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Solvability and Asymptotic Behavior for Some Nonlinear Quadratic Integral Equation Involving Erdélyi-Kober Fractional Integrals on the Unbounded Interval

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Abstract: The paper contains some results on the existence of solutions for a nonlinear Erdélyi-Kober fractional quadratic integral equation with deviating arguments. That result is proved under rather general hypotheses. Our equation contains the famous quadratic integral equation of Chandrasekhar type as a special case. The main tools used in our considerations are the concept of measures of noncompactness and the classical Schauder fixed point principle. The investigations of this equation are placed in the Banach space of real functions, defined, continuous and bounded on an unbounded interval. Moreover, we show that solutions of this integral equation are asymptotically stable. We give some examples for indicating the natural realizations of our results presented in this paper.

Keywords: Erdélyi-Kober integral equation of fractional order, quadratic integral equation, measures of noncompactness, Schauder fixed point theorem, asymptotic behavior.

1 Introduction

Fractional integrals and derivatives are most effective tools to characterize the nonlinear oscillations of earthquakes, real fractal structure of matter, many physical phenomena such as seepage flow in porous media, as well as in fluid dynamic traffic model, and the medium in many physics problems [1,2]. Especially, Erdélyi-Kober fractional integrals are a better approach to describe the medium with non-integer mass dimension, porous media, electrochemistry and viscoelasticity [3,4,5,6,7,8,9,10,11,12].

We recall from [11] that the Erdélyi-Kober fractional integral operator $I_{\zeta}^{\nu,\eta}$, where $\zeta > 0, \eta > 0$ and $\nu \in \mathbb{R}$, for a sufficiently well-behaved function $x(t)$ is given as

$$I_{\zeta}^{\nu,\eta}x(t) = \frac{\zeta}{\Gamma(\eta)}t^{-\zeta(\eta+\nu)}\int_0^t \frac{s^{\zeta(\nu+1)-1}x(s)}{(t^{\zeta}-s^{\zeta})^{1-\eta}}ds. \tag{1}$$

Particularly, if we take $\nu = 0$, Erdélyi-Kober fractional integral operator takes the form

$$I_{\zeta}^{0,\eta}x(t) = \frac{\zeta}{\Gamma(\eta)}t^{-\zeta\eta}\int_0^t \frac{s^{\zeta-1}x(s)}{(t^{\zeta}-s^{\zeta})^{1-\eta}}ds,$$

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or equivalently,

$$t^{\zeta\eta} I_{\zeta}^{0,\eta} x(t) = \frac{\zeta}{\Gamma(\eta)} \int_0^t \frac{s^{\zeta-1} x(s)}{(t^{\zeta}-s^{\zeta})^{1-\eta}} ds.$$

When $\zeta = 1$, the above integral reduces to Riemann-Liouville fractional integral operator. Although there are great number of papers about fractional differential and integral equations involving the Riemann-Liouville fractional operator or the Caputo fractional operator have occurred in the literature (see [13,14,15,?,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30]), only a few papers are on Erdélyi-Kober type integral equation of fractional order are studied by some researchers, see [31,32,33,34].

In this paper, we consider the following Erdélyi-Kober fractional quadratic integral equation with deviating arguments:

$$x(t) = f(t, x(\alpha(t))) + \frac{\zeta g(t, x(\beta(t)))}{\Gamma(\eta)} \int_0^t \frac{s^{\zeta-1} p(t,s) q(s, x(\gamma(s)), (ux)(\theta(s)))}{(t^{\zeta}-s^{\zeta})^{1-\eta}} ds, \tag{2}$$

where $t \in \mathbb{R}_+ = [0,\infty)$, $0 < \eta < 1$, $\zeta > 0$ and $u : BC(\mathbb{R}_+) \to BC(\mathbb{R}_+)$ such that $\|ux - uy\| \le \|x - y\|$ with $\|x\| = \sup\{|x(\acute{t})| : \acute{t} \in \mathbb{R}_+\}$ for $x \in BC(\mathbb{R}_+)$. The space $BC(\mathbb{R}_+)$ is the Banach space consisting of all real functions defined, continuous and bounded on $\mathbb{R}_+$. This space is equipped with the standard norm $\|x\| = \sup\{|x(t)| : t \in \mathbb{R}_+\}$. Moreover, the functions $p : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+, q : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R},\ \alpha, \beta, \gamma, \theta : \mathbb{R}_+ \to \mathbb{R}_+$ and $f, g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ satisfies reliable assumptions which will be given in Section 3.

Particularly, if we take $\zeta = \eta = f(t,x) = 1,\ g(t,x) = x,\ \beta(t) = \gamma(t) = t,\ p(t,s) = \dfrac{t}{t+s}$, and $q(s,x,y) = \phi(s)x$, where ϕ is a continuous function and $\phi(0) = 0$, then integral equation (2) has the following form

$$x(t) = 1 + x(t) \int_0^t \frac{t}{t+s} \phi(s) x(s) ds. \tag{3}$$

The above equation (3) is the Volterra counterpart of the famous quadratic integral equation of Chandrasekhar type appeared in many papers and monographs (see [35,36,37,38,39,40,41] for instance) which is applied in the theories of neutron transport, radiative transfer, traffic theory, and kinetic energy of gases (cf. [36,37,38,42,43,44,45,46]).

The goal here, is to prove the existence of solutions of a nonlinear integral equation (2) in the space of real functions which are defined, bounded and continuous on an unbounded interval. Furthermore, we will find some asymptotic characterization of solutions of integral equation (2). The technique used here is the measure of noncompactness associated with the Schauder fixed point principle to obtain our results.

2 Notations, Definitions and Auxiliary Facts

Let $(E, \|.\|)$ be an infinite dimensional Banach space with the zero element θ'. The symbols $\bar{X}$, Conv X will denote the closure and convex closure of a subset X of E, respectively. Denote by $B(x,r)$ the closed ball centered at x and with radius r. The symbol B_r stands for the ball $B(\theta', r)$. Moreover, let $\mathscr{M}_E$ indicate the family of all nonempty and bounded subsets of E and $\mathscr{N}_E$ indicate its subfamily consisting of all nonempty and relatively compact subsets.

The notion of measure of noncompactness [47] are as follows.

Definition 1. *A mapping $\mu : \mathscr{M}_E \to \mathbb{R}_+$ is said to be a measure of noncompactness in E if it satisfies the following conditions:*

(i) The family $\ker\mu = \{X \in \mathscr{M}_E : \mu(X) = 0\}$ is nonempty and $\ker\mu \subset \mathscr{N}_E$.
(ii) $X \subseteq Y \Rightarrow \mu(X) \le \mu(Y)$.
(iii) $\mu(\bar{X}) = \mu(X)$.
(iv) $\mu(\mathrm{Conv}\, X) = \mu(X)$.
(v) $\mu(\lambda X + (1-\lambda)Y) \le \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0,1]$.
(vi) If (X_n) is a sequence of closed sets from $\mathscr{M}_E$ such that $X_{n+1} \subset X_n$ $(n = 1,2,\dots)$ and if $\lim_{n\to\infty}\mu(X_n) = 0$, then the intersection $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

The family $\ker\mu$ defined in axiom (i) is called the kernel of the measure of noncompactness μ.

Remark.1 Let us mention that the intersection set X_∞ from (vi) is a member of the kernel of the measure of noncompactness μ. Indeed, from the inequality $\mu(X_\infty) \leq \mu(X_n)$ for $n = 1, 2, ...,$ we infer that $\mu(X_\infty) = 0$, so $X_\infty \in \ker \mu$. This property of the intersection set X_∞ will be essential in our study. Further facts concerning measures of noncompactness and their properties may be found in [47].

Next, we gather the construction of the measure of noncompactness in $BC(\mathbb{R}_+)$ which will be applied as main tool of the proof of our main results (see [48,**?**] and some references therein).

Let us fix a nonempty and bounded subset X of $BC(\mathbb{R}_+)$ and numbers $\varepsilon > 0$ and $T > 0$. For arbitrary function $x \in X$, let us denote by $w^T(x,\varepsilon)$ the modulus of continuity of the function x on the interval $[0,T]$, i.e.,

$$w^T(x,\varepsilon) = \sup\{|x(t) - x(s)| : t,s \in [0,T], |t-s| \leq \varepsilon\}.$$

Further, we consider the following quantities

$$w^T(X,\varepsilon) = \sup\{w^T(x,\varepsilon) : x \in X\},$$

$$w_0^T(X) = \lim_{\varepsilon \to 0} w^T(X,\varepsilon),$$

and

$$w_0(X) = \lim_{T \to \infty} w^T(X,\varepsilon).$$

Moreover, if t is fixed number from $\mathbb{R}_+$, let us define

$$X(t) = \{x(t) : x \in X\}$$

and

$$diam X(t) = \sup\{|x(t) - y(t)| : x,y \in X\}.$$

Finally, consider the function μ defined on the family $\mathscr{M}_{BC(\mathbb{R}_+)}$ by the formula

$$\mu(X) = w_0(X) + \lim_{t \to \infty} \sup diam X(t). \tag{4}$$

Then, the function μ is a measure of noncompactness in the space $BC(\mathbb{R}_+)$ (see [47,48]).

Remark.2 The kernel $\ker \mu$ of this measure is the family of all nonempty and bounded sets X such that functions from X are locally equicontinuous on $\mathbb{R}_+$ and the thickness of the bundle generated by functions from X tends to zero at infinity. This property can help us in establishing the behavior of the solutions for the fractional integral equation (2) in the next section.

In order to introduce some other concepts used in the paper let us suppose that Ω is a nonempty subset of the space $BC(\mathbb{R}_+)$. Moreover, let Q be an operator which is defined on Ω with values in $BC(\mathbb{R}_+)$.
Consider the operator equation of the form

$$x(t) = (Qx)(t), t \in \mathbb{R}_+. \tag{5}$$

Below we give the following characterizations for the solutions of the above operator equation (5) on $\mathbb{R}_+$ introduced in the paper [15].

Definition 2. *One says that the solutions of equation (5) are locally attractive if there exists a closed ball $B(x_0,r)$ in the space $BC(\mathbb{R}_+)$ such that for arbitrary solutions $x = x(t)$ and $y = y(t)$ of equation (5) belonging to $B(x_0,r) \cap \Omega$ satisfying*

$$\lim_{t \to \infty} (x(t) - y(t)) = 0. \tag{6}$$

In the case when the limit (6) is uniform with respect to the set $B(x_0,r) \cap \Omega$, i.e., for each $\varepsilon > 0$ there exists $T > 0$ such that

$$|x(t) - y(t)| \leq \varepsilon, \tag{7}$$

for all $x,y \in B(x_0,r) \cap \Omega$ and for $t \geq T$, then the solutions of equation (5) are uniformly locally attractive or asymptotically stable.

3 Main Results

Now we prove the main results of the paper. For that we consider the following assumptions:

(A_1)The functions $f,g:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}$ are continuous and there exist continuous functions $l,m:\mathbb{R}_+\to\mathbb{R}_+$ such that

$$|f(t,x)-f(t,y)|\le l(t)\;|x-y|,$$

$$|g(t,x)-g(t,y)|\le m(t)\;|x-y|,$$

for any $t\in\mathbb{R}_+$ and for all $x,y\in\mathbb{R}$. Moreover, the function $t\to f(t,0)$ is bounded on $\mathbb{R}_+$ such that

$$\overline{f}=\sup\{|f(t,0)|:t\in\mathbb{R}_+\},$$

and the function l is also bounded on $\mathbb{R}_+$. Put $\overline{l}=\sup\{|l(t)|:t\in\mathbb{R}_+\}$.

(A_2)The functions $\alpha,\beta,\gamma,\theta:\mathbb{R}_+\to\mathbb{R}_+$ are continuous and $\alpha(t),\beta(t)\to\infty$ as $t\to\infty$.

(A_3)The function $p:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$ is continuous and there exists a functions $\sigma:\mathbb{R}_+\to\mathbb{R}_+$ being continuous on $\mathbb{R}_+$ such that

$$|p(t,s)|\le\sigma(t),$$

for any $t,s\in\mathbb{R}_+$.

(A_4)The function $q:\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous and there exists a function $\phi:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$ being continuous and nondecreasing in each variable, with $\phi(0,0)=0$ and a constant $\xi\ge 0$ such that

$$|q(t,x_1,y_1)-q(t,x_2,y_2)|\le\xi\;\phi(|x_1-x_2|,|y_1-y_2|),$$

for any $t,s\in\mathbb{R}_+$ and for all $x_i,y_i\in\mathbb{R}$ $(i=1,2)$.

(A_5)The function $u:BC(\mathbb{R}_+)\to BC(\mathbb{R}_+)$ is a nonexpansive mapping, i.e.

$$\|ux-uy\|\le\|x-y\|,$$

for any $x,y\in BC(\mathbb{R}_+)$.

(A_6)The functions $a,b,c,d:\mathbb{R}_+\to\mathbb{R}_+$ defined by

$$a(t)=\xi\;m(t)\sigma(t)\;t^{\zeta\eta},\quad b(t)=m(t)\sigma(t)\;\overline{q}\,t^{\zeta\eta},$$

$$c(t)=\xi\;\sigma(t)\;|g(t,0)|\;t^{\zeta\eta},\quad d(t)=\overline{q}\,\sigma(t)|g(t,0)|\,t^{\zeta\eta},$$

are bounded on $\mathbb{R}_+$, $\overline{q}:\mathbb{R}_+\to\mathbb{R}_+$ is defined by

$$\overline{q}=\max\{|q(t,0,0)|:t\in\mathbb{R}_+\}.$$

Moreover, the functions a and c vanish at infinity, i.e., $\lim_{t\to\infty}a(t)=\lim_{t\to\infty}c(t)=0$.
For brevity, define

$$A=\sup\{a(t):t\in\mathbb{R}_+\},\quad B=\sup\{b(t):t\in\mathbb{R}_+\},$$

$$C=\sup\{c(t):t\in\mathbb{R}_+\},\quad D=\sup\{d(t):t\in\mathbb{R}_+\}.$$

(A_7)There exists a positive solution r_0 satisfying the following inequality

$$(\overline{l}r+\overline{f})\Gamma(1+\eta)+Ar\phi(r,r+\|u0\|)+Br+C\phi(r,r+\|u0\|)+D\le r\Gamma(1+\eta),$$

and the inequality

$$\overline{l}+\frac{A\phi(r_0,r_0+\|u0\|)+B}{\Gamma(1+\eta)}<1,$$

also holds.

Theorem 1. *Under assumptions (A_1-A_7), equation (2) has at least one solution $x=x(t)$ which belongs to the space $BC(\mathbb{R}_+)$. Moreover, all solutions of equation (2) belonging to the ball B_{r_0} are asymptotically stable.*

*Proof.*In order to prove our Theorem 1, we introduce the operator W defined on the space $BC(\mathbb{R}_+)$ in the following way

$$(Wx)(t) = (Fx)(t) + (Gx)(t)\cdot(Vx)(t),$$

where

$$(Fx)(t) = f(t, x(\alpha(t))),$$

$$(Gx)(t) = g(t, x(\beta(t))),$$

$$(Vx)(t) = \frac{\zeta}{\Gamma(\eta)}\int_0^t \frac{s^{\zeta-1}p(t,s)q(s,x(\gamma(s)),(ux)(\theta(s)))}{(t^\zeta - s^\zeta)^{1-\eta}}ds,$$

We split the proof into several steps.

Step 1: We verify that W is continuous operator.
To achieve our aim, we only need to verify that if $x \in BC(\mathbb{R}_+)$ then Wx is continuous on $\mathbb{R}_+$. In fact, for any function $x \in BC(\mathbb{R}_+)$, it is clear that the functions Fx and Gx are continuous on $\mathbb{R}_+$. We only need to show that the same holds also for the function Vx. For an arbitrary $x \in BC(\mathbb{R}_+)$ and fix $T > 0$ and $\varepsilon > 0$. Without loss of generality, we may assume that $0 \le t_1 < t_2 \le T$ with $|t_2 - t_1| \le \varepsilon$, then we obtain

$$\begin{aligned}
&|(Vx)(t_2) - (Vx)(t_1)| \\
&\le \frac{\zeta}{\Gamma(\eta)}\left|\int_0^{t_2} \frac{s^{\zeta-1}p(t_2,s)q(s,x(\gamma(s)),(ux)(\theta(s)))}{(t_2^\zeta - s^\zeta)^{1-\eta}}ds - \int_0^{t_1} \frac{s^{\zeta-1}p(t_2,s)q(s,x(\gamma(s)),(ux)(\theta(s)))}{(t_2^\zeta - s^\zeta)^{1-\eta}}ds\right| \\
&\quad + \frac{\zeta}{\Gamma(\eta)}\left|\int_0^{t_1} \frac{s^{\zeta-1}p(t_2,s)q(s,x(\gamma(s)),(ux)(\theta(s)))}{(t_2^\zeta - s^\zeta)^{1-\eta}}ds - \int_0^{t_1} \frac{s^{\zeta-1}p(t_1,s)q(s,x(\gamma(s)),(ux)(\theta(s)))}{(t_1^\zeta - s^\zeta)^{1-\eta}}ds\right| \\
&\le \frac{\zeta}{\Gamma(\eta)}\left|\int_{t_1}^{t_2} \frac{s^{\zeta-1}p(t_2,s)q(s,x(\gamma(s)),(ux)(\theta(s)))}{(t_2^\zeta - s^\zeta)^{1-\eta}}ds\right| + \frac{\zeta}{\Gamma(\eta)}\left|\int_0^{t_1} \frac{s^{\zeta-1}p(t_2,s)q(s,x(\gamma(s)),(ux)(\theta(s)))}{(t_2^\zeta - s^\zeta)^{1-\eta}}ds\right. \\
&\quad - \int_0^{t_1} \frac{s^{\zeta-1}p(t_1,s)q(s,x(\gamma(s)),(ux)(\theta(s)))}{(t_2^\zeta - s^\zeta)^{1-\eta}}ds + \int_0^{t_1} \frac{s^{\zeta-1}p(t_1,s)q(s,x(\gamma(s)),(ux)(\theta(s)))}{(t_2^\zeta - s^\zeta)^{1-\eta}}ds \\
&\quad \left. - \int_0^{t_1} \frac{s^{\zeta-1}p(t_1,s)q(s,x(\gamma(s)),(ux)(\theta(s)))}{(t_1^\zeta - s^\zeta)^{1-\eta}}ds\right| \\
&\le \frac{\zeta}{\Gamma(\eta)}\int_{t_1}^{t_2} \frac{s^{\zeta-1}|p(t_2,s)||q(s,x(\gamma(s)),(ux)(\theta(s)))|}{(t_2^\zeta - s^\zeta)^{1-\eta}}ds \\
&\quad + \frac{\zeta}{\Gamma(\eta)}\int_0^{t_1} \frac{s^{\zeta-1}|p(t_2,s) - p(t_1,s)||q(s,x(\gamma(s)),(ux)(\theta(s)))|}{(t_2^\zeta - s^\zeta)^{1-\eta}}ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\zeta}{\Gamma(\eta)}\int_0^{t_1} s^{\zeta-1}|p(t_1,s)||q(s,x(\gamma(s)),(ux)(\theta(s)))|\left|\frac{1}{(t_2^{\zeta}-s^{\zeta})^{1-\eta}}-\frac{1}{(t_1^{\zeta}-s^{\zeta})^{1-\eta}}\right|ds\\
\le&\frac{\zeta\sigma(t_2)}{\Gamma(\eta)}\int_{t_1}^{t_2}\frac{s^{\zeta-1}[|q(s,x(\gamma(s)),(ux)(\theta(s)))-q(s,0,0)|+|q(s,0,0)|]}{(t_2^{\zeta}-s^{\zeta})^{1-\eta}}ds\\
&+\frac{\zeta w_1^T(p,\varepsilon)}{\Gamma(\eta)}\int_0^{t_1}\frac{s^{\zeta-1}[|q(s,x(\gamma(s)),(ux)(\theta(s)))-q(s,0,0)|+|q(s,0,0)|]}{(t_2^{\zeta}-s^{\zeta})^{1-\eta}}ds\\
&+\frac{\zeta\sigma(t_1)}{\Gamma(\eta)}\int_0^{t_1}s^{\zeta-1}[|q(s,x(\gamma(s)),(ux)(\theta(s)))|-q(s,0,0)|+|q(s,0,0)|]\left|\frac{1}{(t_1^{\zeta}-s^{\zeta})^{1-\eta}}-\frac{1}{(t_2^{\zeta}-s^{\zeta})^{1-\eta}}\right|ds\\
\le&\frac{\zeta\sigma_T}{\Gamma(\eta)}\int_{t_1}^{t_2}\frac{s^{\zeta-1}[\xi\phi(|x(\gamma(s))|,|(ux)(\theta(s))|)+\overline{q}]}{(t_2^{\zeta}-s^{\zeta})^{1-\eta}}ds+\frac{\zeta w_1^T(p,\varepsilon)}{\Gamma(\eta)}\int_0^{t_1}\frac{s^{\zeta-1}[\xi\phi(|x(\gamma(s))|,|(ux)(\theta(s))|)+\overline{q}]}{(t_2^{\zeta}-s^{\zeta})^{1-\eta}}ds\\
&+\frac{\zeta\sigma_T}{\Gamma(\eta)}\int_0^{t_1}s^{\zeta-1}[\xi\phi(|x(\gamma(s))|,|(ux)(\theta(s))|)+\overline{q}]\cdot[(t_1^{\zeta}-s^{\zeta})^{\eta-1}-(t_2^{\zeta}-s^{\zeta})^{\eta-1}]ds\\
\le&\frac{\sigma_T[\xi\phi(\|x\|,\|ux\|)+\overline{q}]}{\Gamma(1+\eta)}(t_2^{\zeta}-t_1^{\zeta})^{\eta}+\frac{w_1^T(p,\varepsilon)[\xi\phi(\|x\|,\|ux\|)+\overline{q}]}{\Gamma(1+\eta)}[t_2^{\zeta\eta}-(t_2^{\zeta}-t_1^{\zeta})^{\eta}]\\
&+\frac{\sigma_T[\xi\phi(\|x\|,\|ux\|)+\overline{q}]}{\Gamma(1+\eta)}[t_1^{\zeta\eta}-t_2^{\zeta\eta}+(t_2^{\zeta}-t_1^{\zeta})^{\eta}]\\
\le&\frac{2\sigma_T[\xi\phi(\|x\|,\|ux\|)+\overline{q}]}{\Gamma(1+\eta)}(t_2^{\zeta}-t_1^{\zeta})^{\eta}+\frac{w_1^T(p,\varepsilon)[\xi\phi(\|x\|,\|ux\|)+\overline{q}]}{\Gamma(1+\eta)}t_2^{\zeta\eta},
\end{aligned}
$$

where we denote

$$\sigma_T=\max\{\sigma(t):t\in[0,T]\},$$

$$w_1^T(p,\varepsilon)=\sup\{|p(t_2,s)-p(t_1,s)|:s,t_1,t_2\in[0,T],|t_2-t_1|\le\varepsilon\}.$$

Observe that the uniform continuity of the function $p(t,s)$ on the compact set $[0,T]\times[0,T]$, we deduce that $w_1^T(p,\varepsilon)\to 0$ as $\varepsilon\to 0$.

Further, keeping in mind the above estimates, we obtain

$$w^T(Vx,\varepsilon)\le\frac{\xi\phi(\|x\|,\|ux\|)+\overline{q}}{\Gamma(1+\eta)}[2\sigma_T\varepsilon^{\zeta\eta}+w_1^T(p,\varepsilon)T^{\zeta\eta}]. \tag{8}$$

From the inequality (8) together with the above established facts we infer that the function Vx is continuous on the interval $[0,T]$ for any $T>0$. This proceeds the continuity of Vx on $\mathbb{R}_+$.

Step 2: For $x\in\mathbb{R}_+$, boundedness of the function Wx on $\mathbb{R}_+$.
Now, taking a function $x\in BC(\mathbb{R}_+)$, for an arbitrarily fixed $t\in\mathbb{R}_+$, we have

$$
\begin{aligned}
|(Wx)(t)|&\le|(Fx)(t)|+|(Gx)(t)|\cdot|(Vx)(t)|\\
&\le|f(t,x(\alpha(t)))-f(t,0)|+|f(t,0)|+\frac{\zeta}{\Gamma(\eta)}[|g(t,x(\beta(t)))-g(t,0)|+|g(t,0)|]
\end{aligned}
$$

$$
\begin{aligned}
&\times \int_0^t \frac{s^{\zeta-1}|p(t,s)|[|q(s,x(\gamma(s)),(ux)(\theta(s)))-q(s,0,0)|+|q(s,0,0)|]}{(t^\zeta-s^\zeta)^{1-\eta}}ds \\
&\le l(t)|x(\alpha(t))|+\bar{f}+\frac{\zeta}{\Gamma(\eta)}[m(t)|x(\beta(t))|+|g(t,0)|]\int_0^t \frac{s^{\zeta-1}\sigma(t)[\xi\phi(|x(\gamma(s))|,|(ux)(\theta(s))|+\bar{q}]}{(t^\zeta-s^\zeta)^{1-\eta}}ds \\
&\le \bar{l}\|x\|+\bar{f}+\frac{[m(t)\|x\|+|g(t,0)|]}{\Gamma(1+\eta)}\sigma(t)[\xi\phi(\|x\|,\|ux\|)+\bar{q}]t^{\zeta\eta} \\
&\le \bar{l}\|x\|+\bar{f}+\frac{a(t)\|x\|\phi(\|x\|,\|ux\|)+b(t)\|x\|+c(t)\phi(\|x\|,\|ux\|)+d(t)}{\Gamma(1+\eta)}, \qquad (9)
\end{aligned}
$$

by using the imposed assumptions, we have the above inequalities, which shows that the function Wx is bounded on $\mathbb{R}_+$.

Step 3: The operator W transforms the ball B_{r_0} into itself.
Now, let us take

$$\|ux\| \le \|ux-u0\|+\|u0\| \le \|x\|+\|u0\|,$$

and the nondecreasing function ϕ, from the established estimate (9), we obtain

$$\|Wx\| \le \bar{l}\|x\|+\bar{f}+\frac{A\|x\|\phi(\|x\|,\|x\|+\|u0\|)+B\|x\|+C\phi(\|x\|,\|x\|+\|u0\|)+D}{\Gamma(1+\eta)}.$$

From the above estimate and assumption (A_7), we conclude that the operator W transforms the ball B_{r_0} into itself.

Step 4: W is continuous operator on the ball B_{r_0}.
To achieve our aim, it is sufficient to show that $(Fx)(t)=f(t,x(\alpha(t)))$ is continuous on the ball B_{r_0} and

$$(Gx)(Vx)(t)=(Gx)(t)(Vx)(t)=\frac{\zeta g(t,x(\beta(t)))}{\Gamma(\eta)}\int_0^t \frac{s^{\zeta-1}p(t,s)q(s,x(\gamma(s)),(ux)(\theta(s)))}{(t^\zeta-s^\zeta)^{1-\eta}}ds,$$

is also continuous on the ball B_{r_0}.
Taking into account a sequence $\{x_n\}\subset B_{r_0}$ such that $x_n\to x$ with $x\in B_{r_0}$. For this, first of all we have to prove that $Fx_n\to Fx$, for $t\in\mathbb{R}_+$ and in view of assumption (A_1), it follows that

$$
\begin{aligned}
|(Fx_n)(t)-(Fx)(t)| &= |f(t,x_n(\alpha(t)))-f(t,x(\alpha(t)))| \\
&\le l(t)|x_n(\alpha(t))-x(\alpha(t))| \\
&\le \bar{l}\,\|x_n-x\|.
\end{aligned}
$$

Hence, it proves that F is continuous on the ball B_{r_0}.
Next, we have to show that $(Gx_n)\cdot(Vx_n)\to(Gx)\cdot(Vx)$, for $t\in\mathbb{R}_+$ and taking into account of our imposed assumptions, we obtain

$$
\begin{aligned}
|(Gx_n)(Vx_n)(t)-(Gx)(Vx)(t)| &= \left|\frac{\zeta g(t,x_n(\beta(t)))}{\Gamma(\eta)}\int_0^t \frac{s^{\zeta-1}p(t,s)q(s,x_n(\gamma(s)),(ux_n)(\theta(s)))}{(t^\zeta-s^\zeta)^{1-\eta}}ds\right. \\
&\quad \left.-\frac{\zeta g(t,x(\beta(t)))}{\Gamma(\eta)}\int_0^t \frac{s^{\zeta-1}p(t,s)q(s,x(\gamma(s)),(ux)(\theta(s)))}{(t^\zeta-s^\zeta)^{1-\eta}}ds\right| \\
&\le \frac{\zeta|g(t,x_n(\beta(t)))|}{\Gamma(\eta)}\int_0^t \frac{s^{\zeta-1}|p(t,s)||q(s,x_n(\gamma(s)),(ux_n)(\theta(s)))-q(s,x(\gamma(s)),(ux)(\theta(s)))|}{(t^\zeta-s^\zeta)^{1-\eta}}ds
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\zeta|g(t,x_n(\beta(t)))-g(t,x(\beta(t)))|}{\Gamma(\eta)}\int_0^t\frac{s^{\zeta-1}|p(t,s)||q(s,x(\gamma(s)),(ux)(\theta(s)))|}{(t^\zeta-s^\zeta)^{1-\eta}}ds\\
&\le\frac{\zeta|g(t,x_n(\beta(t)))-g(t,0)|+|g(t,0)|}{\Gamma(\eta)}\int_0^t\frac{s^{\zeta-1}\sigma(t)\xi\phi(|x_n(\gamma(s))-x(\gamma(s))|,|(ux_n)(\theta(s))-(ux)(\theta(s))|)}{(t^\zeta-s^\zeta)^{1-\eta}}ds\\
&+\frac{\zeta m(t)|x_n(\beta(t))-x(\beta(t))|}{\Gamma(\eta)}\int_0^t\frac{s^{\zeta-1}\sigma(t)[|q(s,x(\gamma(s)),(ux)(\theta(s)))-q(s,0,0)|+|q(s,0,0)|]}{(t^\zeta-s^\zeta)^{1-\eta}}ds\\
&\le\frac{m(t)|x_n(\beta(t))|+|g(t,0)|}{\Gamma(1+\eta)}[\sigma(t)\xi\phi(\|x_n-x\|,\|ux_n-ux\|)t^{\zeta\eta}]+\frac{m(t)\|x_n-x\|}{\Gamma(1+\eta)}\sigma(t)[\xi\phi(\|x\|,\|ux\|)+\overline{q}]t^{\zeta\eta}\\
&\le\frac{m(t)\|x_n\|+|g(t,0)|}{\Gamma(1+\eta)}[\sigma(t)\xi\phi(\|x_n-x\|,\|x_n-x\|)t^{\zeta\eta}]+\frac{m(t)\|x_n-x\|}{\Gamma(1+\eta)}\sigma(t)[\xi\phi(\|x\|,\|x\|+\|u0\|)+\overline{q}]t^{\zeta\eta}\\
&\le\frac{a(t)r_0+c(t)}{\Gamma(1+\eta)}\phi(\|x_n-x\|,\|x_n-x\|)+\frac{a(t)\phi(r_0,r_0+\|u0\|)+b(t)}{\Gamma(1+\eta)}\|x_n-x\|.
\end{aligned}
$$

Combining this estimates with our assumptions (A_6), we conclude that $|(Gx_n)(Vx_n)(t)-(Gx)(Vx)(t)|\to 0$ when $n\to\infty$. Which shows that W is continuous on the ball B_{r_0}.

Step 5: For $\varphi\ne X\subset B_{r_0}$ and $x,y\in X$, an estimate of $\lim_{t\to\infty}\sup diam(Wx)(t)$.
Then, for $t\in\mathbb{R}_+$ and in view of imposed assumptions, it follows

$$
\begin{aligned}
&|(Wx)(t)-(Wy)(t)|\le|f(t,x(\alpha(t)))-f(t,y(\alpha(t)))|+\frac{\zeta}{\Gamma(\eta)}\Bigg|g(t,x(\beta(t)))\int_0^t\frac{s^{\zeta-1}p(t,s)q(s,x(\gamma(s)),(ux)(\theta(s)))}{(t^\zeta-s^\zeta)^{1-\eta}}ds\\
&-g(t,y(\beta(t)))\int_0^t\frac{s^{\zeta-1}p(t,s)q(s,y(\gamma(s)),(uy)(\theta(s)))}{(t^\zeta-s^\zeta)^{1-\eta}}ds\Bigg|\\
&\le l(t)|x(\alpha(t))-y(\alpha(t))|+\frac{\zeta|g(t,x(\beta(t)))-g(t,y(\beta(t)))|}{\Gamma(\eta)}\int_0^t\frac{s^{\zeta-1}|p(t,s)||q(s,x(\gamma(s)),(ux)(\theta(s)))|}{(t^\zeta-s^\zeta)^{1-\eta}}ds\\
&+\frac{\zeta|g(t,y(\beta(t)))|}{\Gamma(\eta)}\int_0^t\frac{s^{\zeta-1}|p(t,s)||q(s,x(\gamma(s)),(ux)(\theta(s)))-q(s,y(\gamma(s)),(uy)(\theta(s)))|}{(t^\zeta-s^\zeta)^{1-\eta}}ds\\
&\le l(t)diamX(\alpha(t))+\frac{\zeta m(t)|x(\beta(t))-y(\beta(t))|}{\Gamma(\eta)}\int_0^t\frac{s^{\zeta-1}\sigma(t)[|q(s,x(\gamma(s)),(ux)(\theta(s)))-q(s,0,0)|+|q(s,0,0)|]}{(t^\zeta-s^\zeta)^{1-\eta}}ds\\
&+\frac{\zeta[|g(t,y(\beta(t)))-g(t,0)|+|g(t,0)|]}{\Gamma(\eta)}\int_0^t\frac{s^{\zeta-1}\sigma(t)\xi\phi(|x(\gamma(s))-y(\gamma(s))|,|(ux)(\theta(s))-(uy)(\theta(s))|)}{(t^\zeta-s^\zeta)^{1-\eta}}ds\\
&\le l(t)diamX(\alpha(t))+\frac{\zeta m(t)|x(\beta(t))-y(\beta(t))|}{\Gamma(\eta)}\int_0^t\frac{s^{\zeta-1}\sigma(t)[\xi\phi(|x(\gamma(s))|,|(ux)(\theta(s))|)+\overline{q}]}{(t^\zeta-s^\zeta)^{1-\eta}}ds\\
&+\frac{\zeta\sigma(t)\xi[m(t)|y(\beta(t))|+|g(t,0)|]}{\Gamma(\eta)}\int_0^t\frac{s^{\zeta-1}\phi(\|x-y\|,\|ux-uy\|)}{(t^\zeta-s^\zeta)^{1-\eta}}ds
\end{aligned}
$$

$$\leq l(t)diamX(\alpha(t)) + \frac{\zeta m(t)\xi\sigma(t)(|x(\beta(t))|+|y(\beta(t))|)}{\Gamma(\eta)}\int_0^t \frac{s^{\zeta-1}\phi(\|x\|,\|ux\|)}{(t^\zeta - s^\zeta)^{1-\eta}}ds$$

$$+\frac{\zeta m(t)\sigma(t)\overline{q}|x(\beta(t))-y(\beta(t))|}{\Gamma(\eta)}\int_0^t \frac{s^{\zeta-1}}{(t^\zeta - s^\zeta)^{1-\eta}}ds + \frac{\zeta m(t)\xi\sigma(t)|y(\beta(t))|}{\Gamma(\eta)}\int_0^t \frac{s^{\zeta-1}\phi(\|x-y\|,\|x-y\|)}{(t^\zeta - s^\zeta)^{1-\eta}}ds$$

$$+\frac{\zeta\sigma(t)\xi|g(t,0)|}{\Gamma(\eta)}\int_0^t \frac{s^{\zeta-1}\phi(\|x-y\|,\|x-y\|)}{(t^\zeta - s^\zeta)^{1-\eta}}ds$$

$$\leq l(t)diamX(\alpha(t)) + \frac{2m(t)\xi\sigma(t)r_0\phi(r_0,r_0+\|u0\|)t^{\zeta\eta}}{\Gamma(1+\eta)} + \frac{m(t)\sigma(t)\overline{q}t^{\zeta\eta}}{\Gamma(1+\eta)}diamX(\beta(t))$$

$$+\frac{m(t)\xi\sigma(t)r_0\phi(2r_0,2r_0)t^{\zeta\eta}}{\Gamma(1+\eta)} + \frac{\xi\sigma(t)|g(t,0)|\phi(2r_0,2r_0)t^{\zeta\eta}}{\Gamma(1+\eta)}$$

$$\leq l(t)diamX(\alpha(t)) + \frac{2a(t)r_0\phi(r_0,r_0+\|u0\|)}{\Gamma(1+\eta)} + \frac{b(t)}{\Gamma(1+\eta)}diamX(\beta(t)) + \frac{a(t)r_0\phi(2r_0,2r_0)}{\Gamma(1+\eta)} + \frac{c(t)\phi(2r_0,2r_0)}{\Gamma(1+\eta)}.$$

From the above estimate, we derive the following inequality

$$diam(Wx)(t) \leq l(t)diamX(\alpha(t)) + \frac{b(t)}{\Gamma(1+\eta)}diamX(\beta(t)) + \frac{2a(t)r_0}{\Gamma(1+\eta)}\phi(r_0,r_0+\|u0\|) + \frac{a(t)r_0+c(t)}{\Gamma(1+\eta)}\phi(2r_0,2r_0).$$

Keeping in mind assumption (A_6), we have

$$\limsup_{t\to\infty} diam(WX)(t) \leq \left(\overline{l} + \frac{B}{\Gamma(1+\eta)}\right)\limsup_{t\to\infty} diam(X)(t). \tag{10}$$

Step 6: For $\varphi \neq X \subset B_{r_0}$, an estimate of $w_0(Wx)$.
Fix $\varepsilon > 0$ and $x \in X$, for $T > 0$ we choose $t_1, t_2 \in [0,T]$ such that $|t_2 - t_1| \leq \varepsilon$ and assume that $t_1 < t_2$. Then, taking into account of our imposed assumptions and using the previously obtained estimate (8), we get

$$\begin{aligned}
|(Wx)(t_2)-(Wx)(t_1)| &\leq |f(t_2,x(\alpha(t_2)))-f(t_1,x(\alpha(t_1)))| + |(Gx)(t_2)(Vx)(t_2)-(Gx)(t_1)(Vx)(t_1)| \\
&\leq |f(t_2,x(\alpha(t_2)))-f(t_2,x(\alpha(t_1)))| + |f(t_2,x(\alpha(t_1)))-f(t_1,x(\alpha(t_1)))| \\
&\quad + |(Gx)(t_2)(Vx)(t_2)-(Gx)(t_1)(Vx)(t_2)| + |(Gx)(t_1)(Vx)(t_2)-(Gx)(t_1)(Vx)(t_1)| \\
&\leq l(t_2)|x(\alpha(t_2))-x(\alpha(t_1))| + w_1^T(f,\varepsilon) + |(Gx)(t_2)-(Gx)(t_1)||(Vx)(t_2)| \\
&\quad + |(Gx)(t_1)||(Vx)(t_2)-(Vx)(t_1)| \\
&\leq l(t_2)w^T(x,v^T(\alpha,\varepsilon)) + w_1^T(f,\varepsilon) + \frac{\zeta|g(t_2,x(\beta(t_2)))-g(t_1,x(\beta(t_1)))|}{\Gamma(\eta)} \\
&\quad \int_0^{t_2}\frac{s^{\zeta-1}|p(t_2,s)||q(s,x(\gamma(s)),(ux)(\theta(s)))|}{(t_2^\zeta - s^\zeta)^{1-\eta}}ds + |g(t_1,x(\beta(t_1)))|w^T(Vx,\varepsilon) \\
&\leq l(t_2)w^T(x,v^T(\alpha,\varepsilon)) + w_1^T(f,\varepsilon) \\
&\quad + \frac{\zeta[|g(t_2,x(\beta(t_2)))-g(t_2,x(\beta(t_1)))| + |g(t_2,x(\beta(t_1)))-g(t_1,x(\beta(t_1)))|]}{\Gamma(\eta)} \\
&\quad \int_0^{t_2}\frac{s^{\zeta-1}\sigma(t_2)[q(s,x(\gamma(s)),(ux)(\theta(s)))-q(s,0,0)|+|q(s,0,0)|]}{(t_2^\zeta - s^\zeta)^{1-\eta}}ds \\
&\quad + [|g(t_1,x(\beta(t_1)))-g(t_1,0)| + |g(t_1,0)|]w^T(Vx,\varepsilon)
\end{aligned}$$

$$\begin{aligned}
&\leq l(t_2)w^T(x,v^T(\alpha,\varepsilon))+w_1^T(f,\varepsilon)\\
&\quad+\frac{\zeta[m(t_2)|x(\beta(t_2))-x(\beta(t_1))|+w_1^T(g,\varepsilon)]\sigma(t_2)}{\Gamma(\eta)}\int_0^{t_2}\frac{s^{\zeta-1}[\xi\phi(|x(\gamma(s))|,|(ux)(\theta(s))|)+\overline{q}]}{(t_2^{\zeta}-s^{\zeta})^{1-\eta}}ds\\
&\quad+\frac{[m(t_1)|x(\beta(t_1))|+|g(t_1,0)|][\xi\phi(\|x\|,\|ux\|)+\overline{q}]}{\Gamma(1+\eta)}[2\sigma_T\varepsilon^{\zeta\eta}+w_1^T(p,\varepsilon)T^{\zeta\eta}]\\
&\leq l(t_2)w^T(x,v^T(\alpha,\varepsilon))+w_1^T(f,\varepsilon)\\
&\quad+\frac{[m(t_2)w^T(x,v^T(\beta,\varepsilon))+w_1^T(g,\varepsilon)]\sigma(t_2)}{\Gamma(1+\eta)}[\xi\phi(\|x\|,\|ux\|)+\overline{q}]t_2^{\zeta\eta}\\
&\quad+\frac{[m(t_1)\|x\|+|g(t_1,0)|]}{\Gamma(1+\eta)}[\xi\phi(\|x\|,\|ux\|)+\overline{q}][2\sigma_T\varepsilon^{\zeta\eta}+w_1^T(p,\varepsilon)T^{\zeta\eta}]\\
&\leq l(t_2)w^T(x,v^T(\alpha,\varepsilon))+w_1^T(f,\varepsilon)+\frac{a(t_2)\phi(r_0,r_0+\|u0\|)+b(t_2)}{\Gamma(1+\eta)}w^T(x,v^T(\beta,\varepsilon))\\
&\quad+\frac{\xi T^{\zeta\eta}\phi(r_0,r_0+\|u0\|)+\overline{q}T^{\zeta\eta}}{\Gamma(1+\eta)}\sigma_T w_1^T(g,\varepsilon)\\
&\quad+\frac{\overline{m}(T)r_0+\overline{g}(T)}{\Gamma(1+\eta)}[\xi\phi(r_0,r_0+\|u0\|)+\overline{q}][2\sigma_T\varepsilon^{\zeta\eta}+w_1^T(p,\varepsilon)T^{\zeta\eta}],
\end{aligned}\tag{11}$$

where we denote

$$\begin{aligned}
w_1^T(f,\varepsilon)&=\sup\{|f(t_2,x)-f(t_1,x)|:t_1,t_2\in[0,T],|t_2-t_1|\leq\varepsilon,x\in[-r_0,r_0]\},\\
w_1^T(g,\varepsilon)&=\sup\{|g(t_2,x)-g(t_1,x)|:t_1,t_2\in[0,T],|t_2-t_1|\leq\varepsilon,x\in[-r_0,r_0]\},\\
v^T(\alpha,\varepsilon)&=\sup\{|\alpha(t_2)-\alpha(t_1)|:t_1,t_2\in[0,T],|t_2-t_1|\leq\varepsilon\},\\
v^T(\beta,\varepsilon)&=\sup\{|\beta(t_2)-\beta(t_1)|:t_1,t_2\in[0,T],|t_2-t_1|\leq\varepsilon\},\\
\overline{m}(T)&=\max\{m(t):t\in[0,T]\},\\
\overline{g}(T)&=\max\{|g(t,0)|:t\in[0,T]\}.
\end{aligned}$$

Now, using the uniform continuity of the functions $f(t,x)$ and $g(t,x)$ on the set $[0,T]\times[-r_0,r_0]$, we derive $w_1^T(f,\varepsilon)$ and $w_1^T(g,\varepsilon)\to 0$ as $\varepsilon\to 0$. Hence, from the estimate (11), we conclude

$$w_0^T(WX)\leq\left(\overline{l}+\frac{A\phi(r_0,r_0+\|u0\|)+B}{\Gamma(1+\eta)}\right)w_0^T(X).$$

Consequently,

$$w_0(WX)\leq\left(\overline{l}+\frac{A\phi(r_0,r_0+\|u0\|)+B}{\Gamma(1+\eta)}\right)w_0(X).\tag{12}$$

Step 7: W is contraction with respect to the measure of noncompactness μ.
Combining the estimates (10) and (12) and keeping in mind the definition of the measure of noncompactness μ given by the formula (4), we deduce the following inequality

$$\begin{aligned}
\mu(WX)&=w_0(WX)+\lim_{t\to\infty}\sup diam(WX)(t)\\
&\leq\left(\overline{l}+\frac{A\phi(r_0,r_0+\|u0\|)+B}{\Gamma(1+\eta)}\right)w_0(X)+\left(\overline{l}+\frac{B}{\Gamma(1+\eta)}\right)\lim_{t\to\infty}\sup diam(X)(t)\\
&\leq\left(\overline{l}+\frac{A\phi(r_0,r_0+\|u0\|)+B}{\Gamma(1+\eta)}\right)(w_0(X)+\lim_{t\to\infty}\sup diam(X)(t))
\end{aligned}$$

$$\mu(WX)\leq\left(\overline{l}+\frac{A\phi(r_0,r_0+\|u0\|)+B}{\Gamma(1+\eta)}\right)\mu(X).\tag{13}$$

Obviously, in view of assumption (A_7), we have that

$$\bar{l}+\frac{A\phi(r_0,r_0+\|u0\|)+B}{\Gamma(1+\eta)}<1.$$

From the above results, it is clear that WX is a contraction with respect to the measure of noncompactness μ.

Step 8: Construction of a nonempty, bounded, closed and convex set Y.
Then, we will employ a fixed point theorem on the set Y. Further, let us assume that the sequence $(B^n_{r_0})$, where $B^1_{r_0}=\text{Conv}\,W(B_{r_0})$, $B^2_{r_0}=\text{Conv}\,W(B^1_{r_0})$, and so on. Observe that this sequence is decreasing, i.e. $B^{n+1}_{r_0}\subset B^n_{r_0}\subset B_{r_0}$ for $n=1,2,\ldots$, and also the sets of this sequence are nonempty, bounded, closed and convex. Thus, taking into account of estimate (13), we conclude that $\lim_{n\to\infty}\mu(B^n_{r_0})=0$. Further, keeping in mind the axiom (vi) of Definition 1, we infer that the set $Y=\bigcap_{n=1}^{\infty}B^n_{r_0}$ is nonempty, bounded, convex and closed subset of B_{r_0}. Furthermore, in view of Remark 1, we have that $Y\in\ker\mu$.

Particularly,

$$\lim_{t\to\infty}\sup diam\,Y(t)=\lim_{t\to\infty}diam\,Y(t)=0. \tag{14}$$

Also, observe that the operator W transforms the set Y into itself.

Step 9: Continuity of W on the set Y.
Fix $\varepsilon>0$ and take arbitrary functions $x,y\in Y$ such that $\|x-y\|\le\varepsilon$. Follows equation (14) and the fact that $WY\subset Y$, there exists $T>0$ such that for all $x,y\in Y$ and $t\ge T$ we have that $|x(t)-y(t)|\le\varepsilon$.
Since, W transforms Y into itself, we have $Wx,Wy\in Y$. Thus for $t\ge T$ we obtain

$$|(Wx)(t)-(Wy)(t)|\le\varepsilon. \tag{15}$$

Now we have to examine the case $t\in[0,T]$. Taking into account of our assumptions, after some standard computations, we obtain

$$\begin{aligned}
|(Wx)(t)-(Wy)(t)|&\le l(t)|x(\alpha(t))-y(\alpha(t))|\\
&\quad+\frac{\zeta m(t)\sigma(t)|x(\beta(t))-y(\beta(t))|}{\Gamma(\eta)}\int_0^t\frac{s^{\zeta-1}[\xi\phi(|x(\gamma(s))|,|(ux)(\theta(s))|)+\bar{q}]}{(t^\zeta-s^\zeta)^{1-\eta}}ds\\
&\quad+\frac{\zeta[m(t)|y(\beta(t))|+|g(t,0)|]\sigma(t)\xi}{\Gamma(\eta)}\int_0^t\frac{s^{\zeta-1}\phi(|x(\gamma(s))-y(\gamma(s))|,|(ux)(\theta(s))-(uy)(\theta(s))|)}{(t^\zeta-s^\zeta)^{1-\eta}}ds\\
&\le l(t)\varepsilon+\frac{\zeta m(t)\sigma(t)[\xi\phi(r_0,r_0+\|u0\|)+\bar{q}]}{\Gamma(\eta)}\varepsilon\int_0^t\frac{s^{\zeta-1}}{(t^\zeta-s^\zeta)^{1-\eta}}ds\\
&\quad+\frac{\zeta[m(t)r_0+|g(t,0)|]\sigma(t)\xi\phi(\varepsilon,\|ux-uy\|)}{\Gamma(\eta)}\int_0^t\frac{s^{\zeta-1}}{(t^\zeta-s^\zeta)^{1-\eta}}ds\\
&\le l(t)\varepsilon+\frac{\xi m(t)\sigma(t)\phi(r_0,r_0+\|u0\|)t^{\zeta\eta}}{\Gamma(1+\eta)}\varepsilon+\frac{m(t)\sigma(t)\bar{q}t^{\zeta\eta}}{\Gamma(1+\eta)}\varepsilon+\frac{m(t)\sigma(t)\xi r_0\phi(\varepsilon,\|x-y\|)t^{\zeta\eta}}{\Gamma(1+\eta)}\\
&\quad+\frac{\sigma(t)\xi|g(t,0)|\phi(\varepsilon,\|x-y\|)t^{\zeta\eta}}{\Gamma(1+\eta)}\\
&\le l(t)\varepsilon+\frac{a(t)\phi(r_0,r_0+\|u0\|)}{\Gamma(1+\eta)}\varepsilon+\frac{b(t)}{\Gamma(1+\eta)}\varepsilon+\frac{a(t)r_0\phi(\varepsilon,\varepsilon)}{\Gamma(1+\eta)}+\frac{c(t)\phi(\varepsilon,\varepsilon)}{\Gamma(1+\eta)}\\
&\le\left(\bar{l}+\frac{A\phi(r_0,r_0+\|u0\|)+B}{\Gamma(1+\eta)}\right)\varepsilon+\left(\frac{Ar_0+C}{\Gamma(1+\eta)}\right)\phi(\varepsilon,\varepsilon).
\end{aligned} \tag{16}$$

In view of equation (15), (16) and taking into account the assumption (A_6), we conclude that the operator W is continuous on the set Y.

Finally, taking into account all the above obtained facts about the set Y and the operator $W : Y \to Y$ established above and using the classical Schauder fixed point principle we deduce that the operator W has at least one fixed point $x = x(t)$ in the set Y. Hence, the function $x(t)$ is a solution of the Erdélyi-Kober fractional quadratic integral equation (2). Moreover, keeping in mind the fact that $Y \in \ker \mu$ and characterization of sets belonging to $\ker \mu$, we have that all solutions of equation (2) belonging to the ball B_{r_0} are asymptotically stable in the sense of Definition 2.

4 Examples

In this section, we provide two examples to illustrate the usefulness of our main results.
Firstly, we give an example of function $u : BC(\mathbb{R}_+) \to BC(\mathbb{R}_+)$ illustrating the assumption (A_5) of Theorem 1.

*Example 1.*Let a given function $x \in BC(\mathbb{R}_+)$ then

$$(Sx)(\theta(\acute{t})) = \max\{|x(\theta(t))| : 0 \le t \le \acute{t}\},$$

is a function belonging to $BC(\mathbb{R}_+)$.

In this way, It becomes as follows,

$$\begin{aligned}|(Sx)(\theta(\acute{t}))| &= |\max\{|x(\theta(t))| : 0 \le t \le \acute{t}\}| \\ &\le \|x|_{t\in[0,\acute{t}]}\| \\ &\le \|x\|.\end{aligned}$$

From the above, it is clear that the function Sx is bounded on $\mathbb{R}_+$.
Next, we have to show that Sx is continuous function on $\mathbb{R}_+$. To do this, let us take arbitrary numbers $T > 0$ and $\varepsilon > 0$. Fix $T > 0$ and $t_1, t_2 \in [0, T]$ such that $|t_2 - t_1| \le \varepsilon$, assume that $t_1 < t_2$. Since, Sx is nondecreasing function, we obtain

$$\begin{aligned}|(Sx)(\theta(t_2)) - (Sx)(\theta(t_1))| &= (Sx)(\theta(t_2)) - (Sx)(\theta(t_1)) \\ &= \max\{|x(\theta(t))| : 0 \le t \le t_2\} - \max\{|x(\theta(t))| : 0 \le t \le t_1\} \\ &= |x(\theta(\delta_1))| - |x(\theta(\delta_2))|,\end{aligned}$$

where $\delta_1 \le t_2$ and $\delta_2 \le t_1$.

Now, we assume that $\delta_1 \le t_1$ then we have that $(Sx)(\theta(t_2)) = (Sx)(\theta(t_1))$ and $|(Sx)(\theta(t_2)) - (Sx)(\theta(t_1))| = 0$.
If $t_1 < \delta_1 \le t_2$, we obtain

$$\begin{aligned}|(Sx)(\theta(t_2)) - (Sx)(\theta(t_1))| &= |x(\theta(\delta_1))| - |x(\theta(\delta_2))| \\ &\le |x(\theta(\delta_1))| - |x(\theta(t_1))| \\ &\le |x(\theta(\delta_1)) - x(\theta(t_1))|,\end{aligned}$$

we have also $\delta_1 - t_1 \le t_2 - t_1 \le \varepsilon$. Thus,

$$\begin{aligned}|(Sx)(\theta(t_2)) - (Sx)(\theta(t_1))| &\le |x(\theta(\delta_1)) - x(\theta(t_1))| \\ w^T(Sx, v_1^T(\theta,\varepsilon)) &\le w^T(x, v_2^T(\theta,\varepsilon)),\end{aligned}$$

where

$$v_1^T(\theta,\varepsilon) = \sup\{|\theta(t_2) - \theta(t_1)| : t_1, t_2 \in [0, T], |t_2 - t_1| \le \varepsilon\},$$

$$v_2^T(\theta,\varepsilon) = \sup\{|\theta(\delta_1) - \theta(t_1)| : t_1, \delta_1 \in [0, T], |\delta_1 - t_1| \le \varepsilon\}.$$

From the above estimates together with continuity of function x on the interval $[0, T]$, we conclude

$$w_0^T(SX) \le w_0^T(X).$$

In view of the above established facts we conclude that Sx is continuous on $[0,T]$ for any $T>0$. Hence, $Sx \in BC(\mathbb{R}_+)$. Furthermore, for any $x,y \in BC(\mathbb{R}_+)$, we obtain

$$\begin{aligned}|(Sx)(\theta(\acute{t}))-(Sy)(\theta(\acute{t}))| &= \left|\max\{|x(\theta(t))|:0\le t\le \acute{t}\}-\max\{|y(\theta(t))|:0\le t\le \acute{t}\}\right|\\ &\le \left|\|x|_{t\in[0,\acute{t}]}\|-\|y|_{t\in[0,\acute{t}]}\|\right|\\ &\le \|x-y|_{t\in[0,\acute{t}]}\|\\ &\le \|x-y\|.\end{aligned}$$

Hence,

$$\|Sx-Sy\|\le\|x-y\|.$$

Which shows that S is a nonexpansive mapping in $BC(\mathbb{R}_+)$.

Now, we present a numerical example as an application of Theorem 1.

*Example 2.*Consider the following quadratic integral equation with Erdélyi-Kober fractional operator:

$$x(t)=\frac{t^4+\nu\arctan(x(t/7))}{6+11t^4}+\frac{\frac{3}{2}\left(t^{3/2}e^{-2t}+t^{1/2}x(t/4)\right)}{\Gamma(2/3)}\int_0^t\frac{\frac{e^{-3t}}{(1+s)}\ln\left(1+\frac{|x(s/3)|}{2}+\frac{1}{2}\max_{0\le\acute{t}\le s}|x(\acute{t}/5)|\right)}{\sqrt{s}(t^{3/2}-s^{3/2})^{1/3}}ds, \tag{17}$$

where $t\in\mathbb{R}_+$ and ν is a positive constant.
Observe that the above equation can be treated as a particular case of equation (2) if we put $\eta=2/3,\zeta=3/2,\alpha(t)=t/7,\beta(t)=t/4,\gamma(t)=t/3,\theta(t)=t/5$ and

$$\begin{aligned}f(t,x)&=\frac{t^4+\nu\arctan x}{6+11t^4},\\ g(t,x)&=t^{3/2}e^{-2t}+t^{1/2}x,\\ p(t,s)&=\frac{e^{-3t}}{1+s},\\ q(t,x,y)&=\ln\left(1+\frac{|x|+y}{2}\right),\end{aligned}$$

and $ux=Sx$, where S is nonexpansive mapping satisfies assumption (A_5) of Theorem 1 as shown in Example 1.
It is easily seen that the functions $\alpha(t),\beta(t),\gamma(t)$ and $\theta(t)$ satisfy assumption (A_2).

In fact, we have that the functions $f(t,x)$ and $g(t,x)$ are continuous functions on $\mathbb{R}_+\times\mathbb{R}$, for any $t\in\mathbb{R}_+$ and $x,y\in\mathbb{R}$, we get

$$|f(t,x)-f(t,y)|\le\frac{\nu}{6+11t^4}|\arctan x-\arctan y|\le\frac{\nu}{6+11t^4}|x-y|,$$

and

$$|g(t,x)-g(t,y)|=t^{1/2}|x-y|.$$

Thus, from the above we have $l(t)=\dfrac{\nu}{6+11t^4}$ and $m(t)=t^{1/2}$. Clearly, $f(t,0)=\dfrac{t^4}{6+11t^4},\bar{f}=\dfrac{1}{11}$ and $l(t)$ is bounded function on $\mathbb{R}_+$ with $\bar{l}=\dfrac{\nu}{6}$. Hence, assumption (A_1) of Theorem 1 is satisfied.
Further observe that the function $p(t,s)$ satisfies assumption (A_3) with $\sigma(t)=e^{-3t}$. Now, to check that assumption (A_4) is satisfied, for $t\in\mathbb{R}_+$ and $x_i,y_i\in\mathbb{R}$ $(i=1,2)$.

$$\begin{aligned}|q(t,x_1,y_1)-q(t,x_2,y_2)| &= \left|\ln\left(1+\frac{|x_1|+y_1}{2}\right)-\ln\left(1+\frac{|x_2|+y_2}{2}\right)\right|\\ &\le\left|\frac{|x_1|+y_1}{2}-\frac{|x_2|+y_2}{2}\right|\\ &\le\frac{1}{2}[||x_1|-|x_2||+|y_1-y_2|]\\ &\le\frac{1}{2}[|x_1-x_2|+|y_1-y_2|].\end{aligned}$$

Thus, assumption (A_4) of Theorem 1 is satisfied and hence we get $\xi = \frac{1}{2}$ and $\phi(a_1,b_1) = a_1 + b_1$ with $\phi(0,0) = 0$. Moreover, $|g(t,0)| = t^{3/2}e^{-2t}$.
Next, we have to verify that the assumption (A_6) is satisfied, let us note that the functions a,b,c and d appearing in that assumption takes the form

$$a(t) = \frac{1}{2}e^{-3t}t^{3/2}, \quad b(t) = 0,$$

$$c(t) = \frac{1}{2}e^{-5t}t^{5/2}, \quad d(t) = 0.$$

Clearly, $q(t,0,0) = 0$ and $\overline{q} = 0$. It is easily seen that $a(t) \to 0$ as $t \to \infty$ and $A = \left(\frac{1}{2}\right)^{5/2} e^{-3/2}$. Moreover, we have that $B = 0$. Further, it is also easy to check that $c(t) \to 0$ as $t \to \infty$ and $C = \left(\frac{1}{2}\right)^{7/2} e^{-5/2}$, and $D = 0$. Thus, the assumption (A_6) of Theorem 1 is satisfied.
Further, the inequality from the assumption (A_7) of Theorem 1 has the form

$$\left(\frac{\nu}{6}r + \frac{1}{11}\right)\Gamma(5/3) + Ar.2r + C.2r \leq r\Gamma(5/3). \tag{18}$$

Hence, taking into account that $\Gamma(5/3) \simeq 0.902745$, we have that the number $r_0 = 1$ is a solution of the inequality (18) if we take $\nu = 1$.
Moreover, the second inequality of assumption (A_7) of Theorem 1 is also satisfied.
Finally, by Theorem 1, we conclude that equation (17) has at least one solution in the space $BC(\mathbb{R}_+)$ belonging to the ball B_{r_0} provided $\nu = 1$ and all solutions of integral equation (17) which belongs to ball B_{r_0} are asymptotically stable in the sense of Definition 2.

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