf{x})\, \mathbf{e}_i = \frac{{}_{\omega}{}^{c}D_i{}^{a} f(\mathbf{x})}{{}_{\omega}{}^{c}D_{\underline{i}}{}^{a} x_{\underline{i}}} \mathbf{e}_i = {}_{o}{}^{L}D_i{}^{a} f(\mathbf{x})\, \mathbf{e}_i, \tag{95}$$

where, ${}_{\omega}{}^{c}D_i{}^{a}$ are Caputo fractional derivatives of order α and the sub line meaning no contraction. Further, ${}_{o}{}^{L}D_i{}^{\alpha} f(\mathbf{x})$ is Leibnitzs derivative, Eq.(18). Hence, the gradient of the vector x is

$$\nabla^{(\alpha)}\mathbf{x} = \mathbf{I} \tag{96}$$

with I denoting the identity matrix. Consequently for a vector field

$$\mathbf{F}(x_1,x_2,x_3) = \mathbf{e}_1 F_1(x_1,x_2,x_3) + \mathbf{e}_2 F_2(x_1,x_2,x_3) + \mathbf{e}_3 F_3(x_1,x_2,x_3), \tag{97}$$

where $F_i(x_1,x_2,x_3)$ are absolutely integrable, the circulation is defined by:

$$C_L{}^{(\alpha)}(\mathbf{F}) = ({}_{\omega}I_L{}^{(a)},\mathbf{F}) = \int_L (dL,\mathbf{F}) = {}_{\omega}I^{(a)}{}_L\,(F_1 d^{\alpha}x_1) + {}_{\omega}I^{(a)}{}_L\,(F_2 d^{\alpha}x_2) + {}_{\omega}I^{(a)}{}_L\,(F_3 d^{\alpha}x_3). \tag{98}$$

Furthermore, the divergence of a vector F(x) is defined by

$$\nabla^{(a)} \cdot \mathbf{F}(x) = div^{(a)}\mathbf{F}(x) = \frac{{}_{\omega}{}^{c}D_k{}^{a}F_k(\mathbf{x})}{{}_{\omega}{}^{c}D_{\underline{k}}{}^{a}x_{\underline{k}}} = {}_{\omega}{}^{L}D_k{}^{a}F_k(\mathbf{x}), \tag{99}$$

where the sub-line denotes no contraction.
Moreover, the fractional curl $F(curl^{(a)}F(x))$ of a vector F is defined by

$$curl^{(a)}\mathbf{F} = \mathbf{e}_l\, \varepsilon_{lmn} \frac{{}_{\omega}{}^{c}D_m{}^{a}F_n}{{}_{\omega}{}^{c}D_m{}^{a}x_m} = \mathbf{e}_l\, \varepsilon_{lmn}\, {}_{\omega}{}^{L}D_m{}^{a}F_n. \tag{100}$$

A Fractional flux of the vector F expressed in Cartesian coordinates across surface S is a fractional surface integral of the field with

$$\Phi_s{}^{\alpha}(\mathbf{F}) = ({}_{\omega}I_s{}^{\alpha},\mathbf{F}) = {}_{\omega}{}^{(\alpha)} \iint_S (F_1 d^{\alpha}x_2 d^{\alpha}x_3 + F_2 d^{\alpha}x_3 d^{\alpha}x_1 + F_3 d^{\alpha}x_2 d^{\alpha}x_3). \tag{101}$$

A fractional volume integral of a triple fractional integral of a scalar field $f = f(x_1,x_2,x_3)$ is defined by

$${}_{\omega}V_{\Omega}{}^{(a)}[f] = {}_{\omega}I_{\Omega}{}^{(a)}[x_1,x_2,x_3]\, f(x_1,x_2,x_3) = {}_{\omega}{}^{(\alpha)} \iiint_{\Omega} f(x_1,x_2,x_3)\, d^{\alpha}x_1 d^{\alpha}x_2 d^{\alpha}x_3. \tag{102}$$

It should be pointed out that the triple fractional integral is not a volume integral, since the fractional derivative of a variable with respect to itself is different from one. So there is a clear distinction between the simple, double or triple integrals and the line, surface and volume integrals respectively.

15 Fractional Vector Field Theorems

In the present section only an outline of the various fractional field theorems will be presented, pointing out the non-conventional tangent spaces, along with the non-conventional unit normal that should be taken into consideration. Vector field theorems have been appeared by Tarasov [13,22] too. Nevertheless, the missing tangent spaces, do not help in accurate application of those field theorems. That is why we find different definitions of strains, stresses in various places, since the definition of the tangent spaces had not been clarified. Extensive discussion of the Fractional vector field theorems may be found in Lazopoulos [25]. The geometrically correct forms of the various field theorems are given below.

a)Fractional Green's formula

Green's theorem relates a line integral around a simple closed curve ∂B and a double integral over the plane region B with boundary. With positively oriented boundary ∂B, the conventional Greens theorem for a vector field $\mathbf{F} = F_{x_1}\mathbf{e}_1 + F_{x_x}\mathbf{e}_2$ is expressed by:

$$\int\limits_{\partial B} (F_1 dx_1 + F_2 dx_2) = \int\int\limits_{B} (\frac{\partial (F_1)}{\partial x_2} - \frac{\partial (F_2)}{\partial x_1}) dx_1 dx_2. \tag{103}$$

Recalling that

$$d^a\mathbf{x} = (d^a x_1, d^a x_2) = ({}_{\omega}{}^{c}D_{x_1}{}^{a}[x_1]\, dx_1{}^{\alpha}, {}^{c}D_{x_2}{}^{a}[x_2]\, dx_2{}^{\alpha}) \tag{104}$$

and substituting into conventional Green's theorem Eq.(104) we get

$$_{\omega}\int\limits_{\partial W}{}^{(\alpha)} (F_1\, d^{\alpha}x_1 + F_2\, d^{\alpha}x_2) = {}_{\omega}\int{}^{(\alpha)}\int\limits_{W} (\frac{{}_{\omega}{}^{c}D_{x_2}{}^{a}(F_1)}{{}_{\omega}{}^{c}D_{x_2}{}^{a}(x_2)} - \frac{{}_{\omega}{}^{c}D_{x_1}{}^{a}(F_2)}{{}_{\omega}{}^{c}D_{x_1}{}^{a}(x_1)}) d^{\alpha}x_1\, d^{\alpha}x_2. \tag{105}$$

b.Fractional Stokes formula:

Restricting in the consideration of a simple surface W, if we denote its boundary by ∂W and if F is a vector field defined on W, then the conventional Stokes Theorem asserts that

$$\oint\limits_{W} \mathbf{F}\cdot\mathbf{dL} = \oiint\limits_{W} curl\mathbf{F}\cdot\mathbf{dS}. \tag{106}$$

It yields in Cartesian coordinates

$$\begin{aligned}&\int\limits_{\partial W} (F_1 dx_1 + F_2 dx_2 + F_3 dx_3)\\ &= \int\int\limits_{W} \left(\frac{\partial (F_3)}{\partial x_2} - \frac{\partial (F_2)}{\partial F_3}\right) dx_2 dx_3 + \left(\frac{\partial (F_1)}{\partial x_3} - \frac{\partial (F_3)}{\partial x_1}\right) dx_3 dx_1 + \left(\frac{\partial (F_2)}{\partial x_1} - \frac{\partial (F_1)}{\partial x_2}\right) dx_1 dx_2,\end{aligned} \tag{107}$$

where $F(x_1,x_2,x_3) = e_1F_1(x_1,x_2,x_3) + e_2F_2(x_1,x_2,x_3) + e_3F_3(x_1,x_2,x_3)$.

In this case the fractional curl operation is defined by

$$\begin{aligned}curl_w{}^{\alpha}(\mathbf{F}) = e_l\varepsilon_{lmn}{}_{\omega}{}^{L}D_{x_m}{}^{a}(F_n) = \mathbf{e}_1\left({}_{\omega}{}^{L}D_{x_2}{}^{a}F_3 - {}_{\omega}{}^{L}D_{x_3}{}^{a}F_2\right) +\\ \mathbf{e}_2\left({}_{\omega}{}^{L}D_{x_3}{}^{a}F_1 - {}_{\omega}{}^{L}D_{x_1}{}^{a}F_3\right) + \mathbf{e}_3\left({}_{\omega}{}^{L}D_{x_1}{}^{a}F_2 - {}_{\omega}{}^{L}D_{x_2}{}^{a}F_1\right).\end{aligned} \tag{108}$$

Therefore transforming the conventional Stokes theorem into the fractional form we get

$$\begin{aligned}{}_{\omega}{}^{(\alpha)}\oint_{\partial W} (F_1 d^{\alpha}x_1 + F_2 d^{\alpha}x_2 + F_3 d^{\alpha}x_3) = {}_{\omega}{}^{(\alpha)}\iint_{W} \left\{\left({}_{\omega}{}^{L}D_{x_2}{}^{a}F_3 - {}_{\omega}{}^{L}D_{x_3}{}^{a}F_2\right) d^{\alpha}x_2 d^{\alpha}x_3\right.\\ \left. + \left({}_{\omega}{}^{L}D_{x_3}{}^{a}F_1 - {}_{\omega}{}^{L}D_{x_1}{}^{a}F_3\right) d^{\alpha}x_3 d^{\alpha}x_1 + \left({}_{\omega}{}^{L}D_{x_1}{}^{a}F_2 - {}_{\omega}{}^{L}D_{x_2}{}^{a}F_1\right) d^{\alpha}x_1 d^{\alpha}x_2\right\}.\end{aligned} \tag{109}$$

c. Fractional Gauss formula

For the conventional fields theory, let $F = e_1F_1 + e_2\ F_2 + e_3\ F_3$. be a continuously differentiable real-valued function in a domain W with boundary. Then the conventional divergence Gauss theorem is expressed by

$$\iint_{\partial W} \mathbf{F}\cdot\mathbf{dS} = \iiint_{W} div\mathbf{F}dV \tag{110}$$

since

$$\mathbf{d}^{(\alpha)}\mathbf{S} = \mathbf{e}_1 d^{\alpha}x_2 d^{\alpha}x_3 + \mathbf{e}_2 d^{\alpha}x_3 d^{\alpha}x_1 + \mathbf{e}_3 d^{\alpha}x_1 d^{\alpha}x_2, \tag{111}$$

where $d^a x_i$, i=1,2,3 is expressed by Eq.(15)

$$d^{(\alpha)}V = d^{\alpha}x_1 d^{\alpha}x_2 d^{\alpha}x_3. \tag{112}$$

Furthermore, see Eq.(99)

$$div^{(a)}\mathbf{F}(x) = \frac{{}_{\omega}{}^{c}D_k{}^{a}F_k(\mathbf{x})}{{}_{\omega}{}^{c}D_k{}^{a}x_i}\delta_{km}.$$

The Fractional Gauss divergence theorem becomes

$${}_{\omega}{}^{(\alpha)}\iint_{\partial W}\mathbf{F}\cdot\mathbf{d}^{(\alpha)}\mathbf{S} = {}_{\omega}{}^{(\alpha)}\iiint_{w} div^{(\alpha)}\mathbf{F}d^{(\alpha)}V$$

.

16 Conclusions

Correcting the picture of fractional differential of a function, the fractional tangent space of a manifold was defined, introducing also Leibnitzs L-fractional derivative that is the only one having physical meaning. Further, the L-fractional chain rule is introduced, that is necessary for the existence of fractional differential. After establishing the fractional differential of a function, the theory of fractional differential geometry of curves is developed. In addition, the basic forms concerning the first and second differential forms of the surfaces were defined, through the tangent spaces defined earlier, having mathematical meaning without any confusion, contrary to the existing procedures. Further the field theorems have been outlined in an accurate way, that may not cause confusion in their application. The present work will help in discussion of many applications concerning mechanics, quantum mechanics and relativity, that need a clear description based upon the fractional differential geometry.

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