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Some Coding Theorems on Generalized Reyni's Entropy of Order α and Type β

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Abstract: In this paper we define a new average code-word length $L_{\beta}^{\alpha}(P)$ of order α and type β and its relationship with generalized Reyniś entropy $H_{\beta}^{\alpha}(P)$ has been discussed. Using $L_{\beta}^{\alpha}(P)$, some coding theorems for discrete noiseless channel has been proved. The measures defined in this communication are not only new but some known measures are the particular cases of our proposed measures. The noiseless coding theorems for discrete channel proved in this paper are verified by considering Huffman and Shannon-Fano coding schemes on taking empirical data.Also we study the monotonic behavior of $H_{\beta}^{\alpha}(P)$ with respect to parameters α and β. The important properties of $H_{\beta}^{\alpha}(P)$ have also been studied.

Keywords: Shannons entropy, Reynis entropy, Code-word length, Kraft inequality, Coding theorem, Holders inequality, Optimal code length, Huffman codes and Shannon-Fano codes.
AMS classification 94A17, 94A24

1 Introduction

The growth of telecommunication in the early twentieth century led several researchers to study the information control of signals , the seminal work of Shannon [1] based on papers by Nyquists [2,3] and Hartley [4] rationalized these early efforts into a coherent mathematical theory of communication and initiated the area of research now known as information theory. The central paradigm of classical information theory is the engineering problem of the transmission of information over a noisy channel. The most fundamental results of this theory are Shannon's source coding theorem which establishes that on average the number of bits needed to represent the result of an uncertain event is given by its entropy; and Shannon's noisy-channel coding theorem which states that reliable communication is possible over noisy channels provided that the rate of communication is below a certain threshold called the channel capacity. Information theory is a broad and deep mathematical theory, with equally broad and deep applications, amongst which is the vital field of coding theory. Information theory is a new branch of probability and statistics with extensive potential application to communication system. The term information theory does not possess a unique definition. Broadly speaking, information theory deals with the study of problems concerning any system.This includes information processing, information storage and decision making. In a narrow sense, information theory studies all theoretical problems connected with the transmission of information over communication channels. This includes the study of uncertainty (information) measure and various practical and economical methods of coding information for transmission.

2 Shannon's Entropy

Let X is a discrete random variable taking values $x_1, x_2, \ldots, x_n$ with respective probabilities $p_1, p_2, \ldots, p_n$ $p_i \geq 0 \ \forall i = 1, 2, 3, \ldots, n$ and $\sum_{i=1}^{n} p_i = 1$. Shannon [1] gives the following measure of information and call it entropy.

$$H(P) = -\sum_{i=1}^{n} p_i \log_D p_i \tag{1}$$

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The measure (1) serves as a suitable measure of entropy. Let $p_1, p_2, p_3, , p_n$ be the probabilities of n codewords to be transmitted and let their lengths $l_1, l_2, , l_n$ satisfy Kraft [5] inequality,

$$\sum_{i=1}^{n} D^{-l_i} \leq 1 \quad (2)$$

For uniquely decipherable codes, Shannon [1] showed that for all codes satisfying (2),the lower bound of the mean codeword length,

$$L = \sum_{i=1}^{n} p_i l_i \quad (3)$$

lies between H(P) and H(P)+1.Where D is the size of code alphabet.

Generalized coding theorems by considering different information measure under the condition of uniquely decipherability were investigated by several authors; see for instance the papers: On Useful Information of order by Gurdial and F Pessoa [6], A generalized useful information measure and coding theorems by D. S. Hooda and U. S. Bhaker [7], Some results on a generalized useful information measure by A. B. Khan, B. A. Bhat and S. Pirzada [8], A noiseless coding theorem for sources having utilities by G. Longo [9], Noiseless coding theorems corresponding to fuzzy entropies by Om Parkash and P. K. Sharma [10], Application of Holders inequality in information theory by R. P. Singh, R. Kumar and R. K. Tuteja [11] and Generalized entropy of order α and typeβ by J. N. Kapur [12].

In this particular paper we study some noiseless coding theorems by considering generalized Reynis information measure and generalized average code-word length of order α and type β in section 3. The results obtained here are not only new but some information measures are the particular cases of our proposed measure that already exist in the literature. In section 4 we verify the noiseless coding theorems by considering Shannon-Fano coding scheme and Huffman coding scheme on taking empirical data.Also we have also studied the monotonic behavior of $H_\beta^\alpha(P)$ with respect to parameters α and type β in section 5. And the important properties of $H_\beta^\alpha(P)$ have also been studied in section 6.

3 Noiseless Coding Theorems

Define a generalized Reyni's entropy of order α and type β as:

$$H_\beta^\alpha(P) = \frac{\beta}{1-\alpha} \log_D \left[\sum_{i=1}^{n} p_i^{\alpha\beta} \right] \quad (4)$$

Where $0 < \alpha < 1, 0 < \beta \leq 1, p_i \geq 0 \ \forall i = 1,2,3,\ldots,n$ and $\sum_{i=1}^{n} p_i = 1$.

Remarks for (4)

1.When $\beta = 1$ (4) reduces to Reynis entropy,i.e.,

$$H^\alpha(P) = \frac{1}{1-\alpha} \log_D \left[\sum_{i=1}^{n} p_i^\alpha \right] \quad (5)$$

2.When $\beta = 1$ and $\alpha \to 1$ (4) reduces to Shannons [1] entropy i.e.,

$$H(P) = -\sum_{i=1}^{n} p_i \log_D p_i \quad (6)$$

3.When $\beta = 1$, $\alpha \to 1$ and $p_i = \frac{1}{n} \forall i = 1,2,3,\ldots,n$ then (4) reduces to maximum entropy i.e.,

$$H(\frac{1}{n}) = \log_D n \quad (7)$$

Further we define a new generalized code-word length of order α and type β corresponding to (4) and is given by

$$L_\beta^\alpha(P) = \frac{\alpha\beta}{1-\alpha} \log_D \left[\sum_{i=1}^{n} p_i^\beta D^{-l_i \left(\frac{\alpha-1}{\alpha}\right)} \right] \quad (8)$$
$$0 < \alpha < 1, 0 < \beta \leq 1$$

Where D is the size of code alphabet.

Remarks for (8)

1.For $\beta = 1$ (8) reduces to code-word length corresponding to entropy (5) i.e.,

$$L^\alpha(P) = \frac{\alpha}{1-\alpha} \log_D \left[\sum_{i=1}^{n} p_i D^{-l_i \left(\frac{\alpha-1}{\alpha}\right)} \right] \quad (9)$$

2.For $\beta = 1$ and $\alpha \to 1$ (8) reduces to optimal code-word length corresponding to Shannon [1] entropy i.e.,

$$L = \sum_{i=1}^{n} p_i l_i \quad (10)$$

3.For $\beta = 1$ and $l_1 = l_2 = \ = l_n = 1$ then (8) reduces to 1. i.e., $L^\alpha = 1$

Now we found the bounds of (8) in terms of (4) under the condition

$$\sum_{i=1}^{n} D^{-l_i} \leq 1 \quad (11)$$

Which is Kraft [5] inequality, where D is the size of code alphabet.

Theorem 3.1. For all integers $(D > 1)$ the code word lengths $l_1, l_2, , l_n$ satisfies the condition (11), then the generalized code-word length (8) satisfies the inequality

$$L_\beta^\alpha(P) \geq H_\beta^\alpha(P), where 0 < \alpha < 1, 0 < \beta \leq 1. \quad (12)$$

Where equality holds good iff

$$l_i = -\log_D \left[\frac{p_i^{\alpha\beta}}{\sum_{i=1}^{n} p_i^{\alpha\beta}} \right] \quad (13)$$

Proof. By Holder's Inequality we have

$$\sum_{i=1}^{n} x_i y_i \geq \left(\sum_{i=1}^{n} x_i^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} y_i^q \right)^{\frac{1}{q}} \quad (14)$$

For all $x_i, y_i > 0, i = 1,2,3,...,n$ and $\frac{1}{p} + \frac{1}{q} = 1, p < 1(\neq 0), q < 0$ or $q < 1(\neq 0), p < 0$. We see the equality holds iff there exists a positive constant c such that,

$$x_i^p = c y_i^q \quad (15)$$

Making the substitution

$$x_i = p_i^{\frac{\alpha\beta}{\alpha-1}} D^{-l_i}, y_i = p_i^{\frac{\alpha\beta}{1-\alpha}}, p = \frac{\alpha - 1}{\alpha}, q = 1 - \alpha$$

Using these values in (14) we get

$$\sum_{i=1}^{n} D^{-l_i} \geq \left[\sum_{i=1}^{n} p_i^{\beta} D^{-l_i} \left(\frac{\alpha - 1}{\alpha} \right) \right]^{\frac{\alpha}{\alpha-1}} \left[\sum_{i=1}^{n} p_i^{\alpha\beta} \right]^{\frac{1}{1-\alpha}} \quad (16)$$

Now using Kraft [5] inequality we get,

$$\left[\sum_{i=1}^{n} p_i^{\beta} D^{-l_i} \left(\frac{\alpha - 1}{\alpha} \right) \right]^{\frac{\alpha}{\alpha-1}} \left[\sum_{i=1}^{n} p_i^{\alpha\beta} \right]^{\frac{1}{1-\alpha}} \leq 1 \quad (17)$$

Or (17) can be written as

$$\left[\sum_{i=1}^{n} p_i^{\alpha\beta} \right]^{\frac{1}{1-\alpha}} \leq \left[\sum_{i=1}^{n} p_i^{\beta} D^{-l_i} \left(\frac{\alpha - 1}{\alpha} \right) \right]^{\frac{\alpha}{1-\alpha}} \quad (18)$$

Taking logarithms to both sides with base D to equation (18) we get

$$\frac{1}{1-\alpha} \log \left[\sum_{i=1}^{n} p_i^{\alpha\beta} \right] \leq \frac{\alpha}{1-\alpha} \log \left[\sum_{i=1}^{n} p_i^{\beta} D^{-l_i} \left(\frac{\alpha - 1}{\alpha} \right) \right] \quad (19)$$

As $0 < \beta \leq 1$,multiply equation (18) both sides by $\beta > 0$, we get

$$\frac{\beta}{1-\alpha} \log \left[\sum_{i=1}^{n} p_i^{\alpha\beta} \right] \leq \frac{\alpha\beta}{1-\alpha} \log \left[\sum_{i=1}^{n} p_i^{\beta} D^{-l_i} \left(\frac{\alpha - 1}{\alpha} \right) \right] \quad (20)$$

Or equivalently, $L_\beta^\alpha(P) \geq H_\beta^\alpha(P)$, where $0 < \alpha < 1, 0 < \beta \leq 1$. Hence the result.

From equation (13) we have

$$l_i = -\log_D \left[\frac{p_i^{\alpha\beta}}{\sum_{i=1}^{n} p_i^{\alpha\beta}} \right]$$

or equivalently,

$$D^{-l_i} = \frac{p_i^{\alpha\beta}}{\sum_{i=1}^{n} p_i^{\alpha\beta}} \quad (21)$$

Raising both sides to the power $\frac{\alpha - 1}{\alpha}$ to equation (21) and after simplification, we get

$$D^{-l_i} \left(\frac{\alpha - 1}{\alpha} \right) = p_i^{\beta(\alpha-1)} \left[\sum_{i=1}^{n} p_i^{\alpha\beta} \right]^{\frac{1-\alpha}{\alpha}} \quad (22)$$

Multiply equation (22) both sides by p_i^{β} and then summing over $i = 1,2,,n$, and after simplification, we get

$$\left[\sum_{i=1}^{n} p_i^{\beta} D^{-l_i} \left(\frac{\alpha - 1}{\alpha} \right) \right] = \left[\sum_{i=1}^{n} p_i^{\alpha\beta} \right]^{\frac{1}{\alpha}} \quad (23)$$

Taking logarithms both sides with base D to equation (23), then multiply both sides by $\frac{\alpha\beta}{1-\alpha}$, we get
$L_\beta^\alpha(P) = H_\beta^\alpha(P)$ Hence the result.

Theorem 3.2. For every code with lengths $l_1, l_2, , l_n$ satisfies Kraft's inequality L_β^α can be made to satisfy the inequality,

$$L_\beta^\alpha(P) < H_\beta^\alpha(P) + \beta, where 0 < \alpha < 1, 0 < \beta \leq 1. \quad (24)$$

Proof. From the theorem 3.1 we have

$$L_\beta^\alpha(P) = H_\beta^\alpha(P)$$

Holds if and only if

$$D^{-l_i} = \frac{p_i^{\alpha\beta}}{\sum_{i=1}^{n} p_i^{\alpha\beta}}, 0 < \alpha < 1, 0 < \beta \leq 1.$$

Or equivalently we can write

$$l_i = -\log_D p_i^{\alpha\beta} + \log_D \left[\sum_{i=1}^{n} p_i^{\alpha\beta} \right]$$

We choose the code-word lengths $l_i, i = 1,2,,n$ in such a way that they satisfy the inequality,

$$-\log_D p_i^{\alpha\beta} + \log_D \left[\sum_{i=1}^{n} p_i^{\alpha\beta} \right] \leq l_i < -\log_D p_i^{\alpha\beta} + \log_D \left[\sum_{i=1}^{n} p_i^{\alpha\beta} \right] + 1 \quad (25)$$

Consider the interval

$$\delta_i = \left[-\log_D p_i^{\alpha\beta} + \log_D \left[\sum_{i=1}^{n} p_i^{\alpha\beta} \right], -\log_D p_i^{\alpha\beta} + \log_D \left[\sum_{i=1}^{n} p_i^{\alpha\beta} \right] + 1 \right]$$

of length unity.In every δ_i, there lies exactly one positive integer l_i, such that,

$$0 < -\log_D p_i^{\alpha\beta} + \log_D \left[\sum_{i=1}^{n} p_i^{\alpha\beta} \right] \leq l_i < -\log_D p_i^{\alpha\beta} + \log_D \left[\sum_{i=1}^{n} p_i^{\alpha\beta} \right] + 1 \quad (26)$$

We will first show that the sequence $l_1, l_2, , l_n$, thus defined satisfies the Kraft [5] inequality. From the left inequality of (26), we have

$$-\log_D p_i^{\alpha\beta} + \log_D \left[\sum_{i=1}^{n} p_i^{\alpha\beta}\right] \le l_i$$

Or equivalently we can write

$$D^{-l_i} \le \frac{p_i^{\alpha\beta}}{\sum_{i=1}^{n} p_i^{\alpha\beta}} \tag{27}$$

Taking sum over i=1,2,,n, on both sides to the equation (27) we get,$\sum_{i=1}^{n} D^{-l_i} \le 1$,which is Kraft [5] inequality. Now the last inequality of (26) gives

$$l_i < -\log_D p_i^{\alpha\beta} + \log_D \left[\sum_{i=1}^{n} p_i^{\alpha\beta}\right] + 1$$

Or we can write

$$D^{l_i} < \left[\frac{p_i^{\alpha\beta}}{\sum_{i=1}^{n} p_i^{\alpha\beta}}\right]^{-1} D \tag{28}$$

As $0 < \alpha < 1$, then $(1-\alpha) > 0$, and $\left(\frac{1-\alpha}{\alpha}\right\rangle 0)$, raising both sides to the power $\left(\frac{1-\alpha}{\alpha}\right) > 0$, to equation (28), we get

$$D^{l_i}\left(\frac{1-\alpha}{\alpha}\right) < \left[\frac{p_i^{\alpha\beta}}{\sum_{i=1}^{n} p_i^{\alpha\beta}}\right]^{\frac{\alpha-1}{\alpha}} D^{\frac{1-\alpha}{\alpha}} \tag{29}$$

Or (29) can be written as

$$D^{-l_i}\left(\frac{\alpha-1}{\alpha}\right) < p_i^{\beta(\alpha-1)} \left[\sum_{i=1}^{n} p_i^{\alpha\beta}\right]^{\frac{1-\alpha}{\alpha}} D^{\frac{1-\alpha}{\alpha}} \tag{30}$$

Multiply equation (30) both sides by p_i^{β} and then summing over $i = 1, 2, , n$, and after simplification, we get,

$$\sum_{i=1}^{n} p_i^{\beta} D^{-l_i}\left(\frac{\alpha-1}{\alpha}\right) < \left[\sum_{i=1}^{n} p_i^{\alpha\beta}\right]^{\frac{1}{\alpha}} D^{\frac{1-\alpha}{\alpha}} \tag{31}$$

Taking logarithms to both sides with base D to equation (31), we get,

$$\log_D \left[\sum_{i=1}^{n} p_i^{\beta} D^{-l_i}\left(\frac{\alpha-1}{\alpha}\right)\right] < \frac{1}{\alpha}\log_D \left[\sum_{i=1}^{n} p_i^{\alpha\beta}\right] + \frac{1-\alpha}{\alpha} \tag{32}$$

As $0 < \alpha < 1, 0 < \beta \le 1$ then$(1-\alpha) > 0$ and $\left(\frac{\alpha\beta}{1-\alpha}\right) > 0$, multiply both sides equation (32) by $\left(\frac{\alpha\beta}{1-\alpha}\right) > 0$, we get $L_\beta^\alpha(P) < H_\beta^\alpha(P) + \beta$, where $0 < \alpha < 1, 0 < \beta \le 1$.

Thus from above two coding theorems we have shown that $H_\beta^\alpha(P) \le L_\beta^\alpha(P) < H_\beta^\alpha(P) + \beta$, where $0 < \alpha < 1, 0 < \beta \le 1$.

4 Illustration

In this section we illustrate the veracity of the theorems 3.1 and 3.2 by taking empirical data as given in table (4.1) and (4.2) on the lines of Om Parkash and Priyanka Kakkar [13].

Using Huffman coding scheme the values of $H_\beta^\alpha(P), H_\beta^\alpha(P) + \beta, L_\beta^\alpha(P)$ and η for different values of α and β are shown in the following table:

Table(4.1)

Probabilities p_i	Huffman codewords	l_i	α	β	$H_\beta^\alpha(P)$	$L_\beta^\alpha(P)$	$\eta = \frac{H_\beta^\alpha(P)}{L_\beta^\alpha} \times 100$	$H_\beta^\alpha(P)+1$
0.41	1	1	0.9	1	2.279327	2.359635	96.5666	3.279327
0.18	000	3	0.9	0.9	3.944197	4.031190	97.842	4.844197
0.15	001	3	0.8	1	2.309614	2.420838	95.40555	3.309614
0.13	010	3						
0.1	0110	4						
0.03	0111	4						

Now using Shannon-Fano coding scheme the values of $H_\beta^\alpha(P), H_\beta^\alpha(P) + \beta, L_\beta^\alpha(P)$ and η for different values of α and β are shown in the following table:

Table(4.2)

Probabilities p_i	Shannon Fano codewords	l_i	α	β	$H_\beta^\alpha(P)$	$L_\beta^\alpha(P)$	$\eta = \frac{H_\beta^\alpha(P)}{L_\beta^\alpha} \times 100$	$H_\beta^\alpha(P)+1$
0.41	00	2	0.9	1	2.279327	2.409698	94.58972	3.279327
0.18	01	2	0.9	0.9	3.944197	4.046642	97.4684	4.844197
0.15	10	2	0.8	1	2.309614	2.435809	94.81917	3.309614
0.13	110	3						
0.1	1110	4						
0.03	1111	4						

From table (4.1) and (4.2) we infer the following:

1. Theorems 3.1 and 3.2 hold both the cases of Shannon-Fano codes and Huffman codes. i.e. $H_\beta^\alpha(P) \le L_\beta^\alpha(P) < H_\beta^\alpha(P) + 1, 0 < \alpha < 1, 0 < \beta \le 1$.
2. Huffman mean code-word length is less than Shannon-Fano mean code-word length.
3. Coefficient of efficiency of Huffman codes is greater than coefficient of efficiency of Shannon-Fano codes i.e. it is concluded that Huffman coding scheme is more efficient than Shannon-Fano coding scheme.

5 Monotonic Behavior of the New Generalized Information Measure $H_\beta^\alpha(P)$

In this section we study the monotonic behavior of the new generalized information measure $H_\beta^\alpha(P)$ given in (4) with respect to the parameters α and β.

Let $P = (0.41, 0.18, 0.15, 013, 0.10, 0.03)$ be a set of probabilities. Assuming $\beta = 1$. We calculates the values of $H_\beta^\alpha(P)$ for different values of α as shown in the following table:

Table 5.1: Monotonic behaviour of $H_\beta^\alpha(P)$ with respect to α for fixed $\beta = 1$

α	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$H_\beta^\alpha(P)$	2.5470	2.5101	2.4743	2.4394	2.4056	2.3727	2.3407	2.3096	2.2793

Next we draw the graph of the table (5.1) and illustrate from fig. 1 that $H_\beta^\alpha(P)$ is monotonic decreasing with increasing values of α

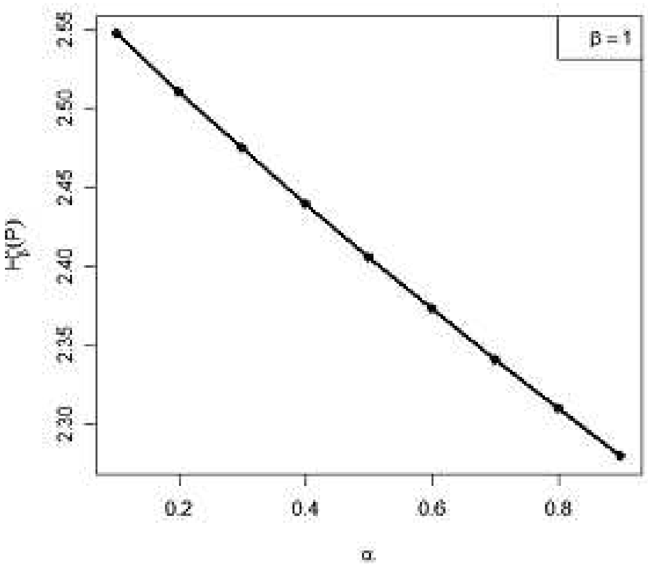

Fig. 1: Monotonic behaviour of $H_\beta^\alpha(P)$ with respect to α for fixed $\beta = 1$

Now assuming $\alpha = 0.5$. We calculates the values of $H_\beta^\alpha(P)$ for different values of β as shown in the following table:

Table 5.2: Monotonic behaviour of $H_\beta^\alpha(P)$ with respect to β for fixed $\alpha = 0.5$

β	0.2	0.4	0.6	0.8	1.0
$H_\beta^\alpha(P)$	0.9169	1.6065	2.0784	2.3419	2.4056

Next we draw the graph of the table (5.2) and illustrate from fig. 2 that $H_\beta^\alpha(P)$ is monotonic increasing with increasing values of β

6 Properties of the New Generalized Information Measure $H_\beta^\alpha(P)$

In this section we will discuss some properties of the new generalized information measure $H_\beta^\alpha(P)$ given in (4)

Property 6.1. $H_\beta^\alpha(P)$ is non-negative.

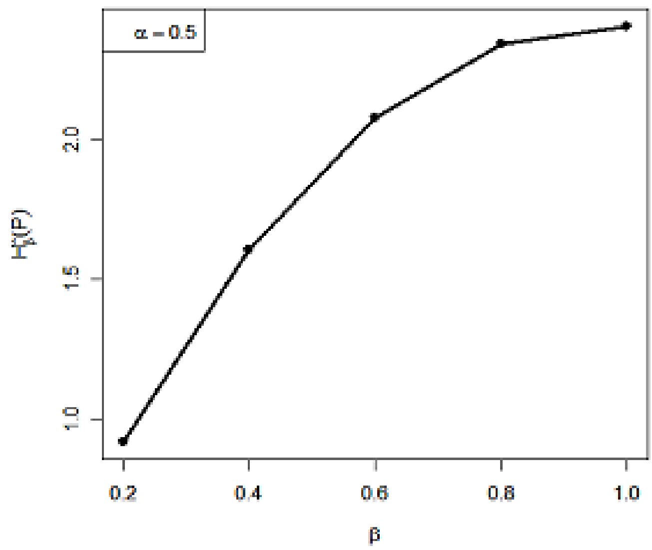

Fig. 2: Monotonic behaviour of $H_\beta^\alpha(P)$ with respect to β for fixed $\alpha = 0.5$

Proof. From (4) we have,

$$H_\beta^\alpha(P) = \frac{\beta}{1-\alpha} \log_D \left[\sum_{i=1}^{n} p_i^{\alpha\beta} \right], 0 < \alpha < 1, 0 < \beta \le 1$$

From table (4.1) and (4.2) it is observed that $H_\beta^\alpha(P)$ is non-negative for given values of α and β

Property 6.2. $H_\beta^\alpha(P)$ is a symmetric function on every $p_i, i = 1, 2, 3, \ldots, n$.

Proof. It is obvious that $H_\beta^\alpha(P)$ is a symmetric function on every $p_i, i = 1, 2, 3, \ldots, n$. i.e.,

$$H_\beta^\alpha(p_1, p_2, \ldots, p_{(n-1)}, p_n) = H_\beta^\alpha(p_n, p_1, p_2, \ldots, p_{(n-1)})$$

Property 6.3. $H_\beta^\alpha(P)$ is maximum when all the events have equal probabilities.

Proof. When $p_i = \frac{1}{n} \forall i = 1, 2, 3, \ldots, n$, $\beta = 1$, and $\alpha \to 1$. Then $H_\beta^\alpha(P) = \log_D n$, which is maximum entropy.

Property 6.4. $H_\beta^\alpha(P)$ satisfies the additivity of the following form:

$$H_\beta^\alpha(P * Q) = H_\beta^\alpha(P) + H_\beta^\alpha(Q)$$

where
$P * Q = (p_1q_1, \ldots, p_1q_m, p_2q_1, \ldots, p_2q_m, \ldots, p_nq_1, \ldots, p_nq_m)$

Proof. Let $H_\beta^\alpha(P * Q) = H_\beta^\alpha(P) + H_\beta^\alpha(Q)$

Talking R.H.S = $H_\beta^\alpha(P)+H_\beta^\alpha(Q)$

$$= \frac{\beta}{1-\alpha}\log_D\left[\sum_{i=1}^{n} p_i^{\alpha\beta}\right] + \frac{\beta}{1-\alpha}\log_D\left[\sum_{j=1}^{m} q_i^{\alpha\beta}\right]$$

$$= \frac{\beta}{1-\alpha}\left[\log_D\left[\sum_{i=1}^{n} p_i^{\alpha\beta}\right] + \log_D\left[\sum_{j=1}^{m} q_j^{\alpha\beta}\right]\right]$$

$$= \frac{\beta}{1-\alpha}\left[\log_D\left[\sum_{i=1}^{n} p_i^{\alpha\beta}\right]\left[\sum_{j=1}^{m} q_j^{\alpha\beta}\right]\right]$$

$$= \frac{\beta}{1-\alpha}\log_D\left[\left(\sum_{i=1}^{n} p_i^{\alpha\beta}\right)\left(\sum_{j=1}^{m} q_j^{\alpha\beta}\right)\right]$$

$$= \frac{\beta}{1-\alpha}\log_D\left[\sum_{i=1}^{n}\sum_{j=1}^{m}\left(p_i^{\alpha\beta}\right)\left(q_j^{\alpha\beta}\right)\right]$$

$$= \frac{\beta}{1-\alpha}\log_D\left[\sum_{i=1}^{n}\sum_{j=1}^{m}(p_i q_j)^{\alpha\beta}\right]$$

$$= H_\beta^\alpha(P*Q)$$

$$= L.H.S.$$

Hence L.H.S = R.H.S.

Property 6.5. $H_\beta^\alpha(P)$ is concave function for $p_1, p_2, ..., p_n$.

Proof. From (4) we have,

$$H_\beta^\alpha(P) = \frac{\beta}{1-\alpha}\log_D\left[\sum_{i=1}^{n} p_i^{\alpha\beta}\right], 0<\alpha<1, 0<\beta\le 1$$

If $\beta=1, \alpha\to 1$, then the first derivative of (4) with respect p_i is given by

$$\left[\frac{d}{dp_i}H_\beta^\alpha(P)\right]_{\substack{\alpha\to 1\\ \beta=1}} = -n - \sum_{i=1}^{n}\log_D p_i$$

and the second derivative is given by

$$\left[\frac{d^2}{dp_i^2}H_\beta^\alpha(P)\right]_{\substack{\alpha\to 1\\ \beta=1}} = -\sum_{i=1}^{n}\left(\frac{1}{p_i}\right) \le 0. \forall p_i \in [0,1], i=1,2,...n$$

Since the second derivative of $H_\beta^\alpha(P)$ with respect to p_i is negative on given interval $p_i \in [0,1] i=1,2,,n$. as $\beta=1$ and $\alpha\to 1$, therefore,

$H_\beta^\alpha(P)$ is concave function for $p_1, p_2,, p_n$.

7 Conclusion

In this paper we define a new generalized entropy measure i.e., $H_\beta^\alpha(P)$ of order α and type β. This measure also generalizes some well-known information measures already existing in the literature of information theory. Also we define a new generalized code-word mean length i.e., $L_\beta^\alpha(P)$ of order α and type β corresponding to $H_\beta^\alpha(P)$, then we characterize $L_\beta^\alpha(P)$ in terms of new generalized entropy measure $H_\beta^\alpha(P)$ of order α and type β and showed that $H_\beta^\alpha(P) \le L_\beta^\alpha < H_\beta^\alpha(P)+1$ where $0<\alpha<1, 0<\beta\le 1$.

Further we have established the noiseless coding theorems proved in this paper with the help of two different techniques by taking experimental data and prove that Huffman coding scheme is more efficient than Shannon-Fano coding scheme. Also we study the monotonic behavior of $H_\beta^\alpha(P)$ with respect to parameters α and type β and the important properties of $H_\beta^\alpha(P)$ have also been studied.

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