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# Extension of Matkowski Contraction Principle in Fuzzy Metric space

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**Abstract:** The main purpose of this paper is to obtain a common fixed point theorem for two systems of Matkowski type maps on fuzzy metric space.

**Keywords:** Fuzzy metric space, fixed point, Cauchy sequence.

## 1 Introduction

Zadeh [5] introduced the concept of fuzzy sets in 1965 and in the next decade Kramosil and Michalek [2] introduced the concept of fuzzy metric space in 1975, which opened an avenue for further development of analysis in such spaces. Consequently in due course of time some metric fixed point results were generalized to fuzzy metric spaces by various authors viz [1,6,10] and others. The concepts of fuzzy metric spaces have been introduced in different ways by many authors. With a view to generalizing the Banach Contraction Principle, Matkowski extended the concept of Banach contraction to a system of equations on a finite product of metric spaces and obtain a fixed point theorem for such system of transformations [3,4,7,11] etc. Arora and Kumar [9] introduced the concept of Matkowski type maps on product of fuzzy metric space and obtain a fixed point theorem for such system of transformation. Matkowski type fixed point theorems are applicable in solving abstract equation on product spaces to convex solution of a system of functional equations and other abstract equation.
In this paper we obtain a common fixed point theorem for two systems of Matkowski types transformation on product of fuzzy metric space.

## 2 Preliminaries

**Definition 21**(*Schweizer and Sklar[13]*) *A binary operation* $* : [0,1]\times[0,1]\to[0,1]$ *is a continuous t-norm if* $*$ *satisfies the following conditions*

*[B.1]*$*$ *is commutative and associative*
*[B.2]*$*$ *is continuous*
*[B.3]*$a*1 = a \quad \forall a\in[0,1]$
*[B.4]*$a*b \le c*d$ *whenever* $a\le c, b\le d$ *and* $a,b,c,d\in[0,1]$.

**Definition 22**(*A. George and P. Veeramani[1]*) *The 3-tuple* $(X,M,*)$ *is called a fuzzy metric space if* $X$ *is an arbitrary non empty set,* $*$ *is an continuous t-norm and* $M$ *is a fuzzy metric in* $X^2\times[0,\infty]\to[0,1]$*,satisfying the following conditions:* $\forall x,y,z\in X$ *and* $t,s>0$.

*[FM.1]*$M(x,y,0)=0$
*[FM.2]*$M(x,y,t)=1 \quad \forall t>0$ *if and only if* $x=y$
*[FM.3]*$M(x,y,t)=M(y,x,t)$
*[FM.4]*$M(x,y,t)*M(y,z,s)\le M(x,z,t+s)$
*[FM.5]*$M(x,y,\cdot):[0,\infty)\to[0,1]$*, is left continuous.*

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*[FM.6]*$\lim_{n\to\infty} M(x,y,t) = 1$.

**Definition 23**(*A. George and P. Veeramani[1]*) *Let $(X,M,*)$ be a fuzzy metric space and let a sequence $x_n \in X$ is said to be converge to $x \in X$ if* $\lim_{n\to\infty} M(x_n,x,t) = 1$.

**Definition 24**(*A. George and P. Veeramani[1]*) *A sequence $x_n \in X$ is called Cauchy sequence if* $\lim_{n\to\infty} M(x_n,x_{n+p},t) = 1$, *for each* $t > 0$ *and* $p = 1,2,3,\ldots$

**Definition 25**(*A. George and P. Veeramani[1]*) *A fuzzy metric space $(X,M,*)$ is said to be complete if every Cauchy sequence in $X$ is convergent in $X$.*
*A fuzzy metric space in which every Cauchy sequence is convergent is called complete. It is called compact if every sequence contains a convergent subsequence.*

**Definition 26**(*A. George and P. Veeramani[1]*) *A self mapping $T : X \to X$ is called fuzzy contractive mapping if $M(Tx,Ty,t) > M(x,y,t)$ for each $x \neq y \in X$ and $t > 0$.*

In all, we generally follow the following notation and definition introduced in Matkowski and Singh et al. [3,4,7,11].
Let $a_{ik}$ be non-negative numbers $i,k = 1,\ldots n$ and $c_{ik}^{(l)}$ be square matrices defined in the following recursive manner

$$c_{ik}^{(0)} = \begin{cases} a_{ik}, & i \neq k \\ 1 - a_{ik}, & i = k \end{cases} \tag{1}$$

and $c_{ik}^{l}$ are defined recursively by

$$c_{ik}^{l+1} = \begin{cases} c_{11}^{l}c_{i+1,k+1}^{l} - c_{i+1,1}^{l}c_{1,k+1}^{l}, & i \neq k \\ c_{11}^{l}c_{i+1,k+1}^{l} + c_{i+1,1}^{l}c_{1,k+1}^{l}, & i = k \end{cases} \tag{2}$$

$i,k = 1,\ldots,n-l-1, l = 0,1,\ldots,n-2$.If $n = 1$, we define $c_{11}^{(0)} = a_{11}$, evidently, $c_{ik}^{l}$ is a $(n-l)\times(n-l)$ square matrix.

**Lemma 27**[9] *Let $c_{ik}^{(0)} \geq 0$ for $i,k = 1,\ldots,n, n \geq 2$ system of inequalities*

$$\sum_{\substack{k=1\\k\neq i}}^{n} c_{ik}^{(0)} r_k \geq c_{ii}^{(0)} r_i, i = 1,\ldots,n. \tag{3}$$

*has a positive solution $r_1,\ldots,r_n$, if and only if*

$$c_{ii}^{l} \geq 0, i = 1,\ldots,n-l, l = 0,\ldots,n-1. \tag{4}$$

## 3 Main Result

**Theorem 31***Let $F_i, G_i : X_i \to X_i, i = 1,\ldots,n$, be two sets of transformations. If there exist non-negative numbers b and $a_{ik}, i,k = 1,\ldots,n$, such that*

$$\begin{aligned} &M_i(T_i(x_1,\ldots,x_n), G_i(\overline{x_1},\ldots,\overline{x_n}),t) \\ &\geq \sum_{k=1}^{n} a_{ik} M_k(x_k,\overline{x_k},t) \\ &+ b[M_i(x_i,F_i(x_1,\ldots,x_n),t) + M_i(\overline{x_i},G_i(\overline{x_1},\ldots,\overline{x_n}),t)] \end{aligned} \tag{5}$$

$\forall x_k, \overline{x_k} \in X_k, i = 1,\ldots,n$
*and the numbers $c_{ik}^{(0)}$ and $c_{ik}^{(l)}$ defined by (1) and (2) fulfils the conditions*

$$c_{ii}^{l} > 0, i = 1,\ldots,n-l, l = 0,1,\ldots,n-1.$$

*Where $0 \leq 2b \leq 1-\nu$ and where*

$$\nu = \max_i (r_i^{-1} \sum_{k=1}^{n} a_{ik} r_k) \tag{6}$$

*Then the system of equations*

$$M_i(F_i x_i, G_i x_i, t) \geq M_i(x_i,x_i,t), i = 1,...,n. \tag{7}$$

$$\Rightarrow F_i(x_1,\ldots,x_n) = x_i = G_i(x_1,\ldots,x_n)$$

*have a unique common solution $x_i,\ldots,x_n$ such that $x_i \in X_i, i = 1,\ldots,n$.*

*Proof.*First of all we note that in view of the homogeneity of the system of inequalities (3), $\nu$ in (6) exist and $0 < \nu < 1$. From Lemma 2.7 and (6) we may choose a system of positive numbers $r_1,...,r_n$, such that

$$\sum_{k=1}^{n} a_{ik} r_k) \geq \nu r_i, i = 1,...,n.$$

Pick $x_i^0 \in X_i$ and choose a sequence $x_i^m \in X_i, i = 1,...,n$, such that

$$M_i(x_i^{2m+1},x_i^{2m+2},t) = M_i(F_i x_i^{2m}, F_i x_i^{2m+1},t) \geq M_i(x_i^{2m},x_i^{2m+1},t)$$

and

$$M_i(x_i^{2m+2}, x_i^{2m+3}, t) =$$
$$M_i(G_i x_i^{2m+1}, G_i x_i^{2m+2}, t) \geq M_i(x_i^{2m+1}, x_i^{2m+2}, t), m = 0, 1, \ldots \quad (8)$$

without loss of generality, we assume that

$$M_i(x_i^0, x_i^1, t) \geq r_i, r_i \leq 1, i = 1, \ldots, n.$$

By (6) and (8)

$$\begin{aligned} M_i(x_i^1, x_i^2, t) &= M_i(F_i(x_i^0, \ldots, x_n^0), G_i(x_i^1, \ldots, x_n^1), t) \\ &\geq \sum_{k=1}^{n} a_{ik} M_k(x_k^0, x_k^1, t) + b[M_i(x_i^0, F_i(x_i^0, \ldots, x_n^0), t) + \\ &M_i(x_i^1, G_i(x_i^1, \ldots, x_n^1), t)] \\ &\geq \sum_{k=1}^{n} a_{ik} r_k + b[M_i(x_i^0, x_i^1, t) + M_i(x_i^1, x_i^2, t)] \end{aligned}$$
$$M_i(x_i^1, x_i^2, t) \geq q r_i, i = 1, \ldots, n.$$

where $0 \leq q = \frac{v+1}{1-b} \leq 1$.
Also from

$$\begin{aligned} M_i(x_i^2, x_i^3, t) =& M_i(F_i(x_i^1, \ldots, x_n^1), G_i(x_i^2, \ldots, x_n^2), t) \\ &\geq \sum_{k=1}^{n} a_{ik} M_k(x_k^1, x_k^2, t) + b[M_i(x_i^1, F_i(x_i^1, \ldots, x_n^1), t) + \\ &M_i(x_i^2, G_i(x_i^2, \ldots, x_n^2), t)] \end{aligned}$$
$$M_i(x_i^2, x_i^3, t) \geq \sum_{k=1}^{n} a_{ik} r_k + b[M_i(x_i^1, x_i^2, t) + M_i(x_i^2, x_i^3, t)]$$
$$M_i(x_i^2, x_i^3, t) \geq q^2 r_i, i = 1, \ldots, n.$$

Inductively

$$M_i(x_i^m, x_i^{m+1}, t) \geq q^m r_i \quad (9)$$

It follows easily from (9) that $x_i^m$ be a sequence in $X_i$.

$$M_{i(x_i^m, x_i^{m+1}, kt) \geq M_i(x_i^0, x_i^1, \frac{t}{k^{n-1}})}$$

for all $m$ and $t > 0$. Thus for any positive integer $p$ we have

$$\begin{aligned} M_i(x_i^m, x_i^{m+p}, t) &\geq M_i(x_i^m, x_i^{m+1}, \frac{t}{p}) * \ldots * M_i(x_i^{m+p-1}, x_i^{m+p}, \frac{t}{p}) \\ &\geq M_i(x_i^0, x_i^1, \frac{t}{pk^n}) * \ldots * M_i(x_i^0, x_i^1, \frac{t}{pk^n}) \end{aligned}$$
$$\lim_m M_i(x_i^m, x_i^{m+p}, t) \geq 1 * \ldots * 1.$$

Therefore ${x_i}^m$ is a Cauchy sequence for each $i = 1, \ldots, n$, hence convergent. Since $M_i(X_i, M_i, *)$ is complete. Let ${x_i}^m$ converges to a point $u_i \in X_i$ such that

$$\lim_{m \to \infty} M_i(F_i x_i^m, F_i u_i, t) \geq \lim_{m \to \infty} M_i(x_i^m, u_i, t) = 1.$$

Now we show that
$u_i = F_i(u_1, \ldots u_n) = G_i(u_1, \ldots u_n), i = 1, \ldots n.$
For each $1 \leq i \leq n$

$$\begin{aligned} &M_i(u_i, F_i(u_1, \ldots u_n), t) \geq M_i(u_i, x_i^{2m+2}, t) + \\ &M_i(x_i^{2m+2}, F_i(u_1, \ldots u_n), t) \\ &\geq M_i(u_i, x_i^{2m+2}, t) + M_i(G_i(x_1^{2m+1}, \ldots, x_n^{2m+1}), F_i(u_1, \ldots u_n), t) \\ &\geq M_i(u_i, x_i^{2m+2}, t) + \sum_{k=1}^{n} a_{ik} M_k(u_k, x_k^{2m+1}, t) + \\ &b[M_i(u_i, F_i(u_1, \ldots u_n), t) \\ &+ M_i(x_i^{2m+1}, G_i(x_1^{2m+1}, \ldots, x_n^{2m+1}), t)] \\ &\geq M_i(u_i, x_i^{2m+2}, t) + \sum_{k=1}^{n} a_{ik} M_k(u_k, x_k^{2m+1}, t) + \\ &b[M_i(u_i, F_i(u_1, \ldots u_n), t) + M_i(x_i^{2m+1}, x_i^{2m+2}, t)]. \end{aligned}$$

Making $m \to \infty$, we obtain for $0 \leq i \leq n$.
$(1-b)M_i(u_i, F_i(u_1, \ldots u_n), t) \geq 0$
$\Rightarrow u_i = F_i(u_1, \ldots, u_n)$
and similarly $u_i = G_i(u_1, \ldots, u_n)$.
To prove the uniqueness of $u_i$. Let $u_i, \overline{u_i}$ two distinct solutions. Then
$u_i = F_i(u_1, \ldots u_n) = G_i(u_1, \ldots u_n)$
and
$\overline{u_i} = F_i(\overline{u_1}, \ldots, \overline{u_n}) = G_i(\overline{u_1}, \ldots, \overline{u_n})$.
We can assume that
$M_i(u_i, \overline{u_i}, t) \geq r_i, i = 1, \ldots, n.$

$$\begin{aligned} M_i(u_i, \overline{u_i}, t) &= M_i(F_i(u_1, \ldots u_n), G_i(\overline{u_1}, \ldots, \overline{u_n}), t) \\ &\geq \sum_{k=1}^{n} a_{ik} M_k(u_k, \overline{u_k}, t) + b[M_i(u_i, F_i(u_1, \ldots, u_n), t) + \\ &M_i(\overline{u_i}, G_i(\overline{u_1}, \ldots, \overline{u_n}), t)] \\ &\geq \sum_{k=1}^{n} a_{ik} M_k(u_k, \overline{u_k}, t) \geq v r_i. \end{aligned}$$

Inductively
$M_i(u_i, \overline{u_i}, t) \geq v^m r_i, i = 1, \ldots, n.$
Therefore

$M_i(u_i, \overline{u_i}, t) = 1, i = 1, \ldots, n.$
This completes the proof.

## 4 Conclusion

The concept of fixed point theorems for systems of Matkowski type transformation in fuzzy metric space have introduced by Arora and Kumar [9]. We have extended the concept of Arora and Kumar [9] for two systems of transformations and proved common fixed point theorem for two systems of Matkowski type transformations in fuzzy metric space.

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