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Slippage of Magnetohydrodynamic Fractionalized Oldroyd-B Fluid in Porous Medium

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Abstract: The present study is analyzed for slippage of velocity field and shear stress for accelerating flows of magnetohydrodynamics Oldroyd-B fluid in presence of porosity. The analytical solutions have been obtained by employing fractional derivative approach on the governing partial differential equations. The expressions of general solutions have been presented in terms of newly defined $\mathbf{M}_q^p$ function. These solutions are verified for initial and boundary conditions. Some special cases of fluid are particularized for ordinary Oldroyd-B, fractionalized and ordinary Maxwell, fractionalized and ordinary second grade and also for Newtonian fluids with respect to fractional parameters $\alpha, \gamma, \lambda_r, \lambda, \beta, \rho, t, \theta, \mu, \nu, p$ and Φ. The influence of rheological and fractional parameters is analyzed through graphical illustrations and depicted in with and without slippage.

Keywords: Slippage, Oldroyd-B fluid, magnetohydrodynamics, porosity, rheological parameters.

1 Introduction

The rheological behavior of Newtonian fluid flows are characterized by classical Navier stokes equations, but the description of non-Newtonian fluid flows is unable to characterized by classical Navier stokes equations. This is due to the appearance of nonlinearity among the classical Navier stokes equations. The variety of significant applications of non-Newtonian fluid flows lie in various sciences and technology. The theoretical and practical analysis of non-Newtonian fluids has diverted interest among scientists, engineers and researchers because of their complex rheological structures of fluids, for instance emulsions, polymer solutions, pastes, blood, slurries, heavy oils and many others lead to shear thinning behavior. The geometry of these fluids flows of is very complicated and difficult to handle due to their nonlinearity among rate of strain and stress. On the other hand, while the mathematical formulation of these fluid flows leads to complex partial differential equations due to their dynamical behavior which has become the challenge among physicists and mathematicians. The general solutions of these complex partial differential equations are not an easy task because of their lengthy and cumbersome calculations [1]-[8].The interlinks and communications between the principles of electromagnetic fields and hydrodynamics is based on magnetohydrodynamics (MHD). It enables the various phenomenon concerned with astrophysics for instance, stellar magnetic fields, galactic fields, sunspots, supernovae remnants, black holes, thermonuclear reactors, relativistic radio jets and active galactic nuclei and various others. In continuation, the investigation of complex rheology of fluid flows in porous medium is really challenge and encountering because of it distinct applications in certain areas for instance, water resources, biological sciences, oil and gas exploration, and irrigation and agricultural sciences etc. Fetecau, C et al. has studied note on an Oldroyd-B fluid for unsteady flow and between two plates [9,10]. Tong et al. used Oldrord-B model in which they investigated viscoelastic fluid in an annular pipe for the Couette and Poiseuille flows [11,12]. An oscillating porous plate for the flow of a viscoelastic second grade fluid is investigated by Hayat et al [13,14]. Meanwhile for Stokes second problem, analytical solution of second grade fluid obtained by Asghar et al. [15]. Most recently, various researchers have investigated with different aspects for unsteady Oldrord-B fluid in hydromagnetic and hydrodynamic flows [16,17,18, 19,20] Motivated by above mentioned study in fluid mechanics, the current analysis is undertaken for the impacts of

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magnetohydrodynamics and porosity with and without slippage of Oldrord-B fluid. The analytical solutions have been obtained for velocity field and shear stress by employing fractional derivative approach. The expressions of general solutions have been presented in terms of newly defined $\mathbf{M}_q^p(z)$ function. These solutions are also verified for initial and boundary conditions. Some limiting cases of fluids have been particularized for ordinary Oldroyd-B, fractionalized and ordinary Maxwell, fractionalized and ordinary Second grade and also for Newtonian fluids. The influence of rheological and pertinent parameters on the motion fluid is analyzed through graphical illustrations and depicted in with and without slippage.

2 The Model

The flow of an incompressible fluid in which the equation of motion and continuity equation for the absence of body forces are given by

$$\nabla V = 0,\ \rho\frac{dv}{dt} - \nabla\mathbf{T} - \rho\mathbf{b} = 0, \tag{1}$$

where ∇ represents the gradient operator,V is the velocity field, T is the Cauchy stress tensor, ρ is the constant density of the fluid, b is the body force field. The incompressible Oldroyd-B fluid for which the Cauchy stress T is given by

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S},\ \mu[A + \lambda_r(\dot{\mathbf{A}} + \mathbf{LA} - \mathbf{AL}^T)] = S + \lambda(\dot{\mathbf{S}} + \mathbf{LS} - \mathbf{SL}^T), \tag{2}$$

where $-p\mathbf{I}$ denotes the indeterminate spherical stress, $\mathbf{S}$ is the extra-stress tensor, $\mathbf{L}$ is the velocity gradient, $\dot{\mathbf{A}}$ is the first Rivlin Ericksen tensor, μ is the dynamic viscosity of the fluid, λ and λ_r are relaxation and retardation times. The constitutive eq. (2) comprises as limiting cases as Maxwell model and the Newtonian fluid model for λ_r and $\lambda = \lambda_r$ respectively. Extra-stress tensor $\mathbf{S}$ and velocity field V are assumed as:

$$V = V(y,t) = u(y,t)i,\ \mathbf{S} = \mathbf{S}(y,t). \tag{3}$$

On the x-coordinate direction i is the unit vector. Such flows can be satisfied for constraint of incompressibility and in the x-direction without pressure gradient, the governing equations of the fractionalized magnetohydrodynamics Oldroyd-B fluid in porous medium are [21,22]

$$\frac{\partial w(y,t)}{\partial t}\left(\lambda^{\alpha}\frac{\partial^{\alpha}}{\partial t^{\alpha}} + 1\right) - \nu\left(\lambda_r^{\gamma}\frac{\partial^{\gamma}}{\partial t^{\gamma}} + 1\right)\frac{\partial^2 w(y,t)}{\partial y^2} + B\left(\lambda^{\alpha}\frac{\partial^{\alpha}}{\partial t^{\alpha}} + 1\right)w(y,t) + \Phi\left(\lambda^{\alpha}\frac{\partial^{\alpha}}{\partial t^{\alpha}} + 1\right)w(y,t) = 0, \tag{4}$$

$$\tau(y,t)\left(\lambda^{\alpha}\frac{\partial^{\alpha}}{\partial t^{\alpha}} + 1\right) - \mu\left(\lambda_r^{\gamma}\frac{\partial^{\gamma}}{\partial t^{\gamma}} + 1\right)\frac{\partial w(y,t)}{\partial y} = 0, \tag{5}$$

where,$\nu = \frac{\mu}{\rho}, B = \frac{\sigma B_0}{\rho}, \Phi = \frac{\phi\mu}{k}$ is the kinematic viscosity, magnetic field and porosity of the fluid respectively [23,24]. Where, α and γ are the parameters of fractionalized calculus such that $0 \le \alpha, \gamma \le 1$ and Caputo fractional operator D_t^{α} defined by [25,26]

$$D_t^p f(t) = \frac{1}{\Gamma(1-p)}\int_0^t \frac{f'(q)}{(t-q)^p}\,dq; \quad 0 \le p < 1. \tag{6}$$

$\Gamma(\bullet)$ is the Gamma function.

3 Statement of the Problem

In this problem, we consider the flow problem of an incompressible magnetohydrodynmics Oldroyd-B fluid in porous medium having fractionalized differential approach over plate that is positioned in the (x,z) plane and perpendicular to the y-axis. Here we assume the presence of slip boundary condition among the velocity and shear rate of the fluid. The fluid and plate both are initially at rest and at $t = 0^+$ the plate starts in variably accelerating in its own plane. Owing to tangential shear, the fluid above the plate is slowly moved having the velocity of the form $(3)_1$ the equations which governs the flow are given by Eqs. (4) and (5) and corresponding initial and boundary conditions are

$$w(y,0) = \frac{\partial w(y,0)}{\partial t} = 0;\quad \tau(y,0) = 0,\ y > 0, \tag{7}$$

$$w(0,t) = UH(t)t^p + \theta \frac{\partial w(y,0)}{\partial t}|_{y=0}; \ t \geq 0, \ . \tag{8}$$

Here $H(t)$ is the Heaviside function. Moreover, the natural conditions

$$w(y,t), \ \frac{\partial w(y,t)}{\partial y} \to 0 \text{ as } y \to \infty \text{ and } t > 0, \tag{9}$$

have to be also satisfied and θ is slippage scale of the flow.

4 Solution of the Problem

4.1 Velocity Field

In order to solve governing equation, applying the Laplace transform to eq. (4), and taking into account the initial and boundary conditions (7) and (8), we find that

$$\left(\frac{\partial^2}{\partial y^2} - \frac{(s+\Phi+B)(\lambda^\alpha s^\alpha+1)}{\nu(\lambda_r^\gamma s^\gamma+1)}\right)\overline{w}(y,s) = 0. \tag{10}$$

Using initial and boundary condition in eq. (10),

$$\overline{w}(y,s) \to 0 \ as \ y \to \infty, \ \overline{w}(0,s) = \frac{U}{s^{p+1}}, \tag{11}$$

where $\overline{w}(y,s)$ be the Laplace transform of $w(y,t)$, computing eqs. (10) and (11), we get

$$\overline{w}(y,s) = \frac{Up!\,Exp\left(-\sqrt{\frac{(s+\Phi+B)(\lambda^\alpha s^\alpha+1)}{\nu(\lambda_r^\gamma s^\gamma+1)}}y\right)}{s^{p+1}\left[1+\theta\sqrt{\frac{(s+\Phi+B)(\lambda^\alpha s^\alpha+1)}{\nu(\lambda_r^\gamma s^\gamma+1)}}\right]}. \tag{12}$$

In order to obtain velocity profile, we apply the discrete inverse Laplace transform. Writing eq. (12) as series form

$$\overline{w}(y,s) = \frac{Up!}{s^{p+1}} + Up!\sum_{\zeta=1}^{\infty}\left(\frac{-\theta}{\sqrt{\nu}}\right)^{\zeta}\sum_{j=0}^{\infty}\frac{(-\Phi-B)^j}{j!}\sum_{\eta=0}^{\infty}\frac{(-\lambda^\alpha)^\eta}{\eta!}\sum_{k=0}^{\infty}\frac{(-\lambda_r^\gamma)^k\Gamma(\frac{\zeta}{2}+1)\Gamma(\frac{\zeta}{2}+k)\Gamma(\frac{\zeta}{2}+1)}{k!\Gamma(-\eta+1+\frac{\zeta}{2})\Gamma(\frac{\zeta}{2})\Gamma(-j+1+\frac{\zeta}{2})},$$

$$\times\frac{1}{s^{-\frac{\zeta}{2}-\gamma k-\alpha\eta+j+p+1}} + Up!\sum_{\zeta=1}^{\infty}\frac{1}{\zeta!}\left(\frac{-y}{\sqrt{\nu}}\right)^{\zeta}\sum_{j=0}^{\infty}\frac{(-\Phi-B)^j}{j!}\sum_{\eta=0}^{\infty}\left(\frac{-\theta}{\sqrt{\nu}}\right)^{\eta}\sum_{k=0}^{\infty}\frac{(-\lambda^\alpha)^k}{k!}\sum_{l=0}^{\infty}\frac{(-\lambda_r^\gamma)^l}{l!}$$

,

$$\times\frac{\Gamma(\zeta+\frac{\eta}{2}+1)\Gamma(l+\zeta+\frac{\eta}{2})\Gamma(\zeta+\frac{\zeta}{2}+1)}{\Gamma(-j+\zeta+\frac{\eta}{2}+1)\Gamma(\zeta+\frac{\eta}{2})\Gamma(-k+\zeta+\frac{\eta}{2}+1)}\frac{1}{s^{-\gamma l-\alpha k+j-\zeta+\frac{\eta}{2}+p+1}} \tag{13}$$

. Inverting eq. (13), by discrete inverse Laplace transform, the suitable expression is

$$w(y,t) = UH(t)t^p + UH(t)p!\sum_{\zeta=1}^{\infty}\left(\frac{-\theta}{\sqrt{\nu}}\right)^{\zeta}\sum_{j=0}^{\infty}\frac{(-\Phi-B)^j}{j!}\sum_{\eta=0}^{\infty}\frac{(-\lambda^\alpha)^\eta}{\eta!}\sum_{k=0}^{\infty}\frac{(-\lambda_r^\gamma)^k\Gamma(\frac{\zeta}{2}+1)\Gamma(\frac{\zeta}{2}+k)\Gamma(\frac{\zeta}{2}+1)}{k!\Gamma(-\eta+1+\frac{\zeta}{2})\Gamma(\frac{\zeta}{2})\Gamma(-j+1+\frac{\zeta}{2})},$$

$$\times\frac{t^{-\frac{\zeta}{2}-\gamma k-\alpha\eta+j+p}}{\Gamma(-\frac{\zeta}{2}-\gamma k-\alpha\eta+j+p+1)} + Up!\sum_{\zeta=1}^{\infty}\frac{1}{\zeta!}\left(\frac{-y}{\sqrt{\nu}}\right)^{\zeta}\sum_{j=0}^{\infty}\frac{(-\Phi-B)^j}{j!}\sum_{\eta=0}^{\infty}\left(\frac{-\theta}{\sqrt{\nu}}\right)^{\eta}\sum_{k=0}^{\infty}\frac{(-\lambda^\alpha)^k}{k!}\sum_{l=0}^{\infty}\frac{(-\lambda_r^\gamma)^l}{l!},$$

$$\times\frac{\Gamma(\zeta+\frac{\eta}{2}+1)\Gamma(l+\zeta+\frac{\eta}{2})\Gamma(\zeta+\frac{\zeta}{2}+1)}{\Gamma(-j+\zeta+\frac{\eta}{2}+1)\Gamma(\zeta+\frac{\eta}{2})\Gamma(-k+\zeta+\frac{\eta}{2}+1)}\frac{t^{-\gamma l-\alpha k+j-\zeta+\frac{\eta}{2}+p}}{\Gamma(-\gamma l-\alpha k+j-\zeta+\frac{\eta}{2}+p+1)}, \tag{14}$$

we obtain the compact form of eq. (14) in terms of $M_q^p(z)$,

$$w(y,t)=UH(t)t^p+UH(t)p\,!\sum_{\zeta=1}^{\infty}\left(\frac{-\theta}{\sqrt{\nu}}\right)^{\zeta}\sum_{\eta=0}^{\infty}\frac{(-\lambda^{\alpha})^{\eta}}{\eta\,!}\sum_{j=0}^{\infty}\frac{(-\Phi-B)^{j}}{j\,!},$$
$$\times\mathbf{M}_4^3\left[-\lambda_r^{\gamma}\Big|_{\left(-\eta+\frac{\zeta}{2}+1,0\right),\left(\frac{\zeta}{2},0\right),\left(-j+\frac{\zeta}{2}+1,0\right),\left(-\frac{\zeta}{2}-\alpha\eta+j+p+1,-\gamma\right)}^{\left(\frac{\zeta}{2}+1,0\right),\left(\frac{\zeta}{2},0\right),\left(\frac{\zeta}{2}+1,0\right)}\right],$$
$$+Up\,!\sum_{\zeta=1}^{\infty}\frac{1}{\zeta!}\left(\frac{-y}{\sqrt{\nu}}\right)^{\zeta}\sum_{j=0}^{\infty}\frac{(-\Phi-B)^{j}}{j\,!}\sum_{\eta=0}^{\infty}\left(\frac{-\theta}{\sqrt{\nu}}\right)^{\eta}\sum_{k=0}^{\infty}\frac{(-\lambda^{\alpha})^{k}}{k\,!},$$
$$\times\mathbf{M}_4^3\left[-\lambda_r^{\gamma}\Big|_{\left(-j+\zeta+\frac{\eta}{2}+1,0\right),\left(\zeta+\frac{\eta}{2},0\right),\left(-k+\zeta+\frac{\eta}{2}+1,0\right),\left(-\zeta-\frac{\eta}{2}-\alpha k+j+p+1,-\gamma\right)}^{\left(\zeta+\frac{\eta}{2}+1,0\right),\left(\zeta+\frac{\eta}{2},1\right),\left(\zeta+\frac{\eta}{2}+1,0\right)}\right], \tag{15}$$

in which $M_q^p(z)$ is the M- function expressed below

$$t^{b_q-1}\sum_{n}^{\infty}\frac{(z)^n\Pi_{j=1}^{p}\Gamma(a_j+A_jn)}{n!\,\Pi_{j=1}^{q}\Gamma(b_j+B_jn)}=\mathbf{M}_q^p\left[z\Big|_{(b_1,B_1),(b_2,B_2),\ldots,(b_q,B_q)}^{(a_1,A_1),(a_2,A_2),\ldots,(a_p,A_p)}\right]. \tag{16}$$

4.2 *Shear Stress*

Apply the Laplace transform to eq. (5) and using the initial condition eq. (7), the expression for shear stress is

$$\overline{\tau}(y,s)=\frac{\mu\partial\overline{w}(y,s)}{\partial y}\frac{(\lambda_r^{\gamma}s^{\gamma}1+)}{(\lambda^{\alpha}s^{\alpha}+1)}. \tag{17}$$

Setting eq. (12) into (17), we find that

$$\overline{\tau}(y,s)=\frac{\mu Up\,!\sqrt{(s+\Phi+B)(\lambda_r^{\gamma}s^{\gamma}+1)}}{\sqrt{\nu}s^{p+1}\sqrt{(\lambda^{\alpha}s^{\alpha}+1)}\left[1+\theta\sqrt{\frac{(s+\Phi+B)(\lambda^{\alpha}s^{\alpha}+1)}{\nu(\lambda_r^{\gamma}s^{\gamma}+1)}}\right]}\,Exp\left(-\sqrt{\frac{(s+\Phi+B)(\lambda^{\alpha}s^{\alpha}+1)}{\nu(\lambda_r^{\gamma}s^{\gamma}+1)}}\right). \tag{18}$$

Solving eq. (18) in more suitable representation in series form

$$\overline{\tau}(y,s)=-\frac{\mu Up\,!}{\sqrt{\nu}}\sum_{\zeta=0}^{\infty}\frac{1}{\zeta\,!}\left(\frac{-y}{\sqrt{\nu}}\right)^{\zeta}\sum_{\eta=0}^{\infty}\left(\frac{-\theta}{\sqrt{\nu}}\right)^{\eta}\sum_{j=0}^{\infty}\frac{(-\lambda^{\alpha})^{j}}{j\,!}\sum_{k=0}^{\infty}\frac{(-\Phi-B)^{k}}{k\,!}\sum_{l=0}^{\infty}\frac{(-\lambda_r^{\gamma})^{l}}{l\,!},$$
$$\times\frac{\Gamma(\frac{\zeta-1+\eta}{2}+1)\Gamma(\frac{\zeta-1+\eta}{2}+l)\Gamma(\frac{\zeta-1+\eta}{2}+1)}{\Gamma(-j+\frac{\zeta-1+\eta}{2}+1)\Gamma(\frac{\zeta-1+\eta}{2})\Gamma(-k+\frac{\zeta-1+\eta}{2}+1)}\frac{1}{s^{-\frac{\zeta-1+\eta}{2}-\gamma\,l-\alpha j+k+p+1}}. \tag{19}$$

Using discrete inverse Laplace transform to eq. (19), we get

$$\tau(y,t)=-\frac{\mu Up\,!}{\sqrt{\nu}}\sum_{\zeta=0}^{\infty}\frac{1}{\zeta\,!}\left(\frac{-y}{\sqrt{\nu}}\right)^{\zeta}\sum_{\eta=0}^{\infty}\left(\frac{-\theta}{\sqrt{\nu}}\right)^{\eta}\sum_{j=0}^{\infty}\frac{(-\lambda^{\alpha})^{j}}{j\,!}\sum_{k=0}^{\infty}\frac{(-\Phi-B)^{k}}{k\,!}\sum_{l=0}^{\infty}\frac{(-\lambda_r^{\gamma})^{l}}{l\,!},$$
$$\times\frac{\Gamma(\frac{\zeta-1+\eta}{2}+1)\Gamma(\frac{\zeta-1+\eta}{2}+l)\Gamma(\frac{\zeta-1+\eta}{2}+1)\,t^{-\frac{\zeta-1+\eta}{2}-\gamma\,l-\alpha j+k+p}}{\Gamma(-j+\frac{\zeta-1+\eta}{2}+1)\Gamma(\frac{\zeta-1+\eta}{2})\Gamma(-k+\frac{\zeta-1+\eta}{2}+1)\Gamma(-\frac{\zeta-1+\eta}{2}-\gamma\,l-\alpha j+k+p+1)}. \tag{20}$$

Eq. (20) has equivalent form as

$$\tau(y,t)=-\frac{\mu Up\,!}{\sqrt{\nu}}\sum_{\zeta=0}^{\infty}\frac{1}{\zeta\,!}\left(\frac{-y}{\sqrt{\nu}}\right)^{\zeta}\sum_{\eta=0}^{\infty}\left(\frac{-\theta}{\sqrt{\nu}}\right)^{\eta}\sum_{j=0}^{\infty}\frac{(-\lambda^{\alpha})^{j}}{j\,!}\sum_{k=0}^{\infty}\frac{(-\Phi-B)^{k}}{k\,!},$$
$$\times\mathbf{M}_4^3\left[-\lambda_r^{\gamma}\Big|_{\left(\frac{\zeta-1+\eta}{2}-j+1,0\right),\left(\frac{\zeta-1+\eta}{2},0\right),\left(\frac{\zeta-1+\eta}{2}-k+1,0\right),\left(\frac{\zeta-1+\eta}{2}-\alpha j+k+p+1,-\gamma\right)}^{\left(\frac{\zeta-1+\eta}{2}+1,0\right),\left(\frac{\zeta-1+\eta}{2},1\right),\left(\frac{\zeta-1+\eta}{2}+1,0\right)}\right]. \tag{21}$$

is obtained.

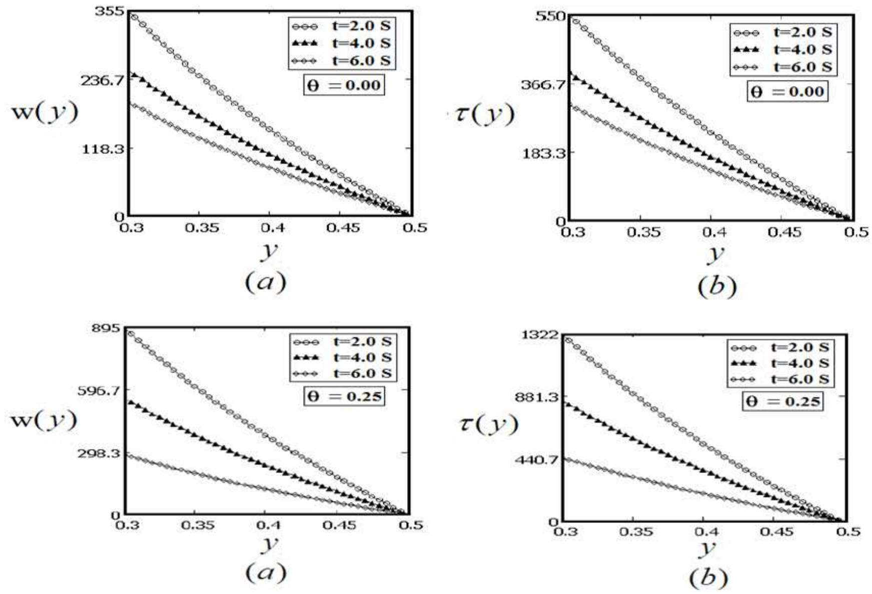

Fig. 1: Profile of velocity field and shear stress with slip and no slip effects

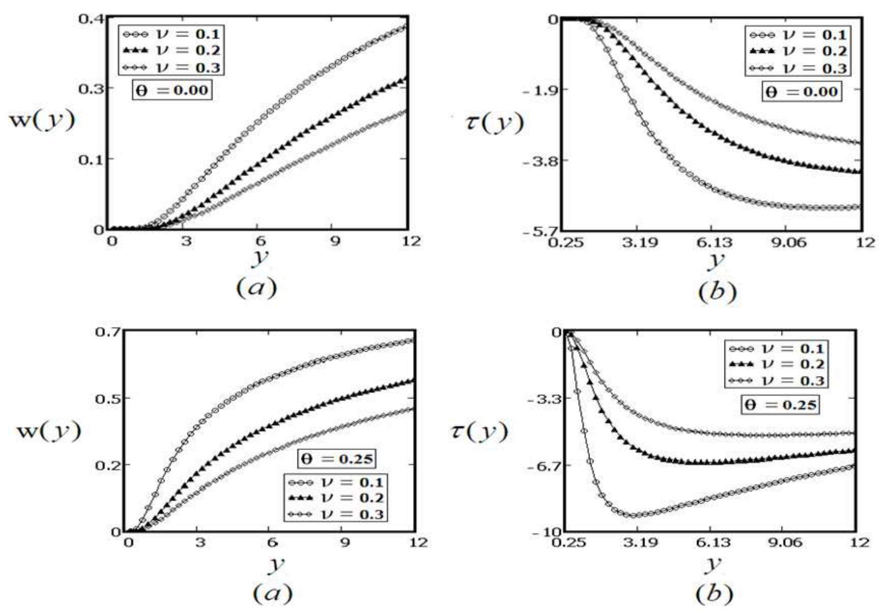

Fig. 2: Profile of velocity field and shear stress with slip and no slip effects.

5 The Limiting Cases

Letting $\alpha \to 1$ and $\gamma \to 1$ into equations (15) and (21) the solutions for Oldroyd-B fluid are in the presence of magnetic field and porosity can be retrieved for ordinary differential operator. Further making $B \to 0$ and $\Phi \to 0$ into equations (15) and (21) the solutions for Oldroyd-B fluid are in the absence of magnetic field and porosity. Some interesting limiting cases are listed below.

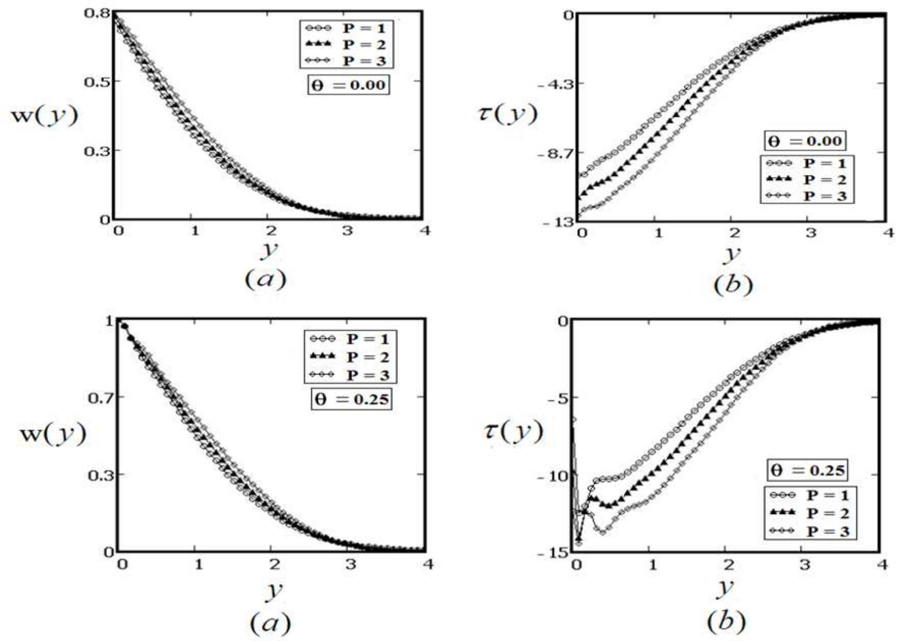

Fig. 3: Profile of velocity field and shear stress with slip and no slip effects.

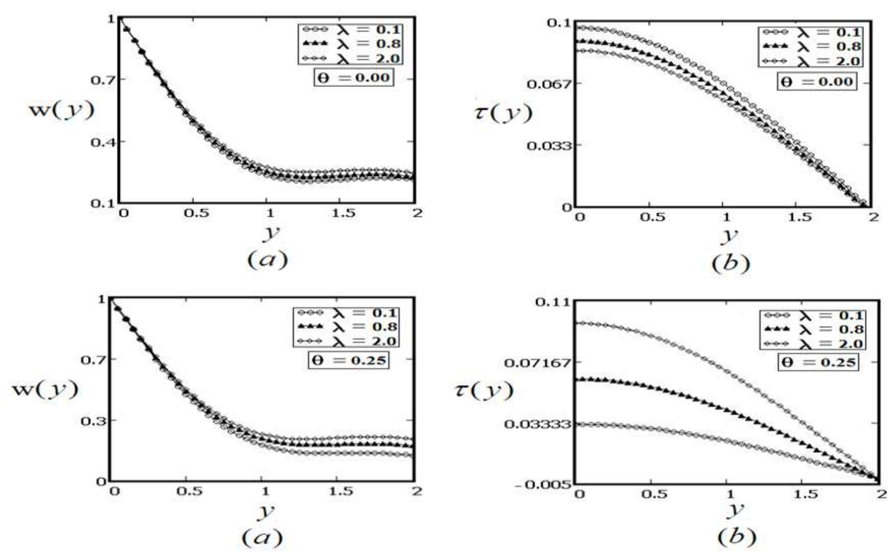

Fig. 4: Profile of velocity field and shear stress with slip and no slip effects.

5.1 Fractionalized MHD Oldroyd-B fluid in porous without slippage

$$w(y,t) = UH(t)t^{p} + UH(t)p\,!\sum_{\zeta=1}^{\infty}\frac{1}{\zeta!}\left(\frac{-y}{\sqrt{\nu}}\right)^{\zeta}\sum_{j=0}^{\infty}\frac{\left(-\Phi-B\right)^{j}}{j\,!}\sum_{k=0}^{\infty}\frac{\left(-\lambda^{\alpha}\right)^{k}}{k\,!},$$

$$\times\mathbf{M}_{4}^{3}\left[-\lambda_{r}^{\gamma}\middle|\begin{matrix}\left(\zeta+\frac{\eta}{2}+1,0\right),\left(\zeta+\frac{\eta}{2},1\right),\left(\zeta+\frac{\eta}{2}+1,0\right)\\ \left(-j+\zeta+\frac{\eta}{2}+1,0\right),\left(\zeta+\frac{\eta}{2},0\right),\left(-k+\zeta+\frac{\eta}{2}+1,0\right),\left(-\zeta-\frac{\eta}{2}-\alpha k+j+p+1,-\gamma\right)\end{matrix}\right], \quad (22)$$

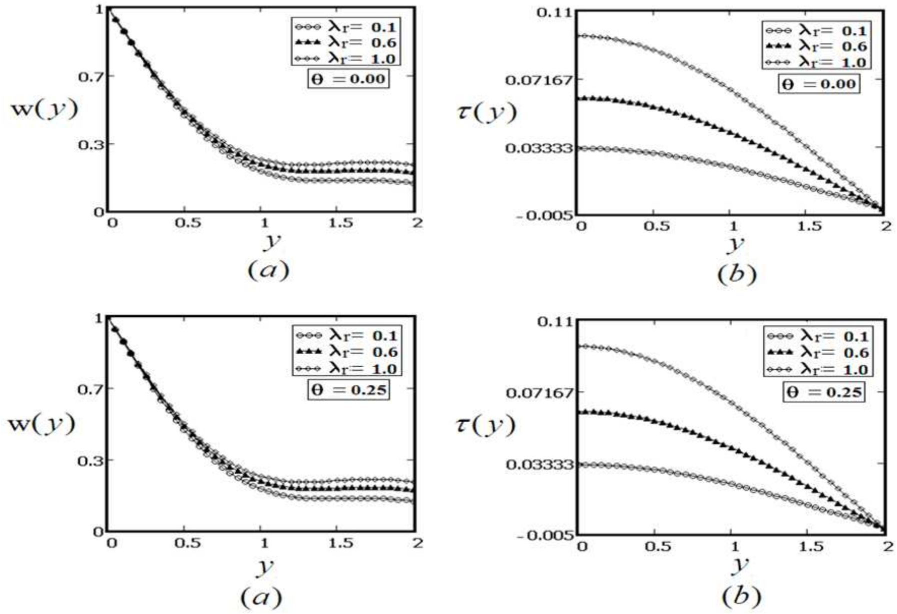

Fig. 5: Profile of velocity field and shear stress with slip and no slip effects.

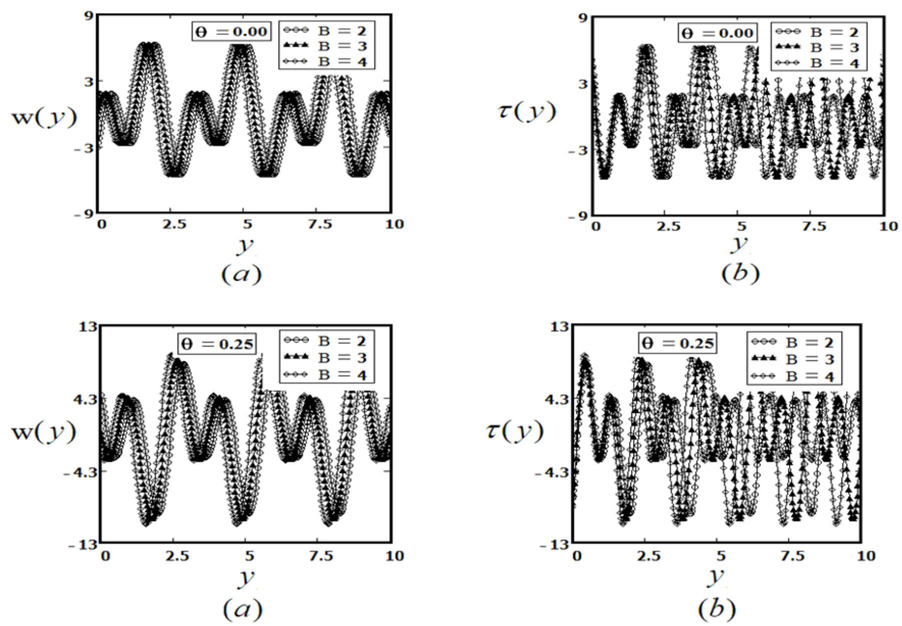

Fig. 6: Profile of velocity field and shear stress with slip and no slip effects.

$$\tau(y,t) = -\frac{\mu U H(t) p\,!}{\sqrt{\nu}} \sum_{\zeta=0}^{\infty} \frac{1}{\zeta\,!}\left(\frac{-y}{\sqrt{\nu}}\right)^{\zeta} \sum_{j=0}^{\infty} \frac{(-\lambda^{\alpha})^{j}}{j\,!} \sum_{k=0}^{\infty} \frac{(-\Phi - B)^{k}}{k\,!},$$

$$\times \mathbf{M}_4^3 \left[-\lambda_r^{\gamma} \left| \begin{matrix} \left(\frac{\zeta-1+\eta}{2}+1,0\right),\left(\frac{\zeta-1+\eta}{2},1\right),\left(\frac{\zeta-1+\eta}{2}+1,0\right) \\ \left(\frac{\zeta-1+\eta}{2}-j+1,0\right),\left(\frac{\zeta-1+\eta}{2},0\right),\left(\frac{\zeta-1+\eta}{2}-k+1,0\right),\left(\frac{\zeta-1+\eta}{2}-\alpha j+k+p+1,-\gamma\right) \end{matrix} \right. \right]. \quad (23)$$

are obtained.

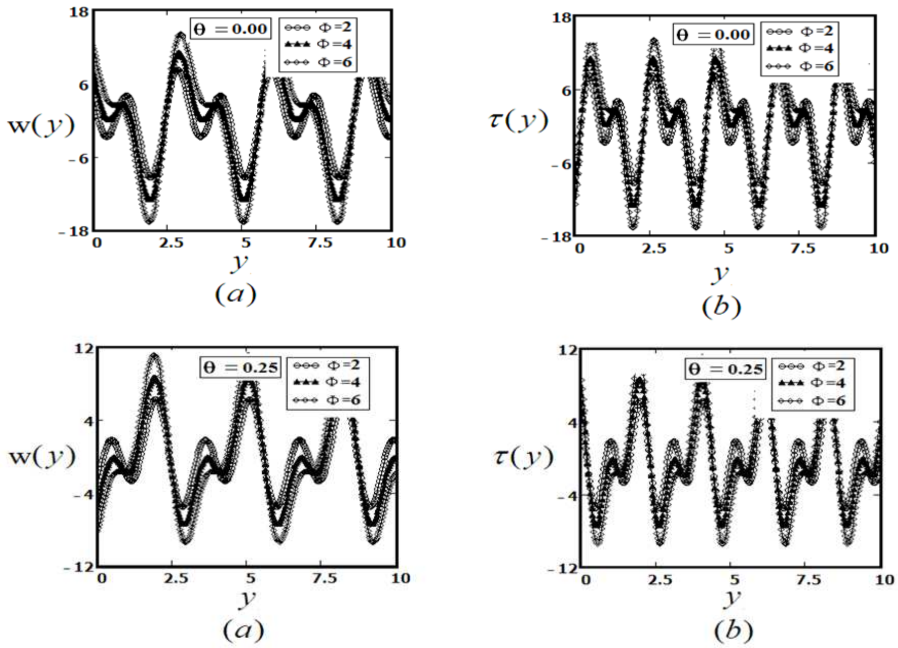

Fig. 7: Profile of velocity field and shear stress with slip and no slip effects.

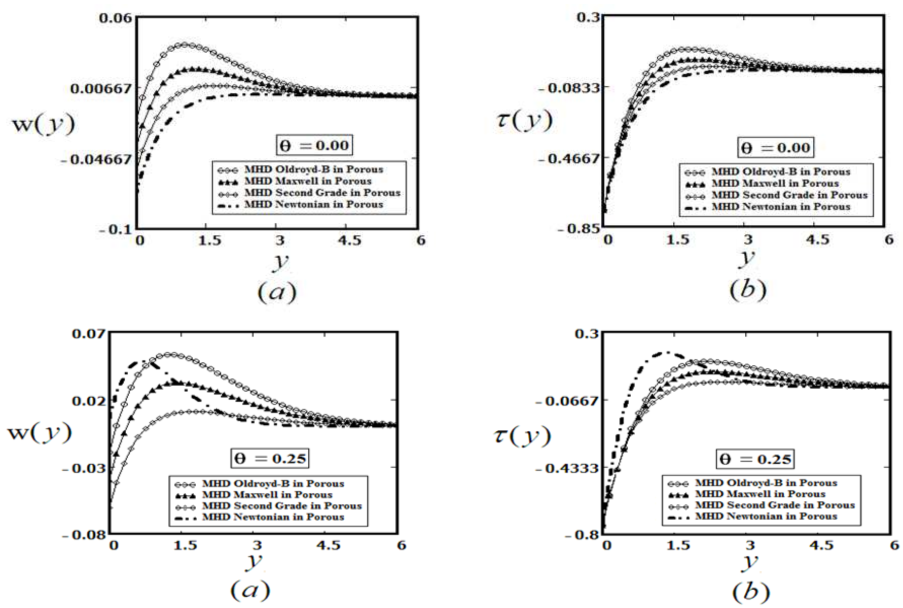

Fig. 8: Comparison of velocity field and shear stress with slip and no slip effects

5.2 Fractionalized MHD Maxwell fluid in porous with slippage

Letting $\lambda_r \to 0$ into equations (15) and (21) the solutions are in the presence of magnetic field and porosity

$$w_M(y,t) = UH(t)t^p + UH(t)p\,!\sum_{\zeta=1}^{\infty}\left(\frac{-\theta}{\sqrt{\nu}}\right)^{\zeta}\sum_{\eta=0}^{\infty}\frac{(-\Phi-B)^{\eta}}{\eta\,!}\mathbf{M}_3^2\left[-\lambda^{\alpha}\left|\begin{matrix}\left(\frac{\zeta}{2}+1,0\right),\left(\frac{\zeta}{2}+1,0\right)\\ \left(-\eta+\frac{\zeta}{2}+1,0\right),\left(\frac{\zeta}{2}+1,1\right),\left(-\frac{\zeta}{2}-\eta+p+1,-\alpha\right)\end{matrix}\right.\right],$$

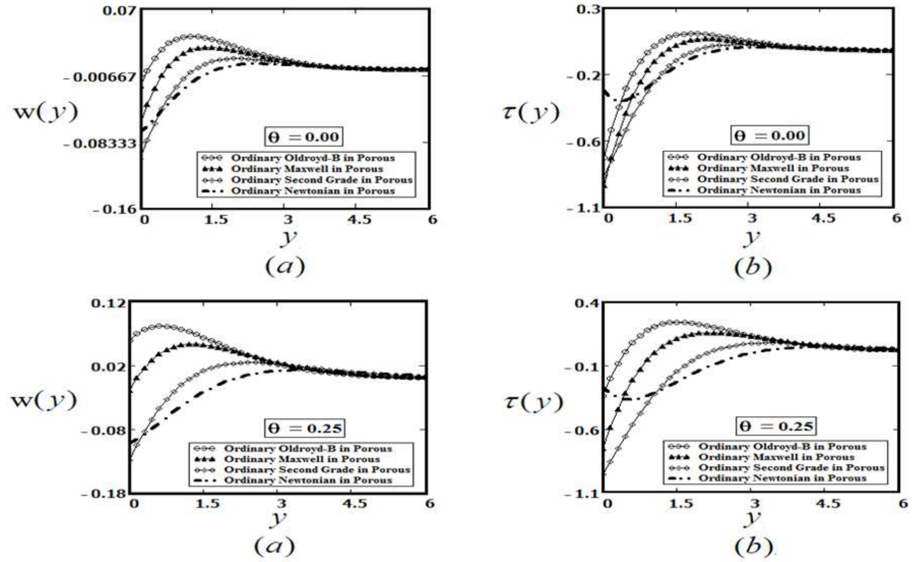

Fig. 9: Comparison of velocity field and shear stress with slip and no slip effects

$$+UH(t)p\,!\sum_{\zeta=1}^{\infty}\frac{1}{\zeta!}\left(\frac{-y}{\sqrt{\nu}}\right)^{\zeta}\sum_{j=0}^{\infty}\frac{(-\Phi-B)^{j}}{j\,!}\sum_{\eta=0}^{\infty}\left(\frac{-\theta}{\sqrt{\nu}}\right)^{\eta}\mathbf{M}_{3}^{2}\left[-\lambda^{\alpha}\left|\begin{matrix}(\zeta+\frac{\eta}{2}+1,0),(\zeta+\frac{\eta}{2}+1,0)\\(-j+\zeta+\frac{\eta}{2}+1,0),(\zeta+\frac{\eta}{2}+1,-1),(-\zeta-\frac{\eta}{2}+j+p+1,-\alpha)\end{matrix}\right.\right],\tag{24}$$

$$\tau_{M}(y,t)=-\frac{\mu UH(t)p\,!}{\sqrt{\nu}}\sum_{\zeta=0}^{\infty}\frac{1}{\zeta\,!}\left(\frac{-y}{\sqrt{\nu}}\right)^{\zeta}\sum_{\eta=0}^{\infty}\left(\frac{-\theta}{\sqrt{\nu}}\right)^{\eta}\sum_{k=0}^{\infty}\frac{(-\Phi-B)^{k}}{k\,!},$$

$$\times\mathbf{M}_{3}^{2}\left[-\lambda^{\alpha}\left|\begin{matrix}\left(\frac{\zeta-1+\eta}{2}+1,0\right),\left(\frac{\zeta-1+\eta}{2}+1,0\right)\\\left(\frac{\zeta-1+\eta}{2}-j+1,0\right),\left(\frac{\zeta-1+\eta}{2}+1,-1\right),\left(-\frac{\zeta-1+\eta}{2}-j+p+1,-\alpha\right)\end{matrix}\right.\right].\tag{25}$$

5.3 Fractionalized MHD Maxwell fluid in porous without slippage

Letting $\lambda_r\to 0$ and $\theta\to 0$ into equations (15) and (21) the solutions for Maxwell fluid are in the presence of magnetic field and porosity

$$w_{M}(y,t)=UH(t)t^{p}+UH(t)p\,!\sum_{\zeta=1}^{\infty}\frac{1}{\zeta!}\left(\frac{-y}{\sqrt{\nu}}\right)^{\zeta}\sum_{j=0}^{\infty}\frac{(-\Phi-B)^{j}}{j\,!}\mathbf{M}_{3}^{2}\left[-\lambda^{\alpha}\left|\begin{matrix}(\zeta+\frac{\eta}{2}+1,0),(\zeta+\frac{\eta}{2}+1,0)\\(-j+\zeta+\frac{\eta}{2}+1,0),(\zeta+\frac{\eta}{2}+1,-1),(-\zeta-\frac{\eta}{2}+j+p+1,-\alpha)\end{matrix}\right.\right],\tag{26}$$

$$\tau_{M}(y,t)=-\frac{\mu UH(t)p\,!}{\sqrt{\nu}}\sum_{\zeta=0}^{\infty}\frac{1}{\zeta\,!}\left(\frac{-y}{\sqrt{\nu}}\right)^{\zeta}\sum_{k=0}^{\infty}\frac{(-\Phi-B)^{k}}{k\,!}\mathbf{M}_{3}^{2}\left[-\lambda^{\alpha}\left|\begin{matrix}\left(\frac{\zeta-1+\eta}{2}+1,0\right),\left(\frac{\zeta-1+\eta}{2}+1,0\right)\\\left(\frac{\zeta-1+\eta}{2}-j+1,0\right),\left(\frac{\zeta-1+\eta}{2}+1,-1\right),\left(-\frac{\zeta-1+\eta}{2}-j+p+1,-\alpha\right)\end{matrix}\right.\right].\tag{27}$$

Letting $\alpha\to 1$ into equations (24) and (25) the solutions for Maxwell fluid are in the presence of magnetic field and porosity can be retrieved for ordinary differential operator. Further making $B\to 0$ and $\Phi\to 0$ into equations (24) and (25) the solutions for Maxwell fluid are in the absence of magnetic field and porosity respectively.

5.4 Fractionalized MHD second grade fluid in porous with slippage

Letting $\lambda \to 0$ into equations (15) and (21) the solutions are in the presence of magnetic field and porosity

$$w_{SG}(y,t) = UH(t)t^p + UH(t)p\,!\sum_{\zeta=1}^{\infty}\left(\frac{-\theta}{\sqrt{\nu}}\right)^{\zeta}\sum_{\eta=0}^{\infty}\frac{\left(-\Phi - B\right)^{\eta}}{\eta\,!}\mathbf{M}_3^2\left[-\lambda_r^{\gamma}\Big|\begin{matrix}\left(\frac{\zeta}{2},0\right),\left(\frac{\zeta}{2}+1,0\right)\\ \left(\frac{\zeta}{2}-\eta+1,0\right),\left(\frac{\zeta}{2},0\right),\left(-\frac{\zeta}{2}+\eta+p+1,\gamma\right)\end{matrix}\right],$$

$$+Up\,!\sum_{\zeta=1}^{\infty}\frac{1}{\zeta!}\left(\frac{-y}{\sqrt{\nu}}\right)^{\zeta}\sum_{j=0}^{\infty}\frac{\left(-\Phi - B\right)^{j}}{j\,!}\sum_{\eta=0}^{\infty}\left(\frac{-\theta}{\sqrt{\nu}}\right)^{\eta}\mathbf{M}_3^2\left[-\lambda_r^{\gamma}\Big|\begin{matrix}\left(\zeta+\frac{\eta}{2}+1,0\right),\left(\zeta+\frac{\eta}{2},1\right)\\ \left(\zeta+\frac{\eta}{2},0\right),\left(\zeta+\frac{\eta}{2}-j+1,0\right),\left(-\zeta-\frac{\eta}{2}+j+p+1,-\gamma\right)\end{matrix}\right], \tag{28}$$

$$\tau_{SG}(y,t) = -\frac{\mu Up\,!}{\sqrt{\nu}}\sum_{\zeta=0}^{\infty}\frac{1}{\zeta\,!}\left(\frac{-y}{\sqrt{\nu}}\right)^{\zeta}\sum_{\eta=0}^{\infty}\left(\frac{-\theta}{\sqrt{\nu}}\right)^{\eta}\sum_{j=0}^{\infty}\frac{\left(-\Phi - B\right)^{j}}{j\,!}\mathbf{M}_3^2\left[-\lambda_r^{\gamma}\Big|\begin{matrix}\left(\frac{\zeta-1+\eta}{2}+1,0\right),\left(\frac{\zeta-1+\eta}{2},1\right)\\ \left(\frac{\zeta-1+\eta}{2}-j+1,0\right),\left(\frac{\zeta-1+\eta}{2},0\right),\left(\frac{\zeta-1+\eta}{2}-j+p+1,-\gamma\right)\end{matrix}\right]. \tag{29}$$

5.5 Fractionalized MHD second grade fluid in porous without slippage

Letting $\theta \to 0$ into equations (30) and (31) the solutions are in the presence of magnetic field and porosity

$$w_{SG}(y,t) = UH(t)t^p + Up\,!\sum_{\zeta=1}^{\infty}\frac{1}{\zeta!}\left(\frac{-y}{\sqrt{\nu}}\right)^{\zeta}\sum_{j=0}^{\infty}\frac{\left(-\Phi - B\right)^{j}}{j\,!}\mathbf{M}_3^2\left[-\lambda_r^{\gamma}\Big|\begin{matrix}\left(\zeta+\frac{\eta}{2}+1,0\right),\left(\zeta+\frac{\eta}{2},1\right)\\ \left(\zeta+\frac{\eta}{2},0\right),\left(\zeta+\frac{\eta}{2}-j+1,0\right),\left(-\zeta-\frac{\eta}{2}+j+p+1,-\gamma\right)\end{matrix}\right], \tag{30}$$

$$\tau_{SG}(y,t) = -\frac{\mu Up\,!}{\sqrt{\nu}}\sum_{\zeta=0}^{\infty}\frac{1}{\zeta\,!}\left(\frac{-y}{\sqrt{\nu}}\right)^{\zeta}\sum_{j=0}^{\infty}\frac{\left(-\Phi - B\right)^{j}}{j\,!}\mathbf{M}_3^2\left[-\lambda_r^{\gamma}\Big|\begin{matrix}\left(\frac{\zeta-1+\eta}{2}+1,0\right),\left(\frac{\zeta-1+\eta}{2},1\right)\\ \left(\frac{\zeta-1+\eta}{2}-j+1,0\right),\left(\frac{\zeta-1+\eta}{2},0\right),\left(\frac{\zeta-1+\eta}{2}-j+p+1,-\gamma\right)\end{matrix}\right]. \tag{31}$$

Letting $\gamma \to 1$ into equations (30) and (31) the solutions for Second grade fluid are in the presence of magnetic field and porosity can be retrieved for ordinary differential operator. Further making $B \to 0$ and $\Phi \to 0$ into equations (30) and (31) the solutions for Second grade fluid are in the absence of magnetic field and porosity. Also Solutions for Newtonian fluid can be easily be found in similar manners which are known in literature.

6 Conclusions

The main aim of this analysis is to present analytic solutions for MHD Oldroyd-B fluid in porous medium with and without slippage. The general solutions are established by using integral transforms (Laplace transforms with inverses) satisfying initial and boundary conditions. The corresponding solutions have been reduced from fractional to ordinary solutions by making $\alpha = 1, \gamma = 1$.These solutions have also been particularized when $\lambda_r = 0, \lambda = 0$ and $\lambda_r = \lambda = 0$ for Maxwell fluid and second grade fluid and Newtonian fluid respectively. In the particular case, the results of Stokes' first problem are obtained when $\lambda_r = 0$ and $\theta = p = 0$ for Maxwell fluid [27, 28]. In order to bring out physical results, impacts of slip and no slip assumptions for various rheological parameters $\alpha, \gamma, \lambda_r, \lambda, \beta, \rho, t, \theta, \mu, \nu, p$ and Φ have been analyzed for fluid motion. The graphs are plotted for velocity profile and shear stress for various pertinent parameters. From all graphs, it is noted that slippage has shown interesting results between plate and fluid. The Major outcomes are:

(i) Fig.1 is plotted to justify impacts of time for the profile of velocity field and shear stress with slip and no slip assumptions.

(ii) Fig.2 is depicted to show the scattering behavior of fluid for kinematic viscosity over the velocity field and shear stress under presence and absence of slip effects.

(iii) When slip is nonzero, the profile of velocity field and shear stress are slighter and smaller as compared with slip effects. This happens in Fig.3, due to the fact that plate start to accelerate variably about its own plane.

(iv) Figs. 4 and 5 have been drawn for relaxation and retardation time, for which profile of velocity field and shear stress is sometimes increasing and sometimes decreasing function of fluid motion with and without slippage.

(v) Fig. 6 has been drawn for magnetic effects on fluid in which fluid motion is helical either slippage is present or absent. This may be the fact that magnetic parameter B decelerates or slows the fluid motion.

(vi) The profile of velocity field and shear stress in fig. 7 depicts that for different values of porosity have brought out the contrast behavior of fluid motion. This is due to the fact that plate is sliding in its plane.

(vii) Figs. 8 and 9 have been drawn for comparison of four models namely fractionalized and ordinary Oldroyd-B, Maxwell, Second Grade and Newtonian fluids with and without slippage, in which Newtonian fluid is slowest in comparison either in fractionalized or ordinary fluids. Among fractionalized and ordinary Oldroyd-B, Maxwell, Second Grade and Newtonian fluid sometimes Newtonian fluid moves slowly.

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