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Computational Method for Fractional Differential Equations Using Nonpolynomial Fractional Spline

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Abstract: In this paper, a new fractional spline method of non-polynomial form have been considered to solve special linear fractional boundary-value problems. Using this fractional spline function a few consistency relations are derived for computing approximations to the solution of the problem. Convergence analysis and error estimates of this methods are discussed. Numerical results are provided to demonstrate the superiority of our methods.

Keywords: Caputo derivative, Non-polynomial spline, Convergence analysis.

1 Introduction

During the past three decades, fractional differential equation has gained importance due to its applicability in diverse fields of science and engineering, such as, control theory, viscoelasticity, diffusion, neurology, and statistics (see [2]). Several forms of fractional differential equations have been proposed in standard models, and there has been significant interest in developing numerical schemes for their solution (see [2,15,16,17,18,19]).

In the last few decades, there has been much effort to develop numerical methods based on non-polynomial spline approximations for the solution of many types of boundary value problems. For example, Akram et al. [8] presented a second-order method using a non- polynomial spline for solving a sixth-order boundary value problem with boundary conditions involving first derivatives. Jalilian et al. [9] established the numerical solutions of Problems in Calculus of Variations using a non-polynomial spline and Islam et al. [10] have solved some special fifth-order boundary value problems. Islam and Tirmizi [11] and Al-Said [12] have solved a system of second-order boundary value problems. Ramadan et al. [13] have solved second-order two-point boundary value problems using polynomial and non-polynomial spline functions. Ramadan et al. [14] have developed nonpolynomial septic spline functions for obtaining smooth approximations to the numerical solution of fourth- order two point boundary value problems occurring in a plate deflection theory. For more details on non-polynomial spline we may also refer to [4,5,6,7].

The main objective of the present paper is to apply a fractional spline of non-polynomial form to develop a new numerical method for obtaining smooth approximations to the solution of the generalized Bagley-Torvik equation of the form [17,22,23,24,25]:

$$\begin{aligned} &D^{2\alpha}y(x)+(\eta D^{\alpha}+\mu)y(x)\\ &=f(x), \quad \alpha=1.5,\ x\in[a,b]. \end{aligned} \tag{1}$$

Subject to boundary conditions:

$$y(a)=y(b)=0 \tag{2}$$

where η,μ are all real constants. The function $f(x)$ is continuous on the interval $[a,b]$ and the operator D^{α} represents the Caputo fractional derivative. When $\alpha=1$, then equation (1) is reduced to the classical second order boundary value problem.

The analytical solution of problem (1) with boundary conditions (2) cannot be determined for any arbitrary choice of η, μ and $f(x)$.We therefore employ numerical methods for obtaining approximate solution to the problem [Equations (1)-(2)].

To show the practical applicability and superiority of our method, some numerical evidence is included and their pertaining approximate solutions are compared with the exact solutions.

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2 Preliminaries and Notations

In this section, we give the definition of Riemann-Liouville fractional integral and fractional derivative with the Caputo fractional derivative and the Grünwald fractional derivative.

Definition 1.[2,3] *The Riemann-Liouville fractional integral of order $\alpha > 0$ is defined by*

$$I^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-\xi)^{\alpha-1} f(\xi) d\xi, \ n-1 < \alpha < n \in \mathbb{N}.$$

where Γ is the gamma function.

Definition 2.[2,3] *The Riemann-Liouville fractional derivative of order $\alpha > 0$ is defined by*

$$D^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x (x-\xi)^{n-\alpha-1} f(\xi) d\xi, \ n-1 < \alpha < n \in \mathbb{N}.$$

Definition 3.[1] *The Caputo fractional derivative of order $\alpha > 0$ is defined by*

$$D_*^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-\xi)^{n-\alpha-1} \frac{d^n}{d\xi^n} f(\xi) d\xi, \ n-1 < \alpha < n \in \mathbb{N}.$$

Definition 4.[1,2,3] *The Grünwald definition for fractional derivative is:*

$$^{G}D^{\alpha} y(x) = \lim_{n\to\infty} \frac{1}{h^{\alpha}} \sum_{k=0}^{n} g_{\alpha,k}\, y(x-kh) \qquad (3)$$

where the Grünwald weights are:

$$g_{\alpha,k} = \frac{\Gamma(k-\alpha)}{\Gamma(-\alpha)\Gamma(k+1)} \qquad (4)$$

3 Nonpolynomial Fractional Spline Method

In this section, we obtain an approximate solution of the fractional differential equation (1)–(2) using non-polynomial fractional spline functions. For this purpose, we introduce a finite set of grid points x_i by dividing the interval $[a,b]$ into n- equal parts.

$$x_i = a + ih, \ x_0 = a, \ x_n = b, \ h = \frac{b-a}{n}, \quad i = 0(1)n. \quad (5)$$

Let $y(x)$ be the exact solution of the equation (1) and S_i be an approximation to $y_i = y(x_i)$ obtained by the segment $P_i(x)$ passing through the points (x_i, S_i) and (x_{i+1}, S_{i+1}) then in each subinterval the fractional spline segment $P_i(x)$ has the form:

$$\begin{aligned} P_i(x) &= a_i + b_i(x-x_i)^{3/2} + c_i \sin_{1.5} k(x-x_i)^{3/2} \\ &\quad + d_i \cos_{1.5} k(x-x_i)^{3/2}, \quad i = 0(1)n. \end{aligned} \qquad (6)$$

where a_i, b_i, c_i and d_i are constants and k is the frequency of the trigonometric functions which will be used to raise the accuracy of the method. For convenience consider the following relations:

$$\begin{aligned} &P_i(x_i) = y_i, \ P_i(x_{i+1}) = y_{i+1}, \ (D^{3/2})^2 P_i(x_i) = M_i, \\ &(D^{3/2})^2 P_i(x_{i+1}) = M_{i+1}, \quad i = 0(1)n-1. \end{aligned} \qquad (7)$$

Via a straightforward calculation we obtain the values of a_i, b_i, c_i and d_i as follows:

$$a_i = y_i + \frac{M_i}{k^2}, \qquad (8)$$

$$b_i = \frac{y_{i+1} - y_i}{h^{1.5}} + \frac{M_{i+1} - M_i}{\theta k}, \qquad (9)$$

$$c_i = \frac{M_i \cos_{1.5}\theta - M_{i+1}}{k^2 \sin_{1.5}\theta}, \qquad (10)$$

$$d_i = -\frac{M_i}{k^2}. \qquad (11)$$

where $\theta = kh^{\alpha}$ and for $i = 0(1)n-1$.

Using the continuity conditions $D^{3/2}P_{i-1}(x_i) = D^{3/2}P_i(x_i)$ we have the following consistency relations:

$$\begin{aligned} &\frac{1}{h^3}(y_{i+1} - 2y_i + y_{i-1}) \\ &= \lambda M_{i+1} + 2\beta M_i + \lambda M_{i-1}, \quad i = 2(1)n-1. \end{aligned} \qquad (12)$$

where

$$\lambda = \frac{1}{\theta^2}\left(\frac{4}{3\sqrt{\pi}}\theta \csc_{\alpha}\theta - 1\right), \quad \text{and}$$

$$\beta = \frac{1}{\theta^2}\left(1 - \frac{4}{3\sqrt{\pi}}\theta \cot_{\alpha}\theta\right),$$

where $\theta = kh^{1.5}$ and

$$M_i = f_i - \mu S_i - \eta D^{3/2} S(x)|_{x=x_i}, \quad i = 0(1)n \qquad (13)$$

with $f_i = f(x_i)$. Now to determine $D^{3/2}S(x)|_{x=x_i}$, $i = 0(1)n$, we use the fact that

$$(w+1)^r = \sum_{k=0}^{\infty} \binom{r}{k} w^k, \quad \text{for each } |w| \le 1, \ p > 0$$

where

$$\binom{r}{k} = \frac{(-1)^k \Gamma(k-r)}{\Gamma(-r)\Gamma(k+1)}$$

If we set $w = -1$ then the above summation will be vanished. From which together with equation (4) we can approximate the fractional term, $D^{\alpha}S(x)|_{x=x_i}$, $i = 0(1)n$, as follows:

$$D^{\alpha} S(x)|_{x=x_i} \approx h^{-\alpha} \sum_{k=0}^{i} g_{\alpha,k} S(x_i - kh), \qquad \text{for } i = 0(1)n$$

where the Grünwald weights $g_{\alpha,k}$ are given in equation (4). Hence we have

$$D^{3/2}S(x)|_{x=x_i} \approx h^{-1.5}\sum_{k=0}^{i} g_{1.5,k}S(x_i-kh), \quad \text{for } i=0(1)n. \tag{14}$$

4 Convergence Analysis

Here we investigate the error analysis of the spline method described in section 3. Let $Y=(y_i)$, $S=(s_i)$, $T=(t_i)$ and $E=(e_i)=Y-S$ be $n-1$ dimensional column vectors. Then, we can write the system given by (13) as follows:

$$PS=h^3BM \tag{15}$$

where the matrices P and B are given below

$$P_{i,j}=\begin{cases} -2, & \text{for } i=j=1(1)n-1 \\ 1, & \text{for } |i-j|=1 \\ 0, & \text{otherwise} \end{cases}$$

The tridiagonal matrix B is given by

$$B=\begin{pmatrix} 2\beta & \lambda & & & \\ \lambda & 2\beta & \lambda & & \\ & \ddots & \ddots & \ddots & \\ & & \lambda & 2\beta & \lambda \\ & & & \lambda & 2\beta \end{pmatrix}.$$

The vector M can be written as:

$$M=F-\mu S-\eta h^{-1.5}GS \tag{16}$$

Where the vectors F and the matrix G are given below respectively:

$$F=\left(f_1 \quad f_2 \quad \dots \quad f_{n-2} \quad f_{n-1}\right)^t, \tag{17}$$

and

$$G=\begin{pmatrix} g_{1.5,0} & & & & \\ g_{1.5,1} & g_{1.5,0} & & & \\ \vdots & \vdots & \ddots & & \\ g_{1.5,n-3} & g_{1.5,n-4} & \cdots & g_{1.5,1} & g_{1.5,0} & \\ g_{1.5,n-2} & g_{1.5,n-3} & \cdots & g_{1.5,2} & g_{1.5,1} & g_{1.5,0} \end{pmatrix},$$

where $g_{1.5,k}$ are the Grünwald weights as given in equation (4) with $\alpha=1.5$.

Substituting from equation (17) into equation (15) we get:

$$(P+\mu h^3B+\eta h^{1.5}BG)S=h^3BF,$$

and

$$(P+\mu h^3B+\eta h^{1.5}BG)Y=h^3BF+T.$$

Hence

$$T=(P+\mu h^3B+\eta h^{1.5}BG)E. \tag{18}$$

Our main purpose now is to derive a bound on $\|E\|$. From the equation (18) we can write the error term as

$$E=(I+\mu h^3P^{-1}B+\eta h^{1.5}P^{-1}BG)^{-1}P^{-1}T.$$

which implies that

$$\|E\|= \left\|(I+\mu h^3P^{-1}B+\eta h^{1.5}P^{-1}BG)^{-1}\right\| \cdot \left\|P^{-1}\right\| \cdot \|T\|. \tag{19}$$

In order to derive the bound on $\|E\|$, the following two lemmas are needed.

Lemma 4.1. [19] If N is a square matrix of order n and $\|N\|<1$, then $(I+N)^{-1}$ exists and

$$\left\|(I+N)^{-1}\right\| < \frac{1}{1-\|N\|}.$$

Lemma 4.2. The matrix $(P+\mu h^3B+\eta h^{1.5}BG)$ given in Eq. (18) is nonsingular if

$$(\mu+2\eta m h^{-1.5})w<1, \text{ where } w=\frac{1}{8}\left((b-a)^2+h^2\right).$$

Proof. Let

$$H=\mu h^2P^{-1}B+\eta h^{0.5}P^{-1}BG. \tag{20}$$

It was shown, in [4], that

$$\|P^{-1}\| \le \frac{h^{-2}}{8}\left((b-a)^2+h^2\right)=wh^{-2}, \tag{21}$$

and from the system B, for $\lambda+\beta=\frac{1}{6}$ and $\lambda \ne \frac{1}{120}$, we have

$$\|B\|=1, \tag{22}$$

and from the system G, we have

$$\|G\|=\sum_{i=0}^{n-2}|g_{\alpha,k}|,$$

which, together with the fact that $g_{1.5,0}=1$ and $g_{1.5,1}=-1.5$, leads to

$$\|G\| \le 2m, \quad \text{for all } (m-1)<1.5<m. \tag{23}$$

Substituting equations (21)–(23) into equation (20) and then using our assumption we obtain

$$\|H\| \le 1. \tag{24}$$

Then by lemma 4.1, $(I+\mu h^2P^{-1}B+\eta h^{2-\alpha}P^{-1}BG)^{-1}$ exists and

$$\|(I+\mu h^3P^{-1}B+\eta h^{1.5}P^{-1}BG)^{-1}\| \le \frac{1}{1-\mu h^3\|P^{-1}\|\|B\|-\eta h^{1.5}\|P^{-1}\|\|B\|\|G\|}. \tag{25}$$

This completes proof of the lemma □.

As a result of the above lemma, the discrete boundary value problem (15) has a unique solution if $(\mu+2\eta mh^{-1.5})w<1$. Expanding (12) in fractional Taylor's series about x_i we obtain

$$\|T\|=\xi_1 h^6 M_4 \tag{26}$$

where

$$M_4=\max_{a\le x\le b}\left|\left(D^{3/2}\right)^4 y(x)\right|$$

Hence using equation (19) we have

$$\|E\|\le \frac{\|P^{-1}\|\|T\|}{1-\mu h^3\|P^{-1}\|\|B\|-\eta h^{1.5}\|P^{-1}\|\|B\|\|G\|} \cong O(h^4). \tag{27}$$

In view of lemma 4.2, we can conclude the following theorem:

Theorem 4.1. Let $y(x)$ be the exact solution of the continuous boundary value problem $(1)-(2)$ and let $y(x_i)$, $i=1(1)n-1$, satisfy the discrete boundary value problem (15). Moreover, if we set $e_i=y_i-s_i$, then $\|E\|\cong O(h^4)$ as given by equation (27), neglecting all errors due to round off.

5 Computational Results

To illustrate our method and to demonstrate its convergence and applicability of our presented methods computationally, we have solved two fractional boundary value problems. All calculations are enforced with MATLAB 12b and MAPLE 15.

Example 5.1. Consider the boundary value problem

$$\left(D^{2\alpha}+\eta D^{\alpha}+\mu\right)y(x)=f(x), \tag{28}$$

where

$$f(x)=\mu x^4(x-1)+\eta x^{4-\alpha}\left(\frac{120x}{\Gamma(6-\alpha)}-\frac{24}{\Gamma(5-\alpha)}\right) +x^{4-2\alpha}\left(\frac{120x}{\Gamma(6-2\alpha)}-\frac{24}{\Gamma(5-2\alpha)}\right), \tag{29}$$

subject to the boundary condition $y(0)=y(1)=0$. The exact solution of this problem is

$$y(x)=x^4(x-1).$$

The numerical solutions and absolute errors for the values $\lambda=\beta=\frac{1}{4}$, $\mu=1,\eta=0.5$ and $\alpha=1.5$ are demonstrated in Table 1. Moreover, the exact and numerical solutions are exhibited in Figure 1 for $\alpha=1.65$.

Example 5.2. Consider the fractional differential equation

$$D^{2\alpha}y(x)+\eta D^{\alpha}y(x)+\mu y(x)=f(x), \tag{30}$$

Table 1: Exact, approximate and absolute error for Example 5.1.

$\alpha=1.5$ and $\lambda=\beta=\frac{1}{4}$			
x	Exact	Approximation	Absolute Error
0	0	0	0
0.125	-0.0002140	-0.0000909	$1.2310000E-04$
0.250	-0.0029297	-0.0000579	$2.8718000E-03$
0.375	-0.0123596	-0.0002619	$1.2097628E-02$
0.500	-0.0312500	-0.0009829	$3.0267089E-02$
0.625	-0.0572200	-0.0020054	$5.5215004E-02$
0.750	-0.0791000	-0.0028265	$7.6274996E-02$
0.875	-0.0732730	-0.0023161	$7.0956586E-02$
1	0	0	0

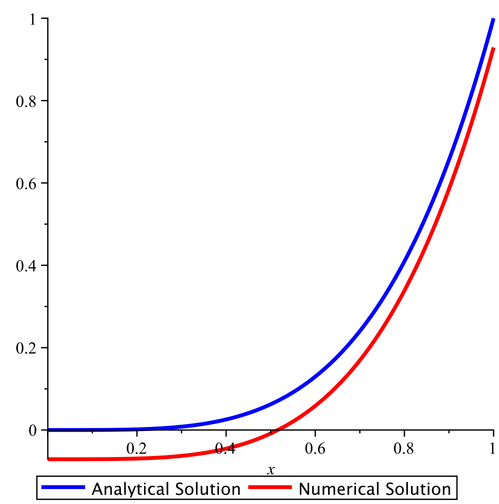

Fig. 1: Exact and approximate solutions of Example 5.1 for $\alpha=1.65$.

where

$$f(x)=\mu x^3(x-1)+120x^{5-\alpha}\left(\frac{\eta}{\Gamma(6-\alpha)}-\frac{x^{-\alpha}}{\Gamma(6-2\alpha)}\right) +5040x^{7-\alpha}\left(\frac{\eta}{\Gamma(8-\alpha)}-\frac{x^{-\alpha}}{\Gamma(8-2\alpha)}\right), \tag{31}$$

with the boundary condition $y(0)=y(1)=0$. The exact solution is

$$y(x)=x^7-x^5.$$

The numerical results obtained, for the values of $\alpha=1.5,\lambda=\frac{1}{6},\beta=\frac{1}{12},\mu=1,\eta=0.5$, and for $0\le x\le 1$, are shown in Table 2, together with absolute errors, to illustrate the accuracy of the proposed method. Also, exact solution and approximate solution for various values of step size $h=\frac{1}{8},\frac{1}{16}$, and $h=\frac{1}{32}$ are represented in Figure 2 for $\alpha=1.70$.

Table 2: Exact, approximate and absolute error for Example 5.2.

$\mu = 1$, $\eta = 0.5$, $\alpha = 1.5$ and $\lambda = \frac{1}{6}, \beta = \frac{1}{3}$			
x	Exact	Approximation	Absolute Error
0	0	0	0
0.125	-0.00003004	-0.00956430	$9.53425985E-03$
0.250	-0.00091552	-0.01995814	$1.90426193E-02$
0.375	-0.00637292	-0.03206469	$2.56917670E-02$
0.500	-0.02343750	-0.04607803	$2.26405371E-02$
0.625	-0.05811452	-0.05944937	$1.33485117E-03$
0.750	-0.10382080	-0.06402991	$3.97908857E-02$
0.875	-0.12021303	-0.04230987	$7.79031563E-02$
1	0	0	0

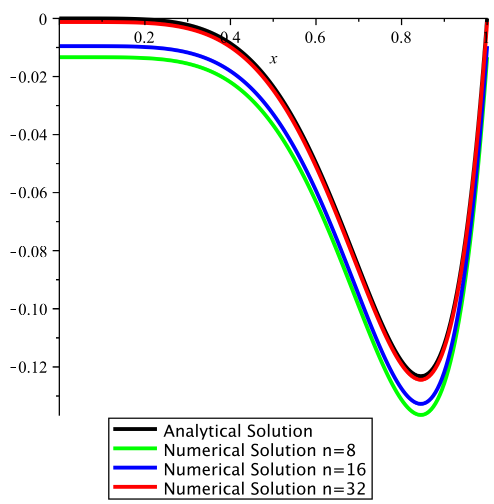

Fig. 2: Exact and approximate solutions of Example 5.2 with variable step size.

Example 5.3. Consider the fractional differential equation [26]:

$$y'''(x) + D^{0.5}y(x) + 2y(x) = 10e^{2x} + \sqrt{2}e^{2x}\mathrm{erf}\left(\sqrt{2x}\right), \quad (32)$$

with the initial conditions

$$y(0) = 1,\ y'(0) = 2,\ y''(0) = 4, \quad (33)$$

where erf is the error function defined by

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$

The exact solution of (32)–(33) is $y(x) = e^{2x}$.

The absolute error $|y(x) - s(x)|$ for different values of n are listed in Table 3.

Table 3: Observed absolute errors in Example 5.3.

n	Our method with $\alpha = \frac{1}{24}$ and $\beta = \frac{11}{24}$	Method in Ref. [26] with $m = 3$
5	$8.4880E-05$	$1.1308E-04$
10	$1.3799E-05$	$7.0160E-06$
15	$3.4479E-06$	$4.3789E-07$

6 Conclusion

The approximate solutions of linear fractional boundary-value problems using the nonpolynomial fractional spline method, show that this method is better in the sense of accuracy and applicability. These have been verified by the absolute errors, $|e_i|$, given in the tables above.

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