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Some Properties of Steenrod Squares on Digital Images

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Abstract: We first present *i*-regularity of two ordered pair of digital simplices, and give the definition of cup-*i* product over digital images by using regularity notion. We study some formulas that can be taken as the basis of inductive definition of cup-*i* product. We define the Steenrod square operations over ordered digital images inspired by analogue in algebraic topology, and then we show that this operation is independent of the ordering on digital images. We study some basic properties of the squarring operations on digital images such as Sq^0 being the identity homomorphism, Sq^1 being the Bockstein homomorphism, Cartan formula, and Adem relations.

Keywords: Digital cohomology group, digital cup product, digital Steenrod squares.

1 Introduction

Digital topology is a very important and essential tool for image analysis as well as computer vision, and its main purpose is to study topological properties of discrete objects those obtained by digitizing the continuous objects.

Cohomology is an algebraic variant of homology, as a result of the dualization in the definition. The homology groups of a space determine its cohomology groups. One of the basic difference between homology and cohomology is that the cohomology groups are contravariant functors while the homology groups are covariant. The contravariance gives a ring structure to the cohomology groups of a space by the cup product. This ring structure is more useful than the additive group structure of cohomology since sometimes group structure is not enough to decide whether two spaces are homeomorphic or not [18].

The term "square" in the phrase *Steenrod square operations* comes from Sq^i (that maps $u \mapsto u^2$) sending a cohomology class u to the 2-fold cup product with itself. The operations Sq^i generate an algebra $\mathscr{A}_2$, called the *Steenrod algebra*, such that $H^*(X;\mathbb{Z}_2)$ is a module over $\mathscr{A}_2$ where X is any topological space.

Many researchers, such as Rosenfeld, Ayala, Bertrand, Kaczynski, Boxer, Karaca and others, have been studying the topology of digital images or just using topological properties of the digital images related to image analysis for several decades.

Gonzalez-Diaz and Real [13] propose a method for computing the cohomology ring of three-dimensional digital binary-valued pictures and they show the computation of the cup product on the cohomology of simple pictures. They [14] give a method for calculating cohomology operations on finite simplicial complexes, and a procedure including the computation of some primary and secondary cohomology operations.

Gonzalez-Diaz et al. [15] present cohomology in the context of structural pattern recognition and introduce an algorithm to compute efficiently the representative cocycles (the basic elements of cohomology) in 2D using a graph pyramid.

Ege and Karaca [12] study on relative homology groups of digital images, give some properties of the Euler characteristics for digital images and present reduced homology groups for digital images. They [11] also give a work that can be used for defining cohomology groups of digital images; they give the Eilenberg-Steenrod axioms and the Universal Coeffcient Theorem for this cohomology theory, and show that the Künneth formula doesn't hold.

Karaca and Burak [20] show that the relative cohomology groups of digital images are determined algebraically by the relative homology groups of digital images, and they express simplicial cup product for

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digital images and use it to establish ring structure of digital cohomology.

Demir and Karaca [9] compute the simplicial homology groups of some digital surfaces. They [10] determine the simplicial cohomology groups of some minimal simple closed curves and a digital surface MSS_6, and give a general algorithm how we make this computation.

At this work, we make a conformation on digital images by using almost the same argument in [27] and [22]. This paper is organized as follows. We recall some basic notions in section 2. The next section is dedicated to i-regularity of digital simplexes, digital version of $\smile_i$ product and its properties. In the last section we introduce the squarring operations on digital images and prove some properties of these operations.

2 Preliminaries

Let $\mathbb{Z}^n$ be the set of lattice points in the n-dimensional Euclidean space where $\mathbb{Z}$ is the set of integers. We say that (X,κ) is a (binary) digital image where $X \subset \mathbb{Z}^n$ and κ is an adjacency relation for the members of X. We use a variety of adjacency relations in the study of digital images.

For a positive integer l with $1 \le l \le n$ and two distinct points $p=(p_1,p_2,...,p_n)$, $q=(q_1,q_2,...,q_n) \in \mathbb{Z}^n$, p and q are c_l*-adjacent* [6] if

(1) there are at most l indices i such that $|p_i - q_i| = 1$; and

(2) for all other indices i such that $|p_i - q_i| \neq 1$, $p_i = q_i$.

Another commonly used notation for c_l-adjacency reflects the number of neighbors $q \in \mathbb{Z}^n$ that a given point $p \in \mathbb{Z}^n$ may have under the adjacency. For example, if $n=1$ we have $c_1 = 2$-adjacency; if $n=2$ we have $c_1 = 4$-adjacency and $c_2 = 8$-adjacency; if $n=3$ we have $c_1 = 6$-adjacency, $c_2 = 18$-adjacency, and $c_3 = 26$-adjacency [6]. Given a natural number l in conditions (1) and (2) with $1 \le l \le n$, l determines each of the κ-adjacency relations of $\mathbb{Z}^n$ in terms of (1) and (2) [16] as follows.

$$\kappa \in \Big\{2n\ (n \ge 1),\ 3^n - 1\ (n \ge 2),\ 3^n - \sum_{t=0}^{r-2} C_t^n 2^{n-t} - 1\ (2 \le r \le n-1, n \ge 3)\Big\} \quad (2.1)$$

The pair (X,κ) is considered in a digital picture $(\mathbb{Z}^n,\kappa,\overline{\kappa},X)$ for $n \ge 1$ in [2,3,5,17], which is called a *digital image* where $(\kappa,\overline{\kappa}) \in \{(\kappa,2n),(2n,3^n-1)\}$. Each of κ and $\overline{\kappa}$ is one of the general κ-adjacency relations. We usually do not permit that κ and $\overline{\kappa}$ both equal $2n$ when $n > 1$, because of the digital connectivity paradox [21]. For instance, $(\kappa,\overline{\kappa}) \in \{(4,8),(8,4)\}$ and $\{(6,18),(6,26),(26,6),(18,6)\}$ are usually considered in $\mathbb{Z}^2$ and $\mathbb{Z}^3$, respectively [5,17,24,25].

A *digital interval* is a set of the form

$$[a,b]_{\mathbb{Z}} = \{z \in \mathbb{Z} \mid a \le z \le b\}$$

where $a,b \in \mathbb{Z}$ with $a < b$.

Let κ be an adjacency relation on $\mathbb{Z}^n$. A κ-neighbor of a lattice point p is κ-adjacent to p. A digital image $X \subset \mathbb{Z}^n$ is κ*-connected* [19] if and only if for every pair of different points $x,y \in X$, there is a set $\{x_0,x_1,...,x_r\}$ of points of a digital image X such that $x=x_0$, $y=x_r$ and x_i and x_{i+1} are κ-neighbors where $i=0,1,...,r-1$. A κ*-component* of a digital image X is a maximal κ-connected subset of X.

Let $X \subset \mathbb{Z}^{n_0}$ and $Y \subset \mathbb{Z}^{n_1}$ be digital images with κ_0 and κ_1-adjacency respectively. Then the function $f: X \to Y$ is called (κ_0,κ_1)*-continuous* [5,25] if for every κ_0-connected subset U of X, $f(U)$ is a κ_1-connected subset of Y. We say that such a function is digitally continuous.

Let X be a digital image with κ-adjacency. If $f:[0,m]_{\mathbb{Z}} \to X$ is a $(2,\kappa)$-continuous function such that $f(0)=x$ and $f(m)=y$, then f is called a *digital path* from x to y in X. If $f(0)=f(m)$ then the κ-path is said to be *closed*, and the function is called a κ*-loop*. Let $f:[0,m-1]_{\mathbb{Z}} \to X$ be a $(2,\kappa)$-continuous function such that $f(i)$ and $f(j)$ are κ-adjacent if and only if $j = i \pm 1 \ mod \ m$. Then the set $f([0,m-1]_{\mathbb{Z}})$ is called a *simple closed* κ*-curve*. A point $x \in X$ is called a κ*-corner*, if x is κ-adjacent to two and only two points $y,z \in X$ such that y and z are κ-adjacent to each other [3]. Moreover, the κ-corner x is called *simple* if y,z are not κ-corners and if x is the only point κ-adjacent to both y,z [2]. X is called a *generalized simple closed* κ*-curve* if what is obtained by removing all simple κ-corners of X is a simple closed κ-curve [3]. If (X,κ) is a κ-connected digital image in $\mathbb{Z}^3$,

$$|X|^x = N_3^*(x) \cap X,$$

where $N_3^*(x) = \{x' \in \mathbb{Z}^3 : x \text{ and } x' \text{ are 26-adjacent}\}$ [3,4]. Generally, if (X,κ) is a κ-connected digital image in $\mathbb{Z}^n$, $|X|^x = N_n^*(x) \cap X$, where

$$N_n^*(x) = \{x' \in \mathbb{Z}^n : x \text{ and } x' \text{ are } c_n\text{-adjacent}\}[17].$$

Let $X \subset \mathbb{Z}^{n_0}$ and $Y \subset \mathbb{Z}^{n_1}$ be digital images with κ_0 and κ_1-adjacency respectively. A function $f: X \to Y$ is a (κ_0,κ_1)*-isomorphism* [7] (called (κ_0,κ_1)-homeomorphism in [4]) if f is (κ_0,κ_1)-continuous, bijective and $f^{-1}: Y \to X$ is (κ_1,κ_0)-continuous, in which case we write $X \approx_{(\kappa_0,\kappa_1)} Y$.

Definition 2.1. [17] Let $c^* := \{x_0,x_1,...,x_n\}$ be a closed κ-curve in $\mathbb{Z}^2$ where $\{\kappa,\overline{\kappa}\} = \{4,8\}$. A point x of the complement $\overline{c^*}$ of a closed κ-curve c^* in $\mathbb{Z}^2$ is said to be in the *interior* of c^* if it belongs to the bounded $\overline{\kappa}$-connected component of $\overline{c^*}$. The set of all interior points of c^* is denoted by $Int(c^*)$.

Definition 2.2. [17] Let (X,κ) be a digital image in $\mathbb{Z}^n$, $n \ge 3$ and $\overline{X} = \mathbb{Z}^n - X$. Then X is called a *closed* κ*-surface* if it satisfies the following.

(1) In case that $(\kappa,\overline{\kappa}) \in \{(\kappa,2n),(2n,3^n-1)\}$, where the κ-adjacency is taken from Definition 2.1 with $\kappa \neq 3^n - 2^n - 1$ and $\overline{\kappa}$ is the adjacency on $\overline{X}$, then

(a) for each point $x \in X$, $|X|^x$ has exactly one κ-component κ-adjacent to x;

(b) $|\overline{X}|^x$ has exactly two $\overline{\kappa}$-components $\overline{\kappa}$-adjacent to x; we denote by C^{xx} and D^{xx} these two components; and

(c) for any point $y \in N_\kappa(x) \cap X$, $N_{\overline{\kappa}}(y) \cap C^{xx} \neq \emptyset$ and $N_{\overline{\kappa}}(y) \cap D^{xx} \neq \emptyset$, where $N_\kappa(x)$ means the κ-neighbors of x.

Further, if a closed κ-surface X does not have a simple κ-point, then X is called simple.

(2) In case that $(\kappa, \overline{\kappa}) = (3^n - 2^n - 1, 2n)$, then

(a) X is κ-connected,

(b) for each point $x \in X$, $|X|^x$ is a generalized simple closed κ-curve.

Further, if the image $|X|^x$ is a simple closed κ-curve, then the closed κ-surface X is called simple.

For a closed κ-surface S_κ, we denote by $\overline{S_\kappa}$ the complement of S_κ in $\mathbb{Z}^n$. Then a point x of $\overline{S_\kappa}$ is said to be *interior* of S_κ if it belongs to the bounded $\overline{\kappa}$-connected component of S_κ. The set of all interior points of S_κ is denoted by $int(S_\kappa)$.

The 3-dimensional digital images MSS^*_{18} and MSS^*_6 which are obtained from the minimal simple closed curves MSC_8 and MSC_4 in $\mathbb{Z}^2$, respectively, are essentially used in establishing the notion of a connected sum [17].

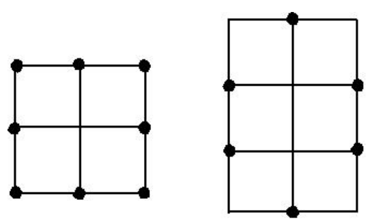

Fig. 1: Minimal simple closed curves MSC_4 and MSC_8.

- $MSS^*_6 := MSS_6 \cup Int(MSS_6)$ where

$$MSS_6 \approx_{(6,6)} (MSC_4 \times [0,2]_{\mathbb{Z}}) \cup (Int(MSC_4) \times \{0,2\})$$

and MSC_4 is 4-isomorphic to the set

$$\{(1,0),(1,1),(0,1),(-1,1),(-1,0),(-1,-1),(0,-1),(1,-1)\}.$$

- $MSS^*_{18} := MSS_{18} \cup Int(MSS_{18})$ where

$$MSS_{18} \approx_{(18,18)} (MSC_8 \times \{1\}) \cup (Int(MSC_8) \times \{0,2\})$$

and MSC_8 is 8-isomorphic to the set

$$\{(0,0),(-1,1),(-2,0),(-2,-1),(-1,-2),(0,-1)\}.$$

Definition 2.3. [17] Let S_{κ_0} be a closed κ_0-surface in $\mathbb{Z}^{n_0}$ and S_{κ_1} be a closed κ_1-surface in $\mathbb{Z}^{n_1}$ for $n_0, n_1 \geq 3$. Consider $A'_{\kappa_0} \subset A_{\kappa_0} \subset S_{\kappa_0}$ such that $A'_{\kappa_0} \approx_{(\kappa_0,8)} Int(MSC^*_8)$, $A'_{\kappa_0} \approx_{(\kappa_0,4)} Int(MSC^*_4)$ or $A'_{\kappa_0} \approx_{(\kappa_0,8)} Int(MSC'^*_8)$. Let $f : A_{\kappa_0} \to f(A_{\kappa_0}) \subset S_{\kappa_1}$ be a (κ_0, κ_1)-isomorphism. Let $S'_{\kappa_i} = S_{\kappa_i} \setminus A'_{\kappa_i}$, $i \in \{0,1\}$. Then the connected sum, denoted by $S_{\kappa_0} \sharp S_{\kappa_1}$, is the quotient space $S'_{\kappa_0} \cup S'_{\kappa_1} / \sim$, where $i : A_{\kappa_0} \setminus A'_{\kappa_0} \to S'_{\kappa_0}$ is the inclusion map and $i(x) \sim f(x)$ for $x \in A_{\kappa_0} \setminus A'_{\kappa_0}$.

Definition 2.4. [26] Let S be a set of nonempty subsets of a digital image (X,κ). The members of S are called simplexes of (X,κ) if the following holds:

(i) If p and q are distinct points of $s \in S$, then p and q are κ-adjacent.

(ii) If $s \in S$ and $\emptyset \neq t \subset s$, then $t \in S$ (note this implies every point p that belongs to a simplex determines a simplex $\{p\}$).

An m-simplex is a simplex S such that $|S| = m+1$.

Let P be a digital m-simplex. If P' is a nonempty proper subset of P, then P' ia called a face of P.

Definition 2.5. [1] Let (X,κ) be a finite collection of digital m-simplices, $0 \leq m \leq d$ for some nonnegative integer d. If the following statements hold, then (X,κ) is called a finite digital simplicial complex:

(1) If P belongs to X, then every face of P also belongs to X.

(2) If $P,Q \in X$, then $P \cap Q$ is either empty or a common face of P and Q.

The dimension of a digital simplicial complex X is the biggest integer m such that X has an m-simplex.

$C^\kappa_q(X)$ is a free abelian group with basis all digital (κ,q)-simplices in X [1].

Corollary 2.6. [8] Let $(X,\kappa) \subset \mathbb{Z}^n$ be a digital simplicial complex of dimension m. Then for all $q > m$, $C^\kappa_q(X)$ is a trivial group.

Let $(X,\kappa) \subset \mathbb{Z}^n$ be a digital simplicial complex of dimension m. The homomorphism $\partial_q : C^\kappa_q(X) \to C^\kappa_{q-1}(X)$ defined by

$$\partial_q(< p_0, p_1, \ldots, p_q >) = \begin{cases} \sum_{i=0}^{q} (-1)^i < p_0, p_1, \ldots, \widehat{p_i}, \ldots, p_q >, & q \leq m; \\ 0, & q > m \end{cases}$$

is called a boundary homomorphism where $\widehat{p_i}$ means deleting the point p_i. Then for all $1 \leq q \leq m$, we have $\partial_{q-1} \circ \partial_q = 0$ [1].

Theorem 2.7. [1] Let $(X,\kappa) \subset \mathbb{Z}^n$ be a digital simplicial complex of dimension m. Then

$$C^\kappa_*(X) : 0 \xrightarrow{\partial_{m+1}} C^\kappa_m(X) \xrightarrow{\partial_m} C^\kappa_{m-1}(X) \xrightarrow{\partial_{m-1}} \ldots \xrightarrow{\partial_1} C^\kappa_0(X) \xrightarrow{\partial_0} 0$$

is a chain complex.

Let (X,κ) be a digital simplicial complex. The group of digital simplicial q-cycles is

$$Z_q^\kappa(X) = Ker\ \partial_q = \{\sigma \in C_q^\kappa(X) | \partial_q(\sigma) = 0\}$$

and the group of digital simplicial q-boundaries is

$$\begin{aligned} B_q^\kappa(X) =& Im\ \partial_{q+1} \\ =& \{\tau \in C_q^\kappa(X) | \partial_{q+1}(\sigma) = \tau \text{ for } \sigma \in C_{q+1}^\kappa(X)\}. \end{aligned}$$

The q^{th} digital simplicial homology group [1] is

$$H_q^\kappa(X) = Z_q^\kappa(X)/B_q^\kappa(X).$$

Definition 2.8. [23] Let $(X,\kappa) \subset \mathbb{Z}^n$ be a digital simlicial complex and C_q^κ be an abelian group whose bases are all (κ,q)-simplexes in X. $C^{*,\kappa}(X) = \{C^{q,\kappa}(X), \delta_q\}_{q\geq 0}$ is the digital cochain complex of X where

$$\begin{aligned} C^{q,\kappa}(X) =& Hom(C_q^\kappa(X), G) \\ =& \{c : C_q^\kappa(X) \to G, c \text{ is a homomorphism}\}. \end{aligned}$$

Here $\delta_q : C^{q,\kappa}(X) \to C^{q+1,\kappa}(X)$ is the *digital cochain homomorphism* and defined as $\delta_q(c)(a) = c(\partial_{q+1}(a))$ for $c \in C^{q,\kappa}(X)$, $a \in C_{q+1}^\kappa(X)$. $Z^{q,\kappa}(X;G)$ is the kernel of δ_q and called group of *digital cocycles* of (X,κ) with coefficients in G, $B^{q,\kappa}(X;G)$ is the image of δ_{q-1} and called group of *digital coboundaries* of (X,κ) with coefficients in G, and (noting that since $\partial^2 = 0$, $\delta^2 = 0$)

$$H^{q,\kappa}(X;G) = Z^{q,\kappa}(X;G)/B^{q,\kappa}(X;G)$$

is called the *digital q^{th} cohomology group* of (X,κ) with coefficients in G. If u is a digital q-cocycle, then $\{u\} \in H^{q,\kappa}(X;G)$ denotes the cohomology class. $\{u\} = \{v\}$ means that $u - v \in B^{q,\kappa}(X;G)$.

Theorem 2.9. [23] If (X,κ) is a singleton digital image, then

$$H^{q,\kappa}(X;G) = \begin{cases} G, & q = 0; \\ 0, & q > 0 \end{cases}$$

where G is an abelian group.

Theorem 2.10. [23] Let (X,κ) be a digital simplicial complex. For any abelian group G, there is exact sequence

$$0 \to Ext(H_{q-1}^\kappa(X), G) \to H^{q,\kappa}(X;G) \to Hom(H_q^\kappa(X), G) \to 0$$

where $H_q^\kappa(X) = H_q^\kappa(X;\mathbb{Z})$. This exact sequence splits; hence

$$H^{q,\kappa}(X;G) \cong Hom(H_q^\kappa(X), G) \oplus Ext(H_{q-1}^\kappa(X), G).$$

Definition 2.11. [23] Let $(X,\kappa) \subset \mathbb{Z}^n$ be a finite digital simplicial complex; A be digital subcomplex of (X,κ) with the same adjacency relation. The p-cochains of (X,κ) which are zero on digital simplexes of A form a subgroup

$$C^{p,\kappa}(X,A;\mathbb{Z}_2) = Hom(C_p^\kappa(X,A), \mathbb{Z}_2)$$

of $C^{p,\kappa}(X;\mathbb{Z}_2)$. Since the digital coboundary of a cochain is zero on A, thus $Z^{p,\kappa}(X,A;\mathbb{Z}_2) = Ker\ \delta$, $B^{p+1,\kappa}(X,A;\mathbb{Z}_2) = Im\ \delta$, and

$$H^{p,\kappa}(X,A;\mathbb{Z}_2) = Z^{p,\kappa}(X,A;\mathbb{Z}_2)/B^{p,\kappa}(X,A;\mathbb{Z}_2)$$

can be defined as usual.

If $f : (Y,\kappa') \to (X,\kappa)$ be a digital simplicial map such that $f(B) \subset A$ where B is a digital subcomplex of (Y,κ'), then f induces the homomorphism below:

$$f^\sharp : C^{p,\kappa}(X,A) \to C^{p,\kappa'}(Y,B).$$

$\delta \circ f^\sharp = f^\sharp \circ \delta$ so that $f^\sharp$ maps digital cocycles to digital cocycles, and digital coboundaries to digital coboundaries. Therefore it is called *digital cochain mapping* and $f^\sharp$ induces a homomorphism

$$f^* : H^{p,\kappa}(X,A) \to H^{p,\kappa'}(Y,B).$$

3 The Cup Product

We present the definition and some properties of digital simplicial cup product by using property of regularity. The proofs of the following theorems are analogues to algebraic topology (see [27]).

Let σ and τ be two digital simplices of dimensions p and q respectively; α be a fixed order in (X,κ), and $i \geq 0$ be a positive integer. The ordered pair (σ,τ) is said to be *i-regular* in α if the following conditions are satisfied:

(-1) The vertices of σ and τ span a $(p+q-i)$-simplex ζ. In this case, σ, τ have $i+1$ common vertices denoted by $V^0, V^1, \cdots, V^i$ in the order α.
(0) V^0 is the first vertex of τ.
(1) V^0, V^1 are adjacent vertices in σ.
(2) V^1, V^2 are adjacent vertices in τ.
. . . . . .
(j+1) V^j, V^{j+1} are adjacent vertices in $\sigma(\tau)$ if j is even (odd).
. . . . . .
(i+1) V^i is the last vertex of $\sigma(\tau)$ if i is even (odd).

If (σ,τ) is i-regular, let σ_0 be the face of σ spanned by vertices precessor of the V^0, let $\sigma_{2j}(0 < 2j \leq i)$ be the face of σ spanned by vertices successor of the V^{2j-1} and precessor of the V^{2j}, and if i is odd, σ_{i+1} be the face of σ spanned by vertices successor of the V^i. Similarly, let $\tau_{2j+1}(1 \leq 2j+1 \leq i+1)$ be the face of τ spanned by vertices successor of the V^{2j} and precessor of the V^{2j+1}, and if i is even, τ_{i+1} be the face of τ spanned by vertices successor of the V^i vertices. By the i-regularity, σ ve τ can be written as joins of subsimplexes:

$$\sigma = \sigma_0.\sigma_2.....\sigma_{2k}\ ,\ \tau = \tau_1.\tau_3.....\tau_{2k+(-1)^i}$$

where $2k = i$ if i is even and $2k = i+1$ if i is odd. Let τ'_{2j+1} be the face of τ_{2j+1} by deleting V^{2j} and V^{2j+1} vertices, and if i is even, τ'_{i+1} be the face of τ_{i+1} by deleting V^i vertex. Then the digital simplex ξ spanned by the vertices of σ and τ can be written as follows:

$$\xi = \sigma_0.\tau'_1.\sigma_2.\tau'_3.... \begin{cases} \tau'_{i+1}, & i \text{ is odd;} \\ \sigma_{i+1}, & i \text{ even.} \end{cases}$$

In the group of digital $(p+q-i)$-cochains, let us define; if (σ,τ) is not i-regular, then $\sigma \sqcup_i \tau = 0$, and if (σ,τ) is i-regular, then $\sigma \sqcup_i \tau = \pm\xi$. If $i=0$, then the sign is ”+”. In general, the sign of permutation is identified by bringing the ordered vertices

$$\sigma_0, \sigma_2, ..., \sigma_{2k}, \tau'_1, \tau'_3, ..., \tau_{2k+(-1)^i}$$

into the order α

$$\sigma_0.\tau'_1.\sigma_2.\tau'_3.....\sigma_j.\tau'_{j+1}....$$

Let G, G' be abelian groups, and G'' be an abelian grup such that there is a bilinear product $g.g' \in G''$ defined for $g \in G$, $g' \in G'$. Let $u^p \in C^{p,\kappa}(X,G)$, $v^q \in C^{q,\kappa}(X,G)$, and $u^p = \sum g_j \sigma_j^p$, $v^q = \sum g'_k \sigma_k^q$ be their unique representations in terms of the distinct digital p and q-simplexes of (X,κ) oriented by the order α.

$$\smile_i: C^{p,\kappa}(X,G) \times C^{q,\kappa}(X,G') \to C^{p+q-i,\kappa}(X,G'')$$

is defined by

$$u^p \smile_i v^q = \sum_{j,k} (g_j g'_k) \sigma_j^p \sqcup_i \sigma_k^q.$$

Since $C^{p,\kappa}(X,G)$, $C^{q,\kappa}(X,G')$ are paired to $C^{p+q-i,\kappa}(X,G'')$, the product $\smile_i$ is bilinear. However we use $\mathbb{Z}$ instead of G and G' through this paper.

Theorem 3.1. $u^p \smile_i v^q = 0$ if $i > p$ or q.

Proof. If the common face of σ_j^p, σ_k^q has the dimension less or equal to $min\{p,q\}$, then (σ_j^p, σ_k^q) is not i-regular. The result holds.□

Remark 3.2. [27] A digital p-cochain is a function $u^p(A^0, \cdots, A^p)$ with $\mathbb{Z}$ valued and is defined on each ordered set of $p+1$ vertices whose union induces a digital simplex. If the vertices do not span a digital p-simplex in the given order of vertices, then it becomes zero. If ξ is a digital $(p+q-i)$-simplex, any i-face of ξ determines a splitting into a product as $\sigma \sqcup_i \tau = \pm\xi$. And then

$$u^p \smile_i v^q(\xi) = \sum \pm u^p(\sigma).v^q(\tau),$$

the sum is taken over those i-faces of ξ such that

$$dim\ \sigma = p \ \text{ and } \ dim\ \tau = q.$$

Example 3.3. Let

$$\begin{aligned} MSS_{18}\sharp MSS_{18} = \{ & c_0 = (1,0,1), c_1 = (1,1,1), \\ & c_2 = (1,2,1), c_3 = (0,3,1), \\ & c_4 = (-1,2,1), c_5 = (-1,1,1), \\ & c_6 = (-1,0,1), c_7 = (0,-1,1), \\ & c_8 = (0,2,2), c_9 = (0,1,2), \\ & c_{10} = (0,0,2), c_{11} = (0,2,0), \\ & c_{12} = (0,1,0), c_{13} = (0,0,0)\}. \end{aligned}$$

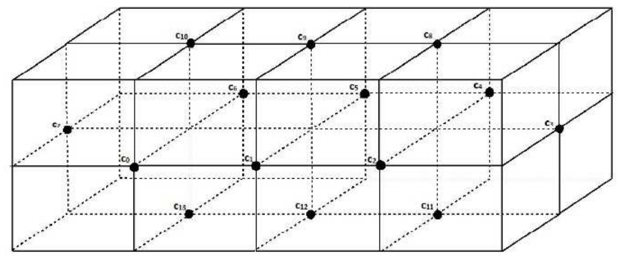

Fig. 2: $MSS_{18}\sharp MSS_{18}$ [14].

Then we can direct $MSS_{18}\sharp MSS_{18}$ by the ordering $c_6 < c_5 < c_4 < c_7 < c_{13} < c_{10} < c_{12} < c_9 < c_{11} < c_8 < c_3 < c_0 < c_1 < c_2$. We have the following simplicial chain complexes:
$C_0^{18}(MSS_{18}\sharp MSS_{18})$ has for the basis

$$\{\langle c_0\rangle, \langle c_1\rangle, ..., \langle c_{13}\rangle\},$$

$C_1^{18}(MSS_{18}\sharp MSS_{18})$ has for the basis

$$\begin{aligned} \{ & e_0 = \langle c_7c_0\rangle, e_1 = \langle c_{10}c_0\rangle, e_2 = \langle c_{13}c_0\rangle, e_3 = \langle c_0c_1\rangle, \\ & e_4 = \langle c_9c_1\rangle, e_5 = \langle c_{12}c_1\rangle, e_6 = \langle c_1c_2\rangle, e_7 = \langle c_8c_2\rangle, \\ & e_8 = \langle c_{11}c_2\rangle, e_9 = \langle c_3c_2\rangle, e_{10} = \langle c_4c_3\rangle, e_{11} = \langle c_8c_3\rangle, \\ & e_{12} = \langle c_{11}c_3\rangle, e_{13} = \langle c_5c_4\rangle, e_{14} = \langle c_4c_8\rangle, e_{15} = \langle c_4c_{11}\rangle, \\ & e_{16} = \langle c_6c_5\rangle, e_{17} = \langle c_5c_9\rangle, e_{18} = \langle c_5c_{12}\rangle, e_{19} = \langle c_6c_7\rangle, \\ & e_{20} = \langle c_6c_{10}\rangle, e_{21} = \langle c_6c_{13}\rangle, e_{22} = \langle c_7c_{10}\rangle, e_{23} = \langle c_7c_{13}\rangle, \\ & e_{24} = \langle c_9c_8\rangle, e_{25} = \langle c_{10}c_9\rangle, e_{26} = \langle c_{12}c_{11}\rangle, e_{27} = \langle c_{13}c_{12}\rangle\}, \end{aligned}$$

and $C_2^{18}(MSS_{18}\sharp MSS_{18})$ has for the basis

$$\begin{aligned} \{ & \sigma_0 = \langle c_7c_{13}c_0\rangle, \sigma_1 = \langle c_7c_{10}c_0\rangle, \sigma_2 = \langle c_8c_3c_2\rangle, \\ & \sigma_3 = \langle c_{11}c_3c_2\rangle, \sigma_4 = \langle c_4c_8c_3\rangle, \sigma_5 = \langle c_4c_{11}c_3\rangle, \\ & \sigma_6 = \langle c_6c_7c_{10}\rangle, \sigma_7 = \langle c_6c_7c_{13}\rangle\}. \end{aligned}$$

Digital simplicial 1-cocycles of $MSS_{18}\sharp MSS_{18}$ are:

$$\begin{aligned} \alpha &= -e_0^* - e_1^* - e_2^* + e_3^* & \lambda &= e_0^* - e_{19}^* + e_{22}^* + e_{23}^* \\ \beta &= -e_3^* - e_4^* - e_5^* + e_6^* & \mu &= e_7^* + e_{11}^* - e_{14}^* - e_{24}^* \\ \gamma &= -e_6^* - e_7^* - e_8^* - e_9^* & \nu &= e_4^* - e_{17}^* + e_{24}^* - e_{25}^* \\ \delta &= e_9^* - e_{10}^* - e_{11}^* - e_{12}^* & \xi &= e_1^* - e_{20}^* - e_{22}^* + e_{25}^* \\ \varepsilon &= e_{10}^* - e_{13}^* + e_{14}^* + e_{15}^* & \pi &= e_8^* + e_{12}^* - e_{15}^* - e_{26}^* \\ \eta &= e_{13}^* - e_{16}^* + e_{17}^* + e_{18}^* & \rho &= e_5^* - e_{18}^* + e_{26}^* - e_{27}^* \\ \theta &= e_{16}^* + e_{19}^* + e_{20}^* + e_{21}^* & \tau &= e_2^* - e_{21}^* - e_{23}^* + e_{27}^*. \end{aligned}$$

For instance

$$\begin{aligned}\alpha \smile \beta =&(-e_0^*-e_1^*-e_2^*+e_3^*)\smile(-e_3^*-e_4^*-e_5^*+e_6^*)\\ =&\,e_0^*\sqcup e_3^*+e_0^*\sqcup e_4^*+e_0^*\sqcup e_5^*-e_0^*\sqcup e_6^*+e_1^*\sqcup e_3^*+\\ &e_1^*\sqcup e_4^*+e_1^*\sqcup e_5^*-e_1^*\sqcup e_6^*+e_2^*\sqcup e_3^*+e_2^*\sqcup e_4^*\\ &+e_2^*\sqcup e_5^*-e_2^*\sqcup e_6^*-e_3^*\sqcup e_3^*-e_3^*\sqcup e_4^*-e_3^*\sqcup e_5^*\\ &+e_3^*\sqcup e_6^*\\ =&(e_0+e_3)+(e_1e_3)+(e_2e_3)\\ =&\,0,\end{aligned}$$

and

$$\begin{aligned}\varepsilon\smile\delta =&(e_{10}^*-e_{13}^*+e_{14}^*+e_{15}^*)\smile(e_9^*-e_{10}^*-e_{11}^*-e_{12}^*)\\ =&e_{10}^*\sqcup e_9^*-e_{10}^*\sqcup e_{10}^*-e_{10}^*\sqcup e_{11}^*-e_{10}^*\sqcup e_{12}^*\\ &-e_{13}^*\sqcup e_9^*+e_{13}^*\sqcup e_{10}^*+e_{13}^*\sqcup e_{11}^*+e_{13}^*\sqcup e_{12}^*\\ &+e_{14}^*\sqcup e_9^*-e_{14}^*\sqcup e_{10}^*-e_{14}^*\sqcup e_{11}^*-e_{14}^*\sqcup e_{12}^*\\ &+e_{15}^*\sqcup e_9^*-e_{15}^*\sqcup e_{10}^*-e_{15}^*\sqcup e_{11}^*-e_{15}^*\sqcup e_{12}^*\\ =&(e_{10}^*+e_9^*)+(e_{13}^*+e_{10}^*)-(e_{14}^*+e_{11}^*)-(e_{15}^*+e_{12}^*)\\ =&\sigma_5.\end{aligned}$$

If we repeat the same procedure for the other digital simplicial 1-cocycles, we get the following table.

$\smile$	α	β	γ	δ	ε	η	θ	λ	μ	ν	ξ	π	ρ	τ
α	0	0	0	0	0	0	0	0	0	0	0	0	0	0
β	0	0	0	0	0	0	0	0	0	0	0	0	0	0
γ	0	0	0	0	0	0	0	0	0	0	0	0	0	0
δ	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ε	0	0	0	σ_5	0	0	0	0	0	0	0	0	0	0
η	0	0	0	0	0	0	0	0	0	0	0	0	0	0
θ	0	0	0	0	0	0	0	0	0	0	$-\sigma_6$	0	0	σ_7
λ	$-\sigma_0+\sigma_1$	0	0	0	0	0	0	0	0	0	0	0	0	0
μ	0	0	σ_2	0	0	0	0	0	0	0	0	0	0	0
ν	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ξ	0	0	0	0	0	0	0	0	0	0	0	0	0	0
π	0	0	$-\sigma_3$	0	0	0	0	0	0	0	0	0	0	0
ρ	0	0	0	0	0	0	0	0	0	0	0	0	0	0
τ	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Let (X,κ), (Y,κ') be digital simplicial complexes, and $f:(Y,\kappa')\to(X,\kappa)$ be a digital simplicial map. f induces a homomorphism $f^\sharp : C^{p,\kappa}(X;\mathbb{Z})\to C^{q,\kappa'}(Y;\mathbb{Z})$ defined as

$$f^\sharp u^p(A^0\cdots A^p)=u^p(f(A^0)\cdots f(A^p)),$$

where $u^p\in C^{p,\kappa}(X;\mathbb{Z})$, and $\sigma'=A^0\cdots A^p$ is a digital p-simplex of (Y,κ'). If $f(\sigma')$ is degenerate, $f^\sharp u^p$ has the value 0 on σ', otherwise $f^\sharp u^p=u^p(f(\sigma'))$. Let δ,δ' be the coboundary operators on (X,κ) and (Y,κ'). $\delta' f^\sharp=f^\sharp\delta$, hence $f^\sharp$ maps digital cocycles into digital cocycles, and digital coboundaries into digital coboundaries. So $f^\sharp$ induces the following homomorphism

$$f^*: H^{p,\kappa}(X;\mathbb{Z})\to H^{q,\kappa'}(Y;\mathbb{Z}).$$

If (Y,κ') is a digital subcomplex of (X,κ), f is the identity map, $u^p=\sum g_j\sigma_j^p$ and $\sigma_j^p\notin(Y,\kappa')$, then $f^*u^p=0$ since $g_j=0$.

If α, α' are orders in (X,κ), (Y,κ') respectively, then f is said to be *order preserving* if $A'\le B'$ in α' implies $f(A')\le f(B')$ in α. If a digital simplicial map and an order are given, then there exists another order such that digital simplicial map is order preserving.

Theorem 3.4. If $f:(Y,\kappa')\to(X,\kappa)$ is an order preserving digital simplicial map, then

$$f^*(u\smile_i v)=f^*(u)\smile_i f^*(v). \tag{3.1}$$

Proof. Suppose that ζ' is a digital $(p+q-i)$-simplex of (Y,κ'), and $\zeta'=\sigma'\smile_i\tau'$ in the order α' where $dim\ \sigma'=p$ and $dim\ \tau'=q$.

If $f(\zeta')$ is degenerate, then either $f(\sigma')$ or $f(\tau')$ is degenerate or $f(\sigma')$ and $f(\tau')$ have more than an i-face in common. In this case, both sides of the equation have the zero value on ζ'.

If $f(\zeta')$ is non-degenerate, then

$$f^*(u\smile_i v)=u\smile_i v(f(\zeta')).$$

Since restriction of f on ζ' is a one-to-one and order preserving map of ζ' on $f(\zeta')$, we get $f(\zeta')=\pm f(\sigma')\smile_i f(\tau')$ and any splitting $f(\zeta')=\pm\sigma\smile_i\tau$ can be attained. Hence

$$\begin{aligned}u\smile_i v(f(\zeta'))&=\sum\pm u(f(\sigma')).v(f(\tau'))\\ &=\sum\pm f^*u(\sigma').f^*v(\tau')\\ &=(f^*u\smile_i f^*v)(\zeta').\square\end{aligned}$$

Let σ be an ordered digital p-simplex and A be a vertex. We can define σA as follows: If the vertices of σ together with A do not span a digital simplex of (X,κ), then σA is the digital 0-simplex in $C^{p+1,\kappa}(X;\mathbb{Z})$. Otherwise σA is the ordered digital $(p+1)$-simplex consisting of the ordered vertices of σ followed by A. If $u^p=\sum g_j\sigma_j^p$, define $u^pA=\sum g_j(\sigma_j^pA)\in C^{p+1,\kappa}(X;\mathbb{Z})$.

Theorem 3.5. If the vertex A follows all vertices of σ^p and τ^q in the order α, then we have

$$\sigma^p\sqcup_i(\tau^qA)=\begin{cases}(\sigma^p\sqcup_i\tau^q)A, & i \text{ even};\\ 0, & i \text{ odd}.\end{cases} \tag{3.2}$$

$$(\sigma^pA)\sqcup_i\tau^q=\begin{cases}0, & i \text{ even};\\ (-1)^{q+1}(\sigma^p\sqcup_i\tau^q)A, & i \text{ odd}.\end{cases} \tag{3.3}$$

$$(\sigma^pA)\sqcup_i(\tau^qA)=(-1)^{q+i+1}(\sigma^p\sqcup_{i-1}\tau^q)A. \tag{3.4}$$

These formulas can be taken as the basis of inductive definition of cup-i product. If we take A be the last vertex of σ, and B be the last vertex of τ in the order α, apply (3.1) if $A<B$, (3.2) if $A>B$, and (3.3) if $A=B$. Since the proof of the theorem is very similar to the algebraic version, we are not going to give the proof here.

Example 3.6.
$A < B$: Given a digital image $X = \{c_0 = (0,0), c_1 = (1,0), c_2 = (1,1)\}$ with 8-adjacency and $c_0 < c_1 < c_2$ ordering; let $\sigma = \langle c_0 c_1\rangle$ and $\tau = \langle c_1 c_2\rangle$ be two digital 1-simplexes with 2-adjacency relation. Here $i = 0$, $A = c_1$, $B = c_2$, and $\sigma \sqcup_0 \tau = \langle c_0 c_1 c_2\rangle$.
$B < A$: Given a digital image $X = \{c_0 = (0,0,0), c_1 = (0,0,1), c_2 = (0,1,0), c_3 = (1,0,0)\}$ with 18-adjacency and $c_0 < c_1 < c_2 < c_3$ ordering; let $\sigma^2 = \langle c_0 c_2 c_3\rangle$ be a digital 2-simplex with 18-adjacency and $\tau^1 = \langle c_1 c_3\rangle + \langle c_2 c_3\rangle$ be a digital 1-simplex with 18-adjacency relation. Here $i = 1$, $A = c_3$, $B = c_2$, and $\sigma \sqcup_1 \tau = \langle c_0 c_1 c_2 c_3\rangle$.
$A = B$: Given a digital image $X = \{c_0 = (0,0), c_1 = (1,0), c_2 = (1,1)\}$ with 8-adjacency and $c_0 < c_1 < c_2$ ordering; let $\sigma^1 = \langle c_0 c_1\rangle$ be a digital 1-simplex with 4-adjacency and $\tau^0 = \langle c_1\rangle$ be a digital 0-simplex. Here $i = 1$, $A = B = c_2$, and $\sigma^1 A \sqcup_1 \tau^0 A = \langle c_0 c_1 c_2\rangle = (\sigma^1 \sqcup_1 \tau^0)A$.

Theorem 3.7. If u and v are p and q-dimensional digital cochains respectively, then

$$\begin{aligned}\delta(u \smile_i v) =& (-1)^{p+q-i} u \smile_{i-1} v + (-1)^{pq+p+q} v \smile_{i-1} u \\ &+ \delta u \smile_i v + (-1)^p u \smile_i \delta v. \end{aligned} \tag{3.5}$$

If u and v are digital cocycles, then the last two terms for $\delta(u \smile_i v)$ become zero. But the first two terms do not have to be zero unless $i = 0$. Thus products of digital cocycles need not be digital cocycles unless $i = 0$.

If $u, v \in Z^{p,\kappa}(X,\mathbb{Z})$ and $w \in C^{p-1,\kappa}(X,\mathbb{Z})$, we get the following statements from the digital coboundary formula (3.5):

$$\delta(u \smile_{i+1} v) = (-1)^{i+1} u \smile_i v + (-1)^p v \smile_i u. \tag{3.6}$$

$$\delta(u \smile_i u) = [(-1)^i + (-1)^p] u \smile_{i-1} u. \tag{3.7}$$

$$\begin{aligned}\delta(w \smile_{i-1} w + w \smile_i \delta w) =& \delta w \smile_i \delta w \\ &- [(-1)^i + (-1)^p](w \smile_{i-2} w + w \smile_{i-1} \delta w).\end{aligned} \tag{3.8}$$

Theorem 3.8. If $p - i$ is odd and $u, v \in Z^{p,\kappa}(X,\mathbb{Z})$, then

$$u \smile_i v + v \smile_i u \sim 0 \tag{3.9}$$
$$\delta(u \smile_i u) = 0 \tag{3.10}$$
$$2u \smile_i u \sim 0 \tag{3.11}$$
$$u \sim 0 \Rightarrow u \smile_i u \sim 0 \tag{3.12}$$
$$u \sim v \Rightarrow u \smile_i u \sim v \smile_i v \tag{3.13}$$
$$(u+v) \smile_i (u+v) \sim u \smile_i u + v \smile_i v. \tag{3.14}$$

Proof.
(3.9) If we use (3.6), we have

$$\delta(u \smile_i v) = (-1)^i u \smile_{i-1} v + (-1)^p v \smile_{i-1} u$$

$$\delta(v \smile_i u) = (-1)^i v \smile_{i-1} u + (-1)^p u \smile_{i-1} v.$$

Hence we conclude that

$$\delta(u \smile_i v + v \smile_i u) = 0 \Rightarrow u \smile_i v + v \smile_i u \sim 0.$$

(3.10) By using (3.7), we have

$$\delta(u \smile_i u) = [(-1)^i + (-1)^p] u \smile_{i-1} u = 0$$

(3.11) If we apply (3.9) with $u = v$, we obtain

$$2\delta(u \smile_i u) = 0 \Rightarrow 2u \smile_i u \sim 0.$$

(3.12) Applying (3.8) with $u = \delta w$, we conclude that

$$\begin{aligned}\delta(w \smile_{i-1} w + w \smile_i \delta w) =& \delta w \smile_i \delta w - [(-1)^i + (-1)^p] \\ &(w \smile_{i-2} w + w \smile_{i-1} \delta w) \\ =& u \smile_i u.\end{aligned}$$

Since $\delta(u \smile_i u) = 0$ from (3.10), we get $u \smile_i u \sim 0$.
(3.13) We know that $u \sim v :\Leftrightarrow u - v \in \delta(x)$. If we apply (3.12) to $u - v$, we have

$$\begin{aligned}(u-v) \smile_i (u-v) =& \delta(x) \smile_i \delta(x) \\ =& (-1)^{2p-i} \delta(x) \smile_{i-1} \delta(x) \\ &+ (-1)^{p^2+2p} \delta(x) \smile_{i-1} \delta(x) \\ &+ \delta(\delta(x)) \smile_i \delta(x) \\ &+ (-1)^p \delta(x) \smile_i \delta(\delta(x)) \\ =& 0.\end{aligned}$$

(3.14) By using bilinearity of $\smile_i$ and apply (3.9), we get

$$(u+v) \smile_i (u+v) = u \smile_i u + v \smile_i v + u \smile_i v + v \smile_i u.$$

Hence

$$(u+v) \smile_i (u+v) \sim u \smile_i u + v \smile_i v. \square$$

Theorem 3.9. If $p - i$ is odd, the operation $u \to u \smile_i u$ maps digital cocycles into digital cocycles, cohomologous digital cocycles into cohomologous digital cocycles, and thus induces the following homomorphism called as i^{th} square

$$Sq_i : H^{p,\kappa}(X;\mathbb{Z}) \to H^{2p-i,\kappa}(X;\mathbb{Z}).$$

Each image under Sq_i has order 2.

Proof. Let $\xi : \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ be the natural homomorphism. Then ξ induces the following homomorphism,

$$\begin{aligned}\xi^* : C^{q,\kappa}(X;\mathbb{Z}) &\to C^{q,\kappa}(X;\mathbb{Z}/2\mathbb{Z}) \\ \sum g'_j \sigma^q_j &\mapsto \xi^*(\sum g'_j \sigma^q_j) = \sum \xi(g'_j)\sigma^q_j.\end{aligned}$$

$\xi^*\delta = \delta\xi^*$ and $f^*\xi^* = \xi^* f^*$ for a digital simplicial map f. The operation ξ^* is called reduction to modulo 2. The relation $u \sim v \ mod\ 2$ means $\xi^* u \sim \xi^* v$.□

Theorem 3.10. If $p - i$ is even and $u, v \in Z^{p,\kappa}(X;\mathbb{Z})$, then the formulas (3.9) to (3.14) all holds mod 2.

The proofs are analogue to those given for $p - i$ is odd.

Theorem 3.11. If $f:(Y,\kappa')\to(X,\kappa)$ is an order preserving digital simplicial map, then $f^*Sq_i=Sq_if^*$ for all i.

Proof. If u is a digital p-cocycle and $p-i$ is odd, then by (3.1) and the properties of f^*, we have

$$\begin{aligned}f^*Sq_i\{u\}&=f^*\{u\smile_i u\}=\{f^*(u\smile_i u)\}=\{f^*u\smile_i f^*u\}\\&=Sq_i\{f^*u\}=Sq_if^*\{u\}.\end{aligned}$$

If $p-i$ is even, we use $f^*\xi^*=\xi^*f^*$ in the proof as follows:

$$\begin{aligned}f^*Sq_i\{u\}&=f^*\{\xi^*(u\smile_i u)\}=\xi^*\{f^*(u\smile_i u)\}\\&=\xi^*\{(f^*u\smile_i f^*u)\}=Sq_i\{f^*u\}=Sq_if^*\{u\}.\Box\end{aligned}$$

We have been worked on the cup products with a fixed order over the digital image (X,κ). Now, our aim is to show that Sq_i is independent of the choice of ordering. To do this, we are going to show that there exists a digital cochain homotopy that indicates products with different orders are equal with this homotopy. We need to consider $X\times I$ space where $I=[0,m]_{\mathbb{Z}}$, and m is a positive integer.

Let (A_0) and (A_1) be two disjoint sets where their vertices are one to one corresponds to vertices of A of the (X,κ). Let $f_0(A)=A_0$ and $f_1(A)=A_1$ where $f_0,f_1:(X,\kappa)\to(X\times I,\kappa')$. The union of (A_0) and (A_1) constitutes the vertices of $(X\times I,\kappa')$. Here, κ' is the adjacency on $X\times I$ which is equal to c_{n_0+1} where we have $c_{n_0}=\kappa$ adjacency on X and $c_1=2$ adjacency on $I=[0,m]_{\mathbb{Z}}$. Let α be the order on (X,κ). If

$$A^0<A^1<\cdots<A^k\le A^{k+1}<\cdots<A^p$$

with respect to order α and these are the vertices of a p or $(p-1)$-digital simplex of (X,κ), a set of the vertices $A_0^0\cdots A_0^kA_1^{k+1}\cdots A_1^p$ are the vertices of $(X\times I,\kappa')$.

Let $f_0,f_1:(X,\kappa)\to(X\times I,\kappa')$ be the digital simplical maps, and $g:(X\times I,\kappa')\to(X,\kappa)$ is defined as $g(A_0)=g(A_1)=A$ for every A where $I=[0,m]_{\mathbb{Z}}$, and m is a positive integer. Then g is a digital simplicial map such that

$$g\circ f_0=g\circ f_1=id_{(X,\kappa)}. \tag{3.15}$$

Let us define $Du\in C^{p-1,\kappa}(X;\mathbb{Z})$ with

$$Du(A^0\cdots A^{p-1})=\sum_{k=0}^{p-1}(-1)^ku(A_0^0\cdots A_0^kA_1^k\cdots A_1^{p-1}) \tag{3.16}$$

where $p>0$, $A^0\cdots A^{p-1}$ is the digital $(p-1)$-simplex in (X,κ) with the order α for $u\in C^{p,\kappa'}(X\times I;\mathbb{Z})$. D is the homomorphism

$$D:C^{p,\kappa'}(X\times I;\mathbb{Z})\to C^{p-1,\kappa}(X;\mathbb{Z}).$$

f_0 and f_1 induce homomorphisms

$$f_0^\sharp,f_1^\sharp:C^{p,\kappa'}(X\times I;\mathbb{Z})\to C^{p,\kappa}(X;\mathbb{Z}).$$

Example 3.12. Let

$$X=\{c_0=(0,0),c_1=(0,1),c_2=(1,1)\}$$

be a digital image in $\mathbb{Z}^2$ with 8-adjacency, and $X\times I=\{p_0=(0,0,0),p_1=(0,0,1),p_2=(0,1,0),p_3=(0,1,1),p_4=(1,1,0),p_5=(1,1,1)\}$ be a digital image in $\mathbb{Z}^3$ with 26-adjacency where $I=[0,1]_{\mathbb{Z}}$. Let $A=\{c_0,c_1,c_2\}$ be the set of vertices of $(X,8)$; let us take

$$\begin{aligned}A_0&=\{p_0,p_2,p_4\}=\{A_0^0,A_0^1,A_0^2\}\text{ and}\\A_1&=\{p_1,p_3,p_5\}=\{A_1^0,A_1^1,A_1^2\}.\end{aligned}$$

$f_0,f_1:(X,8)\to(X\times I,26)$ and $g:(X\times I,26)\to(X,8)$ are digital simplical maps such that $f_0(A)=A_0$, $f_1(A)=A_1$, and $g(A_0)=g(A_1)=A$.

$$\begin{aligned}&D:C^{2,26}(X\times I;\mathbb{Z})\to C^{1,8}(X;\mathbb{Z})\\&\qquad u\mapsto Du(A^0A^1)=\sum_{k=0}^{1}u(A_0^0\cdots A_0^kA_1^k\cdots A_1^1)\\&\qquad\qquad=u(A_0^0A_1^0A_1^1)-u(A_0^0A_0^1A_1^1)\end{aligned}$$

for any $u\in C^{2,26}(X\times I;\mathbb{Z})$ where $u=A_0^0A_0^1A_1^1$.

$$\begin{aligned}\delta Du(A^0A^1A^2)&=\sum_{j=0}^{2}(-1)^jDu(A^0\cdots\widehat{A^j}\cdots A^2)\\&=\sum_{j=0}^{2}(-1)^j[\sum_{k=0}^{j-1}(-1)^ku(A_0^0\cdots A_0^kA_1^k\cdots\widehat{A_1^j}\cdots A_1^2)\\&-\sum_{k=j+1}^{2}(-1)^ku(A_0^0\cdots\widehat{A_0^j}\cdots A_0^kA_1^k\cdots A_1^2)]\\&=u(A_0^1A_1^1A_1^2)-u(A_0^1A_0^2A_1^2)-u(A_0^0A_1^0A_1^2)\\&+u(A_0^0A_0^2A_1^2)+u(A_0^0A_1^0A_1^1)-u(A_0^0A_0^1A_1^1)\end{aligned}$$

$$\begin{aligned}D\delta u(A^0A^1A^2)&=\sum_{k=0}^{2}(-1)^k\delta u(A_0^0\cdots A_0^kA_1^k\cdots A_1^2)\\&=\sum_{k=0}^{2}(-1)^k[\sum_{j=0}^{k}(-1)^ju(A_0^0\cdots\widehat{A_0^j}\cdots A_0^kA_1^k\cdots A_1^2)\\&-\sum_{j=k}^{2}(-1)^ju(A_0^0\cdots A_0^kA_1^k\cdots\widehat{A_1^j}\cdots A_1^2)]\\&=u(A_1^0A_1^1A_1^2)-u(A_0^0A_1^1A_1^2)+u(A_0^0A_0^1A_1^2)\\&-u(A_0^0A_0^0A_1^1)-u(A_0^1A_1^1A_1^2)+u(A_0^0A_1^1A_1^2)\\&-u(A_0^0A_0^1A_1^2)+u(A_0^0A_0^1A_1^1)+u(A_0^1A_0^2A_1^2)\\&-u(A_0^0A_0^2A_1^2)+u(A_0^0A_0^1A_1^2)-u(A_0^0A_0^1A_0^2)\end{aligned}$$

Since

$$u(A_1^0A_1^1A_1^2)-u(A_0^0A_0^1A_0^2)=f_1^*u(A^0A^1A^2)-f_0^*u(A^0A^1A^2),$$

we get $D\delta u=-\delta Du$.

The relations among operations D, $f_0^\sharp$, and $f_1^\sharp$ are

$$Du = f_1^\sharp(u) - f_0^\sharp(u) - D\delta(u)\ ; u \in C^{p,\kappa'}(X\times I;\mathbb{Z}),\ p>0 \quad (3.17)$$

$$0 = f_1^\sharp(u) - f_0^\sharp(u) - D\delta(u)\ ; u \in C^{0,\kappa'}(X\times I;\mathbb{Z}) \quad (3.18)$$

Proof of 3.18.

$$\begin{aligned}\delta Du(A) &= \delta u(A_0A_1) = u(A_1) - u(A_0)\\ &= f_1^\sharp(u)(A) - f_0^\sharp(u)(A).\square\end{aligned}$$

Proof of 3.17. Similar to Example 3.12, $D\delta u = -\delta Du$ on the digital p-simplex $A^0\cdots A^p$. From

$$u(A_1^0\cdots A_1^p) - u(A_0^0\cdots A_0^p) = f_1^\sharp u(A^0\cdots A^p) - f_0^\sharp u(A^0\cdots A^p)$$

and 3.18, we get the result. □

Since $g^\sharp(u)$ is zero on digital simplexes of the (3.16) for any $u\in C^{p,\kappa}(X;\mathbb{Z})$ it follows that

$$Dg^\sharp = 0. \quad (3.19)$$

Let α_0 and α_1 be two orders in (X,κ). Define $X\times I$, f_0, f_1, g with the ordering α_0 as in the product complex. Let

$$g^\sharp : C^{p,\kappa}(X;\mathbb{Z}) \to C^{p,\kappa'}(X\times I;\mathbb{Z})$$

be the digital cochain mapping induced by g. The orders define two cup product $\smile_i^0$, $\smile_i^1$ in (X,κ).

An order (α_0,α_1) is defined in $X\times I$ as follows where $I=[0,m]_{\mathbb{Z}}$: Order (A_0) such that corresponding points in (A) are ordered with α_0, and similarly order (A_1) such that corresponding points in (A) are ordered with α_1. Suppose that a vertex of $(X\times 0,\kappa')$ precedes one of $(X\times m,\kappa')$ on any digital complex in $X\times I$. Then (α_0,α_1) defines cup i product on $(X\times I,\kappa')$. $f_0^\sharp(f_1^\sharp)$ maps $\smile_i$ into $\smile_i^0$ $(\smile_i^1)$ from (3.1) since $f_0(f_1)$ preserves the order $\alpha_0(\alpha_1)$ respectively.

Define a new product on (X,κ) corresponding to α_0 and α_1 as follows:

$$u\vee_i v = D(g^\sharp u \smile_i g^\sharp v)\ ;\ u\in C^{p,\kappa}(X;\mathbb{Z}), v\in C^{q,\kappa}(X;\mathbb{Z}). \quad (3.20)$$

This product is $\vee_i : C^{p,\kappa}(X;\mathbb{Z}) \to C^{p+q-i-1,\kappa}(X;\mathbb{Z})$; $\vee$ is bilinear since D, $g^\sharp$ linear and $\smile$ bilinear. If we apply δ to (3.20), and use (3.17), (3.15), and definition of δ, we get

$$\begin{aligned}\delta(u\vee_i v) =& u\smile_i^1 v - u\smile_i^0 v\\ &-[(-1)^{p+q-i}u\vee_{i-1}v + (-1)^{pq+p+q}v\vee_{i-1}u\\ &+\delta u\vee_i v + (-1)^p u\vee_i \delta v]\end{aligned} \quad (3.21)$$

If $u=v$ is a digital cocycle, then

$$\delta(u\vee_i v) = u\smile_i^1 u - u\smile_i^0 u - [(-1)^i + (-1)^p]u\vee_{i-1}u. \quad (3.22)$$

Theorem 3.13. If the orders α_0,α_1 coincide, then

$$u\vee_i v = 0.$$

Proof. Since $g^\sharp u\smile_i^0 g^\sharp v = g^\sharp(u\smile_i v)$ from (3.1), we have that g is order preserving. If we apply (3.19) to (3.20), we complete the proof.□

Let us consider the relative case. If σ and τ are digital simplexes in $X-A$, then either $\sigma\smile_i\tau$ is zero or a digital simplex of $X-A$. If u and v are zero digital cochains in A, then $u\smile_i v$ is zero. Thus Sq_i can be defined for $H^{p,\kappa}(X,A;\mathbb{Z}_2)$ groups. Hence we get $Sq_i f^* = f^* Sq_i$.

If $w\in C^{p,\kappa}(A;\mathbb{Z}_2)$, we may observe it as an element of $C^{p,\kappa}(X;\mathbb{Z}_2)$ by defining it zero on digital simplexes of $X-A$. Then w has two coboundaries $\delta_A w$ and $\delta_X w$; and

$$\delta_X w = \delta_A w + v$$

where $v\in C^{p+1,\kappa}(X,A;\mathbb{Z}_2)$. If $w\in Z^{p,\kappa}(A;\mathbb{Z}_2)$, $\delta_A w = 0$ so that

$$\delta_X : Z^{p,\kappa}(A;\mathbb{Z}_2) \to Z^{p+1,\kappa}(X,A;\mathbb{Z}_2)$$

homomorphically. Since $0=\delta_X\delta_X w = \delta_X\delta_A w + \delta_X v$ it follows that δ_X maps $B^{p,\kappa}(A;\mathbb{Z}_2)$ to $B^{p+1,\kappa}(X,A;\mathbb{Z}_2)$. Hence δ_X preserves digital cohomology classes and induces a homomorphism

$$\delta^* : H^{p,\kappa}(A;\mathbb{Z}_2) \to H^{p+1,\kappa}(X,A;\mathbb{Z}_2).$$

Because of being $\delta_X f^* = f^*\delta_X$ for a digital simplicial map f, it follows that

$$f^*\delta^* = \delta'^*(f|_B)^*$$

where $\delta'^* : H^{p,\kappa'}(B;\mathbb{Z}_2) \to H^{p+1,\kappa'}(Y,B;\mathbb{Z}_2)$.

Suppose that A contains any digital simplex of X such that vertices in A. If σ and τ are digital simplices on A, thus the product $\sigma\smile_i\tau$ on X and A coincide, and here we use the same ordering with X.

Order the vertices of X such that every vertex of $X-A$ precedes each vertex of A. If $\sigma\in A$ and $\tau\in X-A$, then (σ,τ) is not i-regular since the first vertex of τ is not in σ. Hence if $w\in C^{p,\kappa}(A)$ and $v\in C^{q,\kappa}(X,A)$, then $w\smile_i v = 0$. In particular, if $w\in Z^{p,\kappa}(A)$, then $w\smile_i \delta_X w = 0$. If we apply this to (3.8), we get

$$\delta(w\smile_{i-1} w) = \delta w\smile_i \delta w - [(-1)^i + (-1)^p]w\smile_{i-2} w.$$

And this proves the following statement:

Theorem 3.14. $Sq_i\delta^* = \delta^* Sq_{i-1}$ where $i\geq 1$.

Theorem 3.15. If $i>p$, then $Sq_i\{u^p\} = 0$ where $u\in H^{p,\kappa}(X,A)$.

Proof. Since $u^p\smile_i u^p = 0$ when $i>p$,

$$Sq_i\{u^p\} = \{u^p\smile_i u^p\} = 0.\ \square$$

4 Some Properties of the Steenrod Squares

Now we give some important properties of squarring operation over digital images. The proofs of the following theorems are analogues to algebraic topology (see [22]).

Sq^0 and Sq^1

Let β denote the Bockstein homomorphism attached to the exact coefficient sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}_2 \longrightarrow 0.$$

Then β is a homomorphism

$$\beta : H^*(X,A;\mathbb{Z}_2) \to H^*(X,A;\mathbb{Z}),$$

which raises dimension by one. It is defined on $x \in H^*(X,A;\mathbb{Z}_2)$ as follows: represent the class x by a cocycle c; choose an integral cochain c' which maps to c under reduction mod 2; then $\delta c'$ is divisible by 2 and $\beta x = \frac{1}{2}(\delta c')$ represents βx.

The composition of β and the reduction homomorphism gives a homomorphism

$$\delta_2 : H^{p,\kappa}(X,A;\mathbb{Z}_2) \to H^{p+1,\kappa}(X,A;\mathbb{Z}_2)$$

which we also call "the Bockstein homomorphism"; in fact, it is the Bockstein of the sequence

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2 \longrightarrow 0\ .$$

Definition 4.1. Let us show the homomorphism $Sq^i = Sq_{q-i}$ by

$$Sq^i : H^{q,\kappa}(X;\mathbb{Z}_2) \to H^{q+i,\kappa}(X;\mathbb{Z}_2);\ i = 0,1,\cdots,q.$$

Sq^i is the zero homomorphism for i except $0 \leq i \leq q$.

Lemma 4.2. $\delta_2 Sq^j = \begin{cases} 0, & j \text{ is odd;} \\ Sq^{j+1}, & j \text{ is even.} \end{cases}$

Proof. Given $u \in H^{p,\kappa}(X,A;\mathbb{Z}_2)$, let c be an integral cochain such that the reduction mod 2 of c is in the class u. Then $Sq^j u$ is the class of $(c \smile_{p-j} c)$ by the definition. $\delta c = 2a$ for some integral cochain $a \in C^{p+1,\kappa}(X;A)$. If we write i instead of $(p-j)$, by the coboundary formula

$$\begin{aligned}\delta(c \smile_i c) =&(-1)^{2p-i} c \smile_{i-1} c + (-1)^{p^2+2p} c \smile_{i-1} c \\ &+ \delta c \smile_i c + (-1)^p c \smile_i \delta c \\ =&[(-1)^i + (-1)^p] c \smile_{i-1} c \\ &+ 2a \smile_i c + (-1)^p c \smile_i 2a.\end{aligned}$$

$\delta_2(Sq^j u) = a \smile_i c + c \smile_i a + (s)(c \smile_{i-1} c)$ where the coefficient j is 0 or 1 according to whether j is even or odd, respectively. But the sum of the first two terms is a coboundary, namely,

$$\begin{aligned}\delta(c \smile_{i+1} a) =&(-1)^{2p+1-i} c \smile_i a + (-1)^{p^2+2p} a \smile_i c \\ &+ \delta c \smile_{i+1} a + (-1)^p c \smile_{i+1} \delta a \\ =&a \smile_i c + c \smile_i a\ (mod\ 2)\end{aligned}$$

and the last term represents $(s)Sq^{j+1}u$.
$((s)Sq^{j+1}u \in \{c \smile_{p-j-1} c\};\ p-j = i.)\square$

A special case of the lemma is $\delta_2 Sq^0 = Sq^1$. We want to show that Sq^0 is the identity homomorphism in digital projective plane. Before doing this, let us determine the digital cohomology group of the digital projective plane:

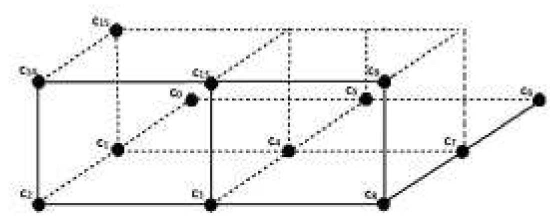

Fig. 3: Digital Projective Plane

Since
- $t = 0$, $H(c,0) = c$
- $t = 1$, $H(c_{12},1) = c_{11}$, $H(c_0,1) = c_5$, $H(c_1,1) = c_4$, $H(c_2,1) = c_3$
- $t = 2$, $H(c_{11},2) = c_{10}$, $H(c_5,2) = c_6$, $H(c_4,2) = c_7$, $H(c_3,2) = c_8$
- $t = 3$, $H(c_{10},3) = c_9$, $H(c_6,3) = c_7$
- $t = 3$, $H(c_9,4) = H(c_7,4) = c_8$

for the digital homotopy map $H : P^{2,6} \times [0,4]_{\mathbb{Z}} \to P^{2,6}$, H is the 6-deformation retract of $P^{2,6}$ [11]. Then P^2 has the same homology group with the one-pointed digital image:

$$H_q^6(P^2;\mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2, & q = 0; \\ 0, & q > 0. \end{cases}$$

By Theorem 2.10, we get

$$\begin{aligned}H^{0,6}(P^2;\mathbb{Z}_2) &\cong Hom(H_0^6(P^2),\mathbb{Z}_2) \oplus Ext(H_{-1}^6(P^2),\mathbb{Z}_2) \\ &\cong Hom(\mathbb{Z}_2,\mathbb{Z}_2) \oplus Ext(0,\mathbb{Z}_2) \\ &\cong \mathbb{Z}_2\end{aligned}$$

when $q = 0$,

$$\begin{aligned}H^{1,6}(P^2;\mathbb{Z}_2) &\cong Hom(H_1^6(P^2),\mathbb{Z}_2) \oplus Ext(H_0^6(P^2),\mathbb{Z}_2) \\ &\cong Hom(0,\mathbb{Z}_2) \oplus Ext(\mathbb{Z}_2,\mathbb{Z}_2) \\ &\cong \mathbb{Z}_2\end{aligned}$$

when $q = 1$, and

$$\begin{aligned}H^{q,6}(P^2;\mathbb{Z}_2) &\cong Hom(H_q^6(P^2),\mathbb{Z}_2) \oplus Ext(H_{q-1}^6(P^2),\mathbb{Z}_2) \\ &\cong Hom(0,\mathbb{Z}_2) \oplus Ext(0,\mathbb{Z}_2) \\ &\cong 0\end{aligned}$$

when $q \geq 2$. Consequently, we have

$$H^{q,6}(P^2;\mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2, & q = 0,1; \\ 0, & q \neq 0,1. \end{cases}$$

By Lemma 4.2,

$$\delta_2(Sq^0(\alpha)) = Sq^1(\alpha) = \alpha \smile \alpha = \alpha^2 \neq 0 \Rightarrow Sq^0(\alpha) \neq 0$$

where α denotes the generator of $H^{1,6}(P^2;\mathbb{Z}_2)$. Since α is the non-zero element of $H^{1,6}(P^2;\mathbb{Z}_2) \cong \mathbb{Z}_2$, it must be $Sq^0(\alpha) = \alpha$. By appliying naturality condition, we get

$$Sq^0\sigma = Sq^0 f^*\alpha = f^*Sq^0\alpha = f^*\alpha = \sigma$$

where if $f : MSC_4 \to P^{2,6}$, then

$$\begin{aligned} f^* : H^*(P^2;\mathbb{Z}_2) &\to H^*(MSC_4;\mathbb{Z}_2) \\ \alpha &\mapsto f^*(\alpha) = \sigma \end{aligned}$$

such that σ is the generator of $H^*(MSC_4;\mathbb{Z}_2)$. Thus being identity homomorphism is true in digital projective plane $P^{2,6}$ for $Sq^0 : H^{q,6}(P^2;\mathbb{Z}_2) \to H^{q,6}(P^2;\mathbb{Z}_2)$.

Cartan Formula

$$Sq^i(x \smile y) = \sum_j Sq^j x \smile Sq^{i-j} y$$

Before proving the Cartan formula, we should better give the following fact.

Proposition 4.3. Let X be a digital image with the κ-adjacency, and

$$\Delta : (X,\kappa) \to (X \times X, \kappa')$$

denote the diagonal map where $\kappa = c_{n_0}$ is the adjacency on X and $\kappa' = c_{n_0+n_0}$ adjacency on $X \times X$. If $x,y \in H^*(X;\mathbb{Z}_2)$, then $x \smile y \in \Delta^*(x \times y)$.

Proof. If $x \in H^{p,\kappa}(X;\mathbb{Z}_2)$, then there exist $u^p = \sum g_i \sigma_i^p \in C^{p,\kappa}(X;\mathbb{Z}_2)$ such that $\overline{u^p} \in \{x\}$. Similarly if $y \in H^{q,\kappa}(X;\mathbb{Z}_2)$, then there exist $v^q = \sum g_j \sigma_j^q \in C^{q,\kappa}(X;\mathbb{Z}_2)$ such that $\overline{v^q} \in \{y\}$. We can write

$$\begin{aligned} x \smile y &= (\sum g_i \sigma_i^p) \smile (\sum g_j \sigma_j^q) \\ &= \sum (g_i g_j) \sigma_i^p \sqcup \sigma_j^q. \end{aligned}$$

If the right side is not a linear $p+q$-simplex, then (u^p, v^q) is not 0-regular. But if the right side is a linear $p+q$-simplex, then (u^p, v^q) is 0-regular and $u^p \smile v^q \in C^{p+q,\kappa}(X,\mathbb{Z}_2)$.

$$\begin{aligned} \Delta^\sharp(u \times v)(A^0,\ldots,A^{p+q}) &= (u \times v)\Delta(A^0,\ldots,A^{p+q}) \\ &= (u^p(\pi_1 \circ \Delta)(A^0,\ldots,A^p) \\ &\quad .(v^q(\pi_2 \circ \Delta)(A^p,\ldots,A^{p+q}) \\ &= (u^p(A^0,\ldots,A^p) \\ &\quad .(v^q(A^p,\ldots,A^{p+q}) \\ &= u^p \smile v^q(A^0,\ldots,A^{p+q}), \end{aligned}$$

here we use $\pi_1 \circ \Delta = id_X = \pi_2 \circ \Delta$. Hence we get $x \smile y \in \Delta^{p+q}(x \times y)$.□

Proof of Cartan Formula. If $x \smile y \in \Delta^*(x \times y)$ for $x,y \in H^*(X;\mathbb{Z}_2)$, then

$$\begin{aligned} Sq^i(x \smile y) &= Sq^i \Delta^*(x \times y) \\ &= \Delta^* Sq^i(x \times y) \\ &= \Delta^* \sum_{j=0}^{i} Sq^j x \times Sq^{i-j} y \\ &= \sum_{j=0}^{i} \Delta^*(Sq^j x \times Sq^{i-j} y) \\ &= \sum_{j=0}^{i} Sq^j x \smile Sq^{i-j} y.\square \end{aligned}$$

Definition 4.4. Let us define $Sq : H^*(X,\mathbb{Z}_2) \to H^*(X,\mathbb{Z}_2)$ with $Sq(u) = \sum Sq^i u$.

The sum given above is finite and this sum does not have to be in $H^{p,\kappa}$ for any p.

Proposition 4.5. Sq is a ring homomorphism.

Proof. By the Cartan formula,

$$Sq(u) \smile Sq(v) = \sum Sq^i u \smile \sum Sq^j v$$

has $Sq^i(u \smile v)$ as its $p+q+i$-dimensional term. Hence $Sq(u \smile v) = Sq(u) \smile Sq(v)$.□

Proposition 4.6. $Sq^i(u^j) = \binom{j}{i} u^{i+j}$ for $u \in H^{1,\kappa}(X;\mathbb{Z}_2)$.

Proof.

If $j = 0$, then $dim\, u^j < i \Rightarrow Sq^i(u^0) = 0 = \binom{0}{i} u^i$.

For $j-1$, let $Sq^i(u^{j-1}) = \binom{j-1}{i} u^{i+j-1}$.

Let's show that the statement is true for j:

$$\begin{aligned} Sq^i(u^j) =& Sq^i(u \smile u^{j-1}) \\ =& \sum_{k=0}^{i} Sq^k(u) \smile Sq^{i-k}(u^{j-1}) \\ =& Sq^0(u) \smile Sq^i(u^{j-1}) + Sq^1 \smile Sq^{i-1}(u^{j-1}) \\ & + Sq^2(u) \smile Sq^{i-2}(u^{j-1}) + \cdots \\ =& Sq^0(u) \smile Sq^i(u^{j-1}) + Sq^1 \smile Sq^{i-1}(u^{j-1}) \\ =& u \smile \binom{j-1}{i} u^{i+j-1} + u^2 \smile \binom{j-1}{i-1} u^{i+j-2} \\ =& \binom{j-1}{i} u^{i+j} + \binom{j-1}{i-1} u^{i+j} \\ =& \left[\binom{j-1}{i} + \binom{j-1}{i-1}\right] u^{i+j} \\ =& \binom{j}{i} u^{i+j}.\square \end{aligned}$$

Adem Relations

The Adem relation has the form

$$R = Sq^a Sq^b + \sum_{c=0}^{[|\frac{a}{2}|]} \binom{b-c-1}{a-2c} Sq^{a+b-c} Sq^c \equiv 0 \ (mod\ 2)$$

where $a < 2b$ and $[|\frac{a}{2}|]$ denotes the greatest integer less or equal to $\frac{a}{2}$.

Example 4.7. $Sq^{2n-1}Sq^n = 0$ for every n.

$$\begin{aligned} n-c-1 \geq 2n-1-2c &\Rightarrow c \geq n \\ &\Rightarrow \binom{n-c-1}{2n-1-2c} = 0;\ \forall c \\ &\Rightarrow Sq^{2n-1}Sq^n = 0. \end{aligned}$$

$$n = 1 \Rightarrow Sq^1 Sq^1 = \sum_{c=0}^{0} \binom{-c}{1-2c} Sq^{2-c} Sq^c = 0$$

$$\begin{aligned} n = 2 \Rightarrow Sq^3 Sq^2 &= \sum_{c=0}^{1} \binom{1-c}{3-2c} Sq^{5-c} Sq^c \\ &= \binom{1}{3} Sq^5 Sq^0 + \binom{0}{1} Sq^4 Sq^1 = 0. \square \end{aligned}$$

Lemma 4.8. Let R be an Adem relation. If $R(y) = 0$ for every class y dimension of p, then $R(z) = 0$ for every class z dimension of $(p-1)$.

Remark [10]: $H^{q,4}(MSC_4;\mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2, & q = 0,1; \\ 0, & q \neq 0,1. \end{cases}$

Proof. Let u be the generator of $H^{1,4}(MSC_4;\mathbb{Z}_2)$. $Sq^i u = 0$ for every $i > 0$:

- If $i = 1$, then

$$\begin{aligned} Sq^1 : H^{1,4}(MSC_4;\mathbb{Z}_2) &\to H^{2,4}(MSC_4;\mathbb{Z}_2) \\ u &\mapsto Sq^1 u = 0 \end{aligned}$$

- If $i > 1$, since $dim\ u = 1$ and $i > dim\ u$, we have $Sq^i u = 0$.

By Cartan formula

$$\begin{aligned} Sq^i(uz) &= \sum_{j=0}^{1} (Sq^j u)(Sq^{i-j} z) \\ &= Sq^0 u Sq^i z + Sq^1 u Sq^{i-1} z \\ &= u Sq^i z + 0 Sq^{i-1} z \\ &= u Sq^i z. \end{aligned}$$

If $dim\ u = 1$ and $dim\ z = p-1$, then

$$dim\ (u \smile_0 z) = dim(uz) = p.$$

Thus $R(uz) = 0$ and

$$\begin{aligned} R(uz) &= Sq^a Sq^b(uz) + \sum_{c=0}^{[|\frac{a}{2}|]} \binom{b-c-1}{a-2c} Sq^{a+b-c} Sq^c(uz) \\ &= u Sq^a Sq^b z + u \sum_{c=0}^{[|\frac{a}{2}|]} \binom{b-c-1}{a-2c} Sq^{a+b-c} Sq^c z \\ &= uR(z) \\ &= 0. \end{aligned}$$

Since $u \neq 0$, we get $R(z) = 0$.$\square$

Lemma 4.9.

$$\binom{p}{q} + \binom{p}{q+1} + \binom{p-1}{q-1} + \binom{p-1}{q+1} \equiv 0 \ (mod\ 2)$$

except the cases $p = q = 0$ and $p = 0$, $q = -1$.

Proof.

$$\begin{aligned} \binom{p-1}{q-1} + \binom{p-1}{q} &= \frac{(p-1)!}{(p-q)!(q-1)!} + \frac{(p-1)!}{(p-q-1)!q!} \\ &= \frac{q(p-1)! + (p-q)(p-1)!}{(p-q)!q!} \\ &= \frac{p(p-1)!}{(p-q)!q!} \\ &= \frac{p!}{(p-q)!q!} = \binom{p}{q} \end{aligned}$$

From this equation, we have

$$\begin{aligned} \binom{p}{q} + \binom{p}{q+1} + \binom{p-1}{q-1} + \binom{p-1}{q+1} &= \binom{p-1}{q-1} + \binom{p-1}{q} + \binom{p-1}{q-1} + \binom{p-1}{q+1} \\ &= 2\binom{p-1}{q-1} + \binom{p-1}{q} + \binom{p}{q+1} + \binom{p-1}{q+1} \\ &= \binom{p}{q+1} + \binom{p}{q+1} \\ &= 2\binom{p}{q+1} \\ &= 0 \ (mod\ 2). \end{aligned}$$

If $p = q = 0$, then

$$\binom{0}{0} + \binom{0}{1} + \binom{-1}{-1} + \binom{-1}{1} = 1+0+0+0 = 1 \ (mod\ 2).$$

If $p = 0, q = -1$, then

$$\binom{0}{-1} + \binom{0}{0} + \binom{-1}{-2} + \binom{-1}{0} = 0+1+0+0 = 1 \ (mod\ 2).$$

$\square$

Lemma 4.10. Let y be a fixed cohomology class such that $R(y) = 0$ for every Adem relation R. Then $R(xy) = 0$ for every one-dimensional cohomology class x and every R.

Proof. Let x be any one-dimensional class and y has the property that $R(y)=0$ for every R.

$$\begin{aligned} dim\,x=1 \Rightarrow Sq^b(xy) &= \sum_{j=0}^{1}(Sq^j x)(Sq^{i-j}y) \\ &= Sq^0 x Sq^b y + Sq^1 x Sq^{b-1} y \\ &= xSq^b y + x^2 Sq^{b-1} y \end{aligned}$$

By using Cartan formula again, we get the formula (1) as follows:

$$\begin{aligned} Sq^a Sq^b(xy) &= Sq^a(xSq^b y + x^2 Sq^{b-1}y) \\ &= Sq^a(xSq^b y) + Sq^a(x^2 Sq^{b-1}y) \\ &= \sum_{k=0}^{1}(Sq^k x)(Sq^{a-k}Sq^b y) \\ &\quad + \sum_{m=0}^{2}(Sq^m x^2)(Sq^{a-m}Sq^{b-1}y) \\ &= Sq^0 x Sq^a Sq^b y + Sq^1 x Sq^{a-1} Sq^b y + \\ &\quad Sq^0 x^2 Sq^a Sq^{b-1} y + Sq^1 x^2 Sq^{a-1} Sq^{b-1} y \\ &\quad + Sq^2 x^2 Sq^{a-2} Sq^{b-1} y \\ &= xSq^a Sq^b y + x^2 Sq^{a-1} Sq^b y + x^2 Sq^a Sq^{b-1} y \\ &\quad + 0 + x^4 Sq^{a-2} Sq^{b-1} y. \end{aligned}$$

Similarly, we get the formula (2) as follows where $s = s(c) = \binom{b-c-1}{a-2c}$:

$$\begin{aligned} \sum(s)Sq^{a+b-c}Sq^c(xy) &= x\sum(s)Sq^{a+b-c}Sq^c y \\ &\quad + x^2 Sq^{a+b-c-1}Sq^c y \\ &\quad + x^2\sum(s)Sq^{a+b-c}Sq^{c-1}y \\ &\quad + x^4\sum(s)Sq^{a+b-c-2}Sq^{c-1}y. \end{aligned}$$

The first terms match in the formulas (1) and (2):

$$\begin{aligned} xSq^a Sq^b y + x\sum(s)Sq^{a+b-c}Sq^c y &= x(Sq^a Sq^b y + \\ &\quad \sum(s)Sq^{a+b-c}Sq^c y) \\ &= xR(y);\ \text{since } R(y)=0 \\ &= 0 \end{aligned}$$

$a<2b$ implies $(a-2)<2(b-1)$, and hence the fourth terms also match: since $R(y)=0$ for every R,

$$R(a-2,b-1)=0.$$

$$\begin{aligned} Sq^{a-2}Sq^{b-1}y &= \sum_c \binom{b-c-2}{a-2-2c} Sq^{a+b-c-3}Sq^c y \\ &= \sum_{c'} \binom{b-c'-1}{a-2c'} Sq^{a+b-c'-2}Sq^{c'-1}y \end{aligned}$$

where $c'=c+1$. Since $R(y)=0$ for every R, by using $R(a-1,b)$ we can change the left-hand side with $Sq^{a-1}Sq^b$ and hence we get

$$\begin{aligned} Sq^a Sq^b y + \sum \binom{b-c-1}{a-2c-1} Sq^{a+b-c}Sq^c y = \\ \sum(s)Sq^{a+b-c-1}Sq^c y + \sum(s)Sq^{a+b-c}Sq^{c-1}. \end{aligned}$$

We have three cases:
Case 1:
$a=2b-2 \Rightarrow a-2c = 2b-2-2c = 2(b-c-1)$.

$$\begin{aligned} k \neq 0 \Rightarrow (s) = \binom{k}{2k} = 0; \\ c \neq b-1 \Rightarrow RHS = Sq^a Sq^{b-1}y + Sq^{a+1}Sq^{b-2}y \end{aligned}$$

$$\begin{aligned} k \neq 1 \Rightarrow \binom{b-c-1}{a-2c-1} = \binom{k}{2k-1} = 0; \\ c \neq b-2 \Rightarrow LHS = Sq^a Sq^{b-1}y + Sq^{a+1}Sq^{b-2}y \end{aligned}$$

So RHS=LHS.

Case 2: The proof is the similar for $a=2b-1$.

Case 3: If $a<2b-2$, then by $R(a,b-1)$

$$Sq^a Sq^{b-1}y = \sum_c \binom{b-c-2}{a-2c} Sq^{a+b-c-1}Sq^c y.$$

Also

$$\begin{aligned} \sum(s)Sq^{a+b-c}Sq^{c-1}y &= \sum_c \binom{b-c-1}{a-2c} Sq^{a+b-c}Sq^{c-1}y \\ &= \sum_{c'} \binom{b-c'-2}{a-2c'-2} Sq^{a+b-c'-1}Sq^{c'}y \end{aligned}$$

where $c'=c-1$.

$$\begin{aligned} \binom{b-c-2}{a-2c} + \binom{b-c-1}{a-2c-1} &\equiv \binom{b-c-1}{a-2c} \\ &\quad + \binom{b-c-2}{a-2c-2}\ (mod\ 2). \square \end{aligned}$$

5 Conclusion

The aim of this paper is to study properties of Steenrod squares on digital images. In order to do this we first define the digital cup product by using the regularity notion. Then we present the properties of the squarring operations such as naturality, identity homomorphism (Sq^0), Bockstein homomorphism (Sq^1), Cartan formula, and Adem relations. We hope that this work will be useful for the researchers studying on image processing.

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