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Fixed Point Theorem in Fuzzy Metric Space Through Weak Compatibility

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Abstract: Fuzzy metric space is first defined by Kramosil and Michalek in 1975. Many authors modified Fuzzy metric space and proved fixed point results in Fuzzy metric space. Singh and Chauhan were first introduced the concept of compatible mappings of Fuzzy metric space and proved the common fixed point theorem in 2000. Cho et al. were introduced the concept of weak compatible mapping. In this paper, a fixed point theorem for six self-mappings is presented by using the concept of weak compatible maps which are the generalized result.

Keywords: Common fixed points, fuzzy metric space, compatible maps and weak compatible maps.

1 Introduction

The concept of Fuzzy sets was initially investigated by Zadeh [7] as a new way to represent vagueness in everyday life. Subsequently, it was developed by many authors and used in various fields. To use this concept in Topology and Analysis, several researchers have defined Fuzzy metric space in various ways. In this paper we deal with the Fuzzy metric space defined by Kramosil and Michalek [6] and modified by George and Veeramani [1]. Recently, Grebiec[8] has proved fixed-point results for Fuzzy metric space. In the sequel, Singh and Chauhan [3] introduced the concept of compatible mappings of Fuzzy metric space and proved the common fixed point theorem. Jain and Singh [2] proved a fixed point theorem for six self maps in a fuzzy metric space. In this paper, a fixed point theorem for six self maps has been established using the concept of weak compatibility of pairs of self maps in fuzzy metric space, which generalizes the result of Cho [9].

For the sake of completeness, we recall some definitions and known results in Fuzzy metric space.

2 Preliminaries

Definition 2.1. A binary operation $*:[0,1]\times[0,1]\to[0,1]$ is called a t-norm if $([0,1],*)$ is an abelian topological monoid with unit 1 such that $a*b\leq c*d$ whenever $a\leq c$ and $b\leq d$ for $a,b,c,d\in[0,1]$.

Example of t-norms are $a*b=ab$ and $a*b=min\{a,b\}$.

Definition 2.2. The 3- tuple $(X,M,*)$ is said to be a Fuzzy metric space if X is an arbitrary set, $*$ is an continuous t - norm and M is a Fuzzy set in $X^2\times[0,\infty)$ satisfying the following conditions : for all $x,y,z\in X$ and $s,t>0$
$M(x,y,0)=0$,
$M(x,y,t)=1$ for all $t>0$ iff $x=y$,
$M(x,y,t)=M(y,x,t)$,
$M(x,y,t)*M(y,z,s)\leq M(x,y,t+s)$,
$M(x,y,\cdot):[0,\infty)\to[0,1]$ is left continuous

Note that $M(x,y,t)$ can be considered as the degree of nearness between x and y with respect to t . We identity $x=y$ with $M(x,y,t)=1$ for all $t>0$. The following example shows that every metric space induces a Fuzzy metric space.

Example 2.1. Let (x,d) be a metric space. Define $a*b=min\{a,b\}$ and $M(x,y,t)=\frac{t}{t+d(x,y)}$ for all $x,y\in X$ and $t>$

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0. Then $(X,M,*)$ is a Fuzzy metric space. It is called the Fuzzy metric space induced by d.

Definition 2.3. A sequence $\{x_n\}$ in a Fuzzy metric space $(X,M,*)$ is said to be a Cauchy sequence if and only if for each $\varepsilon > 0, t > 0$, there exists $n_0 \in N$ such that $M(x_n,x_m,t) > 1-\varepsilon$ for all $n,m \geq n_0$. The sequence $\{x_n\}$ is said to converge to a point x in X if and only if for each $\varepsilon > 0, t > 0$ there exists $n_0 \in N$ such that $M(x_n,x,t) > 1-\varepsilon$ for all $n,m \geq n_0$. A Fuzzy metric space $(X,M,*)$ is said to be complete if every Cauchy sequence in it converges to a point in it.

Definition 2.4. Self mappings A and S of a Fuzzy metric space $(X,M,*)$ are said to be compatible if and only if $M(ASx_n,SAx_n,t) \to 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $Sx_n, Ax_n \to p$ for some p in X as $n \to \infty$.

Definition 2.5. Self maps A and S of a Fuzzy metric space $(X,M,*)$ are said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, i.e. if $Ap = Sp$ for some $p \in X$ then $ASp = SAp$.

Proposition 2.1. Self-mapping A and B of a Fuzzy metric space $(X,M,*)$ are compatible .

proof Suppose $Ap = Sp$, for some p in X. Consider a sequence $\{p_n\} = p$. Now, $\{Ap_n\} \to Ap$ and $\{Sp_n\} \to Sp(Ap)$. As A and S are compatible we have $M(ASp_n,SAp_n,t) \to 1$ for all $t > 0$ as $n \to \infty$. Thus $ASp_n = SAp_n$ and we get that (A,S) is weakly compatible. The following is an example of pair of self-maps in a Fuzzy metric space which are weakly compatible but not compatible.

Example 2.2 Let $(X,M,*)$ be a Fuzzy metric space where $X = [0,2]$. t- norm is defined by $a*b = min\{a,b\}$ for all $a,b \in [0,1]$ and $M(x,y,t) = e^{-\frac{|x-y|}{t}}$ for all $x,y \in X$. Define self maps A and S on X as follows:

$$A_x = \begin{cases} 2-x & \text{if } 0 \leq x \leq 1 \\ 2 & \text{if } 1 \leq x \leq 2 \end{cases} \quad and \quad S_x = \begin{cases} x & \text{if } 0 \leq x < 1 \\ 2 & \text{if } 1 \leq x \leq 2. \end{cases}$$

Taking

$$x_n = 1 - \frac{1}{n},\ n = 1,2,3...$$

. Then

$$x_n \to, x_n < 1 \ and \ 2 - x_n > 2 \ for\ all.$$

Also

$$Ax_n, Sx_n \to 1 \ as\ n \to \infty$$

$$M(ASx_n,SAx_n,t) = e^{-\frac{1}{2}} \neq 1 \ as\ n \to \infty.$$

Hence the pair (A,S) is not compatible. Also set of coincidence points of A and S is $[1,2]$. Now for any $x \in [1,2]$, $Ax = Sx = 2$ and $AS(x) = A(2) = 2 = S(2) = SA(x)$. Thus A and S are weakly compatible but not compatible. From the above example, it is obvious that the concept of weak compatibility is more general than that of compatibility.

Proposition 2.2. In a fuzzy metric space $(X,M,*)$ limit of a sequence is unique.

Lemma 2.1. Let $(X,M,*)$ be a fuzzy metric space. Then for all $x,y \in X$, $M(x,y,.)$ is a non-decreasing function.

Lemma 2.2. Let $(X,M,*)$ be a fuzzy metric space. If there exists $k \in (0,1)$ such that for all $x,y \in X, M(x,y,kt) \geq M(x,y,t), \forall t > 0$, then $x = y$.

Lemma 2.3. Let $\{x_n\}$ be a sequence in a fuzzy metric space $(X,M,*)$. If there exists a number $k \in (0,1)$ such that $M(x_{n+2},x_{n+1},kt) \geq M(x_{n+1},x_n,t). \forall t > 0$ and $n \in N$. Then $\{x_n\}$ is a Cauchy sequence in X.

Lemma 2.4. The only t-norm $*$ satisfying $r*r \geq r$ for all $r \in [0,1]$ is the minimum t-norm, that is $a*b = min\{a,b\}$ for all $a,b \in [0,1]$.

3 Main Result

Theorem 3.1. Let $(X,M,*)$ be a complete fuzzy metric space and let A, B, S, T, P and Q be mappings from X into itself such that the following conditions are satisfied:
(a) $P(X) \subset ST(X)$, $Q(X) \subset AB(X)$;
(b) $AB = BA, ST = TS, PB = BP, QT = TQ$;
(c) either AB or P is continuous;
(d) (P,AB) is compatible and (Q,ST) is weakly compatible;
(e) there exists $q \in (0,1)$ such that for every $x,y \in X$ and $t > 0$

$M(Px,Qy,qt) \geq M(ABx,STy,t) * M(Px,ABx,t) * M(Qy,STy,t) * M(Px,STy,t)$

Then A, B, S, T, P and Q have a unique common fixed point in X.

Proof. Let $x_0 \in X$. From (a) there exist $x_1,x_2 \in X$ such that $Px_0 = STx_1$ and $Qx_1 = ABx_2$. Inductively, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that $Px_{2n-2} = STx_{2n-1} = y_{2n-1}$ and $Qx_{2n-1} = ABx_{2n} = y_{2n}$ for $n = 1,2,3,\ldots$.
Step 1. Put $x = x_{2n}$ and $y = x_{2n+1}$ in (e), we get $M(Px_{2n},Qx_{2n+1},qt) \geq M(ABx_{2n},STx_{2n+1},t) * M(Px_{2n},ABx_{2n},t) * M(Qx_{2n+1},STx_{2n+1},t) * M(Px_{2n},STx_{2n+1},t)$
$= M(y_{2n},y_{2n+1},t) * M(y_{2n+1},y_{2n},t) * M(y_{2n+2},y_{2n+1},t) * M(y_{2n+1},y_{2n+1},t) \geq M(y_{2n},y_{2n+1},t) * M(y_{2n+1},y_{2n+2},t)$.
From lemma 2.1 and 2.2, we have
$M(y_{2n+1},y_{2n+2},qt) \geq M(y_{2n},y_{2n+1},t)$.
Similarly, $M(y_{2n+2},y_{2n+3},qt) \geq M(y_{2n+1},y_{2n+2},t)$.
Thus, $M(y_{n+1},y_{n+2},qt) \geq M(y_n,y_{n+1},t)$ for $n = 1,2,\ldots$
$M(y_n,y_{n+1},t) \geq M(y_n,y_{n+1},\frac{t}{q})$
$\geq M(y_{n-2},y_{n-1},\frac{t}{q^2})$
....
$\geq M(y_1,y_2,\frac{t}{q^n}) \to 1$ as $n \to \infty$,
and hence $M(y_n,y_{n+1},t) \to 1$ as $n \to \infty$ for any $t > 0$. For each $\varepsilon > 0$ and $t > 0$, we can choose $n_0 \in N$ such that

$M(y_n, y_{n+1}, t) > 1-\varepsilon$ for all $n > n_0$. For $m, n \in N$, we suppose $m > n$.Then we have
$M(y_n, y_m, t) \geq M(y_n, y_{n+1}, \frac{t}{m-n}) * M(y_{n+1}, y_{n+2}, \frac{t}{m-n}) * \ldots * M(y_{m-1}, y_m, \frac{t}{m-n})$
$\geq (1-\varepsilon) * (1-\varepsilon) * \ldots * (1-\varepsilon)(m-n)$ times
$\geq (1-\varepsilon)$ and hence $\{y_n\}$ is a Cauchy sequence in X. Since $(X, M, *)$ is complete, $\{y_n\}$ converges to some point $z \in X$. Also its subsequences converges to the same point $z \in X$ i.e,

$$\{Qx_{2n+1}\} \to z \text{ and } \{STx_{2n+1}\} \to z \tag{1}$$

$$\{Px_{2n}\} \to z \text{ and } \{ABx_{2n}\} \to z. \tag{2}$$

Case I. Suppose AB is continuous. Since AB is continuous, we have $(AB)^2x_{2n} \to ABz$ and $ABPx_{2n} \to ABz$. As (P, AB) is compatible pair, then $PABx_{2n} \to ABz$.
Step 2. Put $x = ABx_{2n}$ and $y = x_{2n+1}$ in (e), we get
$M(PABx_{2n}, Qx_{2n+1}, qt) \geq M(ABABx_{2n}STx_{2n+1}, t) * M(PABx_{2n}, ABABx_{2n}, t) * M(Qx_{2n+1}, STx_{2n+1}, t) * M(PABx_{2n}, STx_{2n+1}, t)$.
Taking $n \to \infty$, we get
$M(ABz, z, qt) \geq M(ABz, z, t) * M(ABz, ABz, t) * M(z, z, t) * M(ABz, z, t)$.
$\geq M(ABz, z, t) * M(ABz, z, t)$.
i.e. $M(ABz, z, qt) \geq M(ABz, z, t)$.
Therefore, by using lemma 2.2, we get

$$ABz = z. \tag{3}$$

Step 3 Put $x = z$ and $y = x_{2n+1}$ in (e), we have
$M(Pz, Qx_{2n+1}, qt) \geq M(ABz, STx_{2n+1}, t) * M(Pz, ABz, t) * M(Qx_{2n+1}, STx_{2n+1}, t) * M(Pz, STx_{2n+1}, t)$.
Taking $n \to \infty$ and using equation (1), we get
$M(Pz, z, qt) \geq M(z, z, t) * M(Pz, z, t) * M(z, z, t) * M(Pz, z, t)$
$\geq M(Pz, z, t) * M(Pz, z, t)$
i.e. $M(Pz, z, qt) \geq M(Pz, z, t)$.
Therefore, by using lemma 2.2, we get $Pz = z$. Therefore, $ABz = Pz = z$.

Step 4. Putting $x = Bz$ and $y = x_{2n+1}$ in condition (e), we get
$M(PBz, Qx_{2n+1}, qt) \geq M(ABBz, STx_{2n+1}, t) * M(PBz, ABBz, t) * M(Qx_{2n+1}, STx_{2n+1}, t) * M(PBz, STx_{2n+1}, t)$.
As $BP = PB, AB = BA$, so we have $P(Bz) = B(Pz) = Bz$ and $(AB)(Bz) = (BA)(Bz) = B(ABz) = Bz$. Taking $n \to \infty$ and using (1), we get
$M(Bz, z, qt) \geq M(Bz, z, t) * M(Bz, Bz, t) * M(z, z, t) * M(Bz, z, t)$
$\geq M(Bz, z, t) * M(Bz, z, t)$ i.e. $M(Bz, z, qt) \geq M(Bz, z, t)$.
Therefore, by using lemma 2.2, we get $Bz = z$ and also we have $ABz = z.Az = z$. Therefore,

$$Az = Bz = Pz = z. \tag{4}$$

Step 5 As $P(X) \subset ST(X)$, there exists $u \in X$ such that $z = Pz = STu$. Putting $x = x_{2n}$ and $y = u$ in (e), we get
$M(Px_{2n}, Qu, qt) \geq M(ABx_{2n}, STu, t) * M(Px_{2n}, ABx_{2n}, t) * M(Qu, STu, t) * M(Px_{2n}, STu, t)$.
Taking $n \to \infty$ and using (1) and (2), we get
$M(z, Qu, qt) \geq M(z, z, t) * M(z, z, t) * M(Qu, z, t) * M(z, z, t)$
$\geq M(Qu, z, t)$ i.e. $M(z, Qu, qt)M(z, Qu, t)$.
Therefore, by using lemma 2.2, we get $Qu = z$. Hence $STu = z = Qu$. Since (Q, ST) is weak compatible therefore, we have $QSTu = STQu$. Thus $Qz = STz$.
Step 6. Putting $x = x_{2n}$ and $y = z$ in (e), we get
$M(Px_{2n}, Qz, qt) \geq M(ABx_{2n}, STz, t) * M(Px_{2n}, ABx_{2n}, t) * M(Qz, STz, t) * M(Px_{2n}, STz, t)$.
Taking $n \to \infty$ and using (2) and step 5, we get
$M(z, Qz, qt) \geq M(z, Qz, t) * M(z, z, t) * M(Qz, Qz, t) * M(z, Qz, t)$
$\geq M(z, Qz, t) * M(z, Qz, t)$ i.e.$M(z, Qz, qt) \geq M(z, Qz, t)$.
Therefore, by using lemma 2.2, we get $Qz = z$. Step 7. Putting $x = x_{2n}$ and $y = Tz$ in (e), we get
$M(Px_{2n}, QTz, qt) \geq M(ABx_{2n}, STTz, t) * M(Px_{2n}, ABx_{2n}, t) * M(QTz, STTz, t) * M(Px_{2n}, STTz, t)$.
As $QT = TQ$ and $ST = TS$, we have $QTz = TQz = Tz$ and $ST(Tz) = T(STz) = TQz = Tz$.
Taking $n \to \infty$, we get
$M(z, Tz, qt) \geq M(z, Tz, t) * M(z, z, t) * M(Tz, Tz, t) * M(z, Tz, t)$
$\geq M(z, Tz, t) * M(z, Tz, t)$ i.e. $M(z, Tz, qt) \geq M(z, Tz, t)$.
Therefore, by using lemma 2.2, we get $Tz = z$. Now $STz = Tz = z$implies $Sz = z$. Hence

$$Sz = Tz = Qz = z. \tag{5}$$

Combining (4) and (5), we get$Az = Bz = Pz = Qz = Tz = Sz = z$. Hence, z is the common fixed point of A, B, S, T, P and Q . Case II. Suppose P is continuous. As P is continuous, $P^2x_{2n} \to Pz$ and $P(AB)x_{2n} \to Pz$. As (P, AB) is compatible, we have $(AB)Px_{2n} \to Pz$.

Step 8. Putting $x = Px_{2n}$ and $y = x_{2n+1}$ in condition (e), we have
$M(PPx_{2n}, Qx_{2n+1}, qt) \geq M(ABPx_{2n}, STx_{2n+1}, t) * M(PPx_{2n}, ABPx_{2n}, t) * M(Qx_{2n+1}, STx_{2n+1}, t) * M(PPx_{2n}, STx_{2n+1}, t)$.
Taking $n \to \infty$, we get
$M(Pz, z, qt) \geq M(Pz, z, t) * M(Pz, Pz, t) * M(z, z, t) * M(Pz, z, t)$
$\geq M(Pz, z, t) * M(Pz, z, t)$ i.e. $M(Pz, z, qt) \geq M(Pz, z, t)$.
Therefore by using lemma 2.2, we have $Pz = z$. Further, using steps 5, 6, 7, we get $Qz = STz = Sz = Tz = z$.

Step 9. As $Q(X) \subset AB(X)$, there exists $w \in X$ such that $z = Qz = ABw$. Put $x = w$ and $y = x_{2n+1}$ in (e), we have
$M(Pw, Qx_{2n+1}, qt) \geq M(ABw, STx_{2n+1}, t) * M(Pw, ABw, t) * M(Qx_{2n+1}, STx_{2n+1}, t) * M(Pw, STx_{2n+1}, t)$.
Taking $n \to \infty$, we get
$M(Pw, z, qt) \geq M(z, z, t) * M(Pw, z, t) * M(z, z, t) * M(Pw, z, t)$
$\geq M(Pw, z, t) * M(Pw, z, t)$ i.e. $M(Pw, z, qt) * M(Pw, z, t)$.

Therefore, by using lemma 2.2, we get $Pw = z$.
Therefore, $ABw = Pw = z$. As (P,AB) is compatible, we have $Pz = ABz$. Also, from step 4, we get $Bz = z$.
Thus, $Az = Bz = Pz = z$ and we see that z is the common fixed point of the six maps in this case also. Uniqueness: Let u be another common fixed point of A,B,S,T,P and Q.
Then $Au = Bu = Pu = Qu = Su = Tu = u$. Put $x = z$ and $y = u$ in (e), we get
$M(Pz,Qu,qt) \geq M(ABz,STu,t) * M(Pz,ABz,t) * M(Qu,STu,.t) * M(Pz,STu,t)$.
Taking $n \to \infty$, we get
$M(z,u,qt) \geq M(z,u,t) * M(z,z,t) * M(u,u,t) * M(z,u,t)$
$\geq M(z,u,t) * M(z,u,t)$ i.e. $M(z,u,qt) \geq M(z,u,t)$.
Therefore by using lemma 2.2, we get $z = u$. Therefore z is the unique common fixed point of self-maps A,B,S,T,P and Q.

Remark 3.1. If we take $B = T = I$ then condition (b) of theorem 3.1, is satisfied trivially.

Corollary 3.1. Let $(X,M,*)$ be a complete fuzzy metric space and let A,S,P and Q be mappings from X into itself such that the following conditions are satisfied:
(a) $P(X) \subset S(X)$, $Q(X) \subset A(X)$;
(b) either A or P is continuous;
(c) (P,A) is compatible and (Q,S) is weakly compatible;
(d) there exists $q \in (0,1)$ such that for every x,yX and $t > 0$
(e) $M(Px,Qy,qt) \geq M(Ax,Sy,t) * M(Px,Ax,t) * M(Qy,Sy,t) * M(Px,Sy,t)$.
Then A,S,P and Q have a unique common fixed point in X.

Remark 3.2. In view of remark 3.1, corollary 3.1 is a generalization of the result of Cho [9] in the sense that condition of compatibility of the pairs of self-maps has been restricted to weak compatibility and only one map of the first pair is needed to be continuous.

4 Conclusion

In this paper, a fixed point theorem for six self-mappings is presented by using the concept of weak compatibility which is the generalized result.

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