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Oscillatory Behavior of Solutions for Forced Second Order Nonlinear Functional Integro-Dynamic Equations on Time Scales

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Abstract: In this paper, we deal with the oscillatory behavior of forced second order nonlinear functional integro-dynamic equations of the form

$$(r(t)x^{\Delta}(t))^{\Delta} = e(t) \pm p(t)x^{\gamma}(\tau(t)) + \int_0^t k(t,s)f(s,x(\tau(s)))\Delta s,$$

and

$$(r(t)x^{\Delta}(t))^{\Delta} = e(t) + p(t)x(\tau(t)) - \int_0^t k(t,s)f(s,x(\tau(s)))\Delta s,$$

on time scales $\mathbb{T}$, where $r(t)$, $p(t)$ and $e(t)$ are right dense continuous (rd-continuous) functions on $\mathbb{T}$. Oscillation behavior of these equations dose not studied before. Our results improve and extend some results established by Grace et al. [13]. We also give some examples to illustrate our main results.

Keywords: Oscillation, integro-dynamic equations, time scales.
Mathematics Subject Classification (2010): 34N05, 34C10, 45D05.

1 Introduction

In recent years, there has been an increasing interest in studying the oscillation and nonoscillation of dynamic equations on time scales Hilger introduced the theory of time scale which was expected to unify continuous and discrete calculus. We refer the reader to the books [8,9], papers [1-3], [5-7], and the references cited therein.

Research on oscillation theory for integro-dynamic equations is limited due to lack of techniques available on time scales (see [4] and [11-13]).

The main goal of this paper is to establish some new criteria for the oscillatory behavior of forced second order nonlinear functional integro-dynamic equations on time scales $\mathbb{T}$ of the form

$$(r(t)x^{\Delta}(t))^{\Delta} = e(t) \pm p(t)x^{\gamma}(\tau(t)) + \int_0^t k(t,s)f(s,x(\tau(s)))\Delta s, \tag{1.1}$$

and

$$(r(t)x^{\Delta}(t))^{\Delta} = e(t) + p(t)x(\tau(t)) - \int_0^t k(t,s)f(s,x(\tau(s)))\Delta s. \tag{1.2}$$

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2 Some Preliminaries on time scales

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. On any time scale $\mathbb{T}$, the forward and backward jump operators are defined by

$$\sigma(t) = \inf\{s \in \mathbb{T}, s > t\} \quad \text{and} \quad \rho(t) = \sup\{s \in \mathbb{T}, s < t\}.$$

A point $t \in \mathbb{T}$ is said to be left-dense if $\rho(t) = t$, right-dense if $\sigma(t) = t$, left-scattered if $\rho(t) < t$, and right-scattered if $\sigma(t) > t$. The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) = \sigma(t) - t$. The set $\mathbb{T}^k$ is defined by $\mathbb{T}^k = \mathbb{T} - m$ if $\mathbb{T}$ has a left-scattered maximum m, Otherwise, $\mathbb{T}^k = \mathbb{T}$.

A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided that it is continuous at right-dense points of $\mathbb{T}$ and its left-sided limits exist at left-dense points of $\mathbb{T}$. The set of all rd-continuous functions is denoted by $C_{rd}(\mathbb{T},\mathbb{R})$. By $C^1_{rd}(\mathbb{T},\mathbb{R})$, we mean the set of functions whose delta derivative belong to $C_{rd}(\mathbb{T},\mathbb{R})$.

A function $f : \mathbb{T} \to \mathbb{R}$ is regressive provided that

$$1 + \mu(t) f(t) \neq 0 \quad \text{for all} \quad t \in \mathbb{T}^k,$$

holds. The set of all regressive and rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by

$$\mathscr{R} = \mathscr{R}(\mathbb{T}) = \mathscr{R}(\mathbb{T},\mathbb{R}).$$

If $q \in \mathscr{R}$, then we define the exponential function $e_q(t,s)$ by

$$e_q(t,s) = exp(\int_s^t \xi_{\mu(\tau)}(q(\tau))\Delta\tau) \qquad \text{for} \qquad s,t \in \mathbb{T},$$

where the cylinder function $\xi_h(z)$ is defined by

$$\xi_h(z) = \frac{1}{h} Log(1 + zh).$$

For a function $f : \mathbb{T} \to \mathbb{R}$ (the range $\mathbb{R}$ of f may be actually replaced by any Banach space), the delta derivative f^Δ is defined by

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t},$$

provided f is continuous at t and t is right-scattered. If t is not right-scattered, then the delta derivative $f^\Delta(t)$ is defined by

$$f^\Delta(t) = \lim_{s \to t^+} \frac{f(\sigma(t)) - f(t)}{t - s} = \lim_{s \to t^+} \frac{f(t) - f(s)}{t - s}$$

provided that this limit exists.

A function $f : [a,b] \to \mathbb{R}$ is said to be differentiable if its derivative exists. The derivative f^Δ and the shift f^σ of a function f are related by

$$f^\sigma = f(\sigma(t)) = f(t) + \mu(t) f^\Delta(t).$$

The delta derivative rules of the product and the quotient of two differentiable functions f and g are given by

$$(f.g)^\Delta(t) = f^\Delta(t)g(t) + f^\sigma(t)g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g^\sigma(t)$$

$$(\frac{f}{g})^\Delta(t) = \frac{f^\Delta(t)g(t) - f(t)g^\Delta(t)}{g(t)g^\sigma(t)}, \ gg^\sigma \neq 0.$$

The integration by parts formula reads

$$\int_a^b f(t)g^\Delta(t)\Delta t = [f(t)g(t)]_a^b - \int_a^b f^\Delta(t)g^\sigma(t)\Delta t$$

or,

$$\int_a^b f^\sigma(t)g^\Delta(t)\Delta t = [f(t)g(t)]_a^b - \int_a^b f^\Delta(t)g(t)\Delta t$$

and the infinite integral is defined by

$$\int_b^\infty f(s)\Delta s = \lim_{t \to \infty} \int_b^t f(s)\Delta s.$$

3 Basic Lemmas

Lemma 3.1(*[14]*) *If X and Y are nonnegative, then*

$$X^\lambda - (1-\lambda)Y^\lambda - \lambda X Y^{\lambda-1} \leq 0, \ \lambda < 1,$$

and

$$X^\lambda + (\lambda-1)Y^\lambda - \lambda X Y^{\lambda-1} \geq 0, \ \lambda > 1,$$

with equality holding iff $X = Y$.

Lemma 3.2(*[10]*) *Let $y, f \in C_{rd}(\mathbb{T},\mathbb{R})$, $z \in \mathscr{R}^+(\mathbb{T},\mathbb{R})$, $z \geq 0$ and $\alpha \in \mathbb{R}$.*
If

$$y(t) \leq \alpha + \int_{t_0}^t [f(s) \int_{t_0}^s z(\xi)y(\xi)\Delta\xi]\Delta s \text{ for all } t \in \mathbb{T},$$

then

$$y(t) \leq \alpha e_p(t,t_0) \text{ for all } t \in \mathbb{T},$$

where $p(t) = f(t)\int_{t_0}^t z(s)\Delta s$.

4 Main results

In this section, we give some new oscillation criteria for equations (1.1) and (1.2). We begin by introducing the class of functions $\Im$ which will be used in the proof of the first part of this section. Let $D = \{(t,s) \in \mathbb{T} \times \mathbb{T} : t > s \geq t_0\}$, $D_0 = \{(t,s) \in \mathbb{T} \times \mathbb{T} : t \geq s \geq t_0\}$. A function $H \in C_{rd}(D,\mathbb{R})$ belongs to the class $\Im$, if it satiesfies the following conditions:

(C_1) $H(t,t) = 0$, $t \geq t_0$, $H(t,s) > 0$ on D_0,
(C_2) H has a non positive continuous Δ-partial derivative $H^{\Delta_s}(t,s)$ and a non negative continuous second-order Δ-partial derivative $H^{\Delta^2_{s^2}}(t,s)$.
(C_3) $H^{\Delta_s}(t,t) = 0$, $\lim_{t\to\infty} \frac{H^{\Delta_s}(t,t_0)}{H(t,t_0)} = O(1)$.

4.1 Oscillation criteria for Eq. (1.1):

In the following, we establish oscillation criteria for Eq. (1.1) subject to the following conditions:

(H_1) $r, e, p : \mathbb{T} \to \mathbb{R}$ and $k : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ are rd-continuous, $k(t,s) \geq 0$ for $t > s$ and there exist rd-continuous functions $a, m : \mathbb{T} \to (0,\infty)$ such that

$$k(t,s) \leq a(t)m(s) \text{ for all } t \geq s.$$

(H_2) γ is a quotient of odd positive integers such that $0 < \gamma < 1$.

(H_3) $f : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is continuous and there exist rd-continuous function $q : \mathbb{T} \to (0,\infty)$ and real number β with $0 < \beta < 1$ such that

$$0 < xf(t,x) \leq q(t)|x|^{\beta+1} \text{ for } x \neq 0,\ t \geq 0. \tag{4.1}$$

(H_4) $\tau : \mathbb{T} \to \mathbb{T}$ and $\lim_{t\to\infty} \tau(t) = \infty$.

In the following, we denote

$$A(t) = e(t) + (1-\beta)\beta^{\beta/(1-\beta)} a(t) \int_0^t g^{\beta/(\beta-1)}(s) m^{1/(1-\beta)}(s) q^{1/(1-\beta)}(s)\Delta s, \tag{4.2}$$

where $g : [0,\infty)_{\mathbb{T}} \to (0,\infty)$ is a given rd- continuous function.

A solution $x(t)$ of (1.1) or (1.2) is said to be oscillatory if, for every $t_0 > 0$, we have

$$\inf_{t\geq t_0} x(t) < 0 < \sup_{t\geq t_0} x(t).$$

Otherwise, it is said to be nonoscillatory .

Theorem 4.1 *Assume that $\tau(t) \leq t$, H_1-H_4 hold and there exists a kernel function $H(t,s)$ such that*

$$(H^{\Delta_s}(t,s)r(s))^{\Delta_s} \geq 0, \tag{4.3}$$

$$\limsup_{t\to\infty} \frac{1}{H(t,t_1)} \int_{t_1}^t [H(t,\sigma(s))a(s) \int_{t_1}^s \tau(u)g(u)\Delta u]\Delta s < \infty, \tag{4.4}$$

$$\limsup_{t\to\infty} \frac{1}{H(t,t_1)} [\int_{t_1}^t H(t,\sigma(s))[A(s) - k_2 a(s)]\Delta s + \int_{t_1}^{\tau(t)} G(t,s)\Delta s] = \infty, \tag{4.5}$$

$$\liminf_{t\to\infty} \frac{1}{H(t,t_1)} [\int_{t_1}^t H(t,\sigma(s))[A(s) - k_2 a(s)]\Delta s - \int_{t_1}^{\tau(t)} G(t,s)\Delta s] = -\infty, \tag{4.6}$$

where,

$$G(t,s) = [(\gamma-1)\gamma^{\gamma/(1-\gamma)}[(H^{\Delta_s}(t,\sigma^*(s))r(\sigma^*(s)))^{\Delta_s}(\sigma^*(s))^{\Delta}]^{\gamma/(\gamma-1)}(H(t,\sigma(\tau^*(s)))p^*(\tau^*(s))(\tau^*(s))^{\Delta})^{1/(1-\gamma)}],$$

$\sigma^*(s)$ and $\tau^*(s)$ are the inverse functions of $\sigma(s)$ and $\tau(s)$ respectively, and $p^*(t) = \max\{\pm p(t), 0\}$. Then every solution $x(t) = O(t)$ of Eq. (1.1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1). We may assume that $x(t) > 0$, $x(\tau(t)) > 0$ for all $t \geq t_1 > 0$. Using H_1 and H_3 in (1.1), we have

$$\begin{aligned} e(t) &= (r(t)x^{\Delta}(t))^{\Delta} \mp p(t)x^{\gamma}(\tau(t)) \\ &\quad - \int_0^{t_1} k(t,s)f(s,x(\tau(s)))\Delta s \\ &\geq (r(t)x^{\Delta}(t))^{\Delta} - p^*(t)x^{\gamma}(\tau(t)) \\ &\quad - \int_0^{t_1} k(t,s)f(s,x(\tau(s)))\Delta s \\ &\quad - \int_{t_1}^{t} k(t,s)f(s,x(\tau(s)))\Delta s, \end{aligned} \tag{4.7}$$

where $p^*(t) = \max\{\pm p(t), 0\}$. Hence

$$\begin{aligned} e(t) &\geq (r(t)x^{\Delta}(t))^{\Delta} - p^*(t)x^{\gamma}(\tau(t)) \\ &\quad - a(t)\int_0^{t_1} m(s)|f(s,x(\tau(s)))|\Delta s - a(t)\int_{t_1}^{t} m(s)f(s,x(\tau(s)))\Delta s. \end{aligned} \tag{4.8}$$

Setting,

$$k_1 = \max\{|f(t,x(\tau(t)))|,\ t \in [0,t_1]_{\mathbb{T}}\} < \infty \quad \text{and} \quad k_2 = -k_1 \int_0^{t_1} m(s)\Delta s,$$

then,

$$\begin{aligned} e(t) &\geq (r(t)x^{\Delta}(t))^{\Delta} - p^*(t)x^{\gamma}(\tau(t)) + k_2 a(t) \\ &\quad - a(t)\int_{t_1}^{t} m(s)q(s)x^{\beta}(\tau(s))\Delta s \\ &\geq (r(t)x^{\Delta}(t))^{\Delta} - p^*(t)x^{\gamma}(\tau(t)) + k_2 a(t) \\ &\quad + a(t)\int_{t_1}^{t} [g(s)x(\tau(s)) \\ &\quad - m(s)q(s)x^{\beta}(\tau(s))]\Delta s - a(t)\int_{t_1}^{t} g(s)x(\tau(s))\Delta s. \end{aligned} \tag{4.9}$$

Using Lemma 3.1, we get

$$g(s)x(\tau(s)) - m(s)q(s)x^{\beta}(\tau(s)) \geq (\beta-1)\beta^{\beta/(1-\beta)} g^{\beta/(\beta-1)}(s) m^{1/(1-\beta)}(s) q^{1/(1-\beta)}(s). \tag{4.10}$$

Now, From (4.10) in (4.9), we obtain

$$A(t) \geq (r(t)x^{\Delta}(t))^{\Delta} - p^*(t)x^{\gamma}(\tau(t)) + k_2 a(t) - a(t)\int_{t_1}^{t} g(s)x(\tau(s))\Delta s. \tag{4.11}$$

Multiplying (4.11) by $H(t,\sigma(s))$ and integrating from t to t_1, we have

$$\begin{aligned} \int_{t_1}^{t} H(t,\sigma(s))A(s)\Delta s &\geq \int_{t_1}^{t} H(t,\sigma(s))(r(s)x^{\Delta}(s))^{\Delta}\Delta s - \int_{t_1}^{t} H(t,\sigma(s))p^*(s)x^{\gamma}(\tau(s))\Delta s \\ &\quad + k_2\int_{t_1}^{t} H(t,\sigma(s))a(s)\Delta s - \int_{t_1}^{t} [H(t,\sigma(s))a(s)\int_{t_1}^{s} g(u)x(\tau(u))\Delta u]\Delta s. \end{aligned} \tag{4.12}$$

Using integration by parts two times, we have

$$\begin{aligned} \int_{t_1}^{t} H(t,\sigma(s))(r(s)x^{\Delta}(s))^{\Delta}\Delta s &= -H(t,t_1)r(t_1)x^{\Delta}(t_1) - \int_{t_1}^{t} H^{\Delta_s}(t,s)r(s)x^{\Delta}(s)\Delta s \\ &= -H(t,t_1)r(t_1)x^{\Delta}(t_1) + H^{\Delta_s}(t,t_1)r(t_1)x(t_1) \\ &\quad + \int_{t_1}^{t} (H^{\Delta_s}(t,s)r(s))^{\Delta_s}x(\sigma(s))\Delta s \\ &= A(t,t_1) + \int_{t_1}^{t} (H^{\Delta_s}(t,s)r(s))^{\Delta_s}x(\sigma(s))\Delta s, \end{aligned} \tag{4.13}$$

where $A(t,t_1) = -H(t,t_1)r(t_1)x^{\Delta}(t_1) + H^{\Delta_s}(t,t_1)r(t_1)x(t_1)$.

From (4.13) in (4.12), we get

$$\begin{aligned}\int_{t_1}^{t} H(t,\sigma(s))A(s)\Delta s \geq & A(t,t_1)+\int_{t_1}^{t}(H^{\Delta s}(t,s)r(s))^{\Delta s}x(\sigma(s))\Delta s\\ & -\int_{t_1}^{t} H(t,\sigma(s))p^*(s)x^{\gamma}(\tau(s))\Delta s+k_2\int_{t_1}^{t} H(t,\sigma(s))a(s)\Delta s\\ & -\int_{t_1}^{t}[H(t,\sigma(s))a(s)\int_{t_1}^{s} g(u)x(\tau(u))\Delta u]\Delta s\\ & =A(t,t_1)+\int_{\sigma(t_1)}^{\sigma(t)}(H^{\Delta s}(t,\sigma^*(s))r(\sigma^*(s)))^{\Delta s}x(s)(\sigma^*(s))^{\Delta}\Delta s\\ & -\int_{\tau(t_1)}^{\tau(t)} H(t,\sigma(\tau^*(s)))p^*(\tau^*(s))x^{\gamma}(s)(\tau^*(s))^{\Delta}\Delta s\\ & +k_2\int_{t_1}^{t} H(t,\sigma(s))a(s)\Delta s-\int_{t_1}^{t}[H(t,\sigma(s))a(s)\int_{t_1}^{s} g(u)x(\tau(u))\Delta u]\Delta s. \end{aligned} \tag{4.14}$$

Since $\tau(t)\leq t$, then

$$\begin{aligned}\int_{t_1}^{t} H(t,\sigma(s))A(s)\Delta s \geq & A(t,t_1)+\int_{\sigma(t_1)}^{\tau(t)}(H^{\Delta s}(t,\sigma^*(s))r(\sigma^*(s)))^{\Delta s}x(s)(\sigma^*(s))^{\Delta}\Delta s\\ & -\int_{\tau(t_1)}^{\tau(t)} H(t,\sigma(\tau^*(s)))p^*(\tau^*(s))x^{\gamma}(s)(\tau^*(s))^{\Delta}\Delta s\\ & +k_2\int_{t_1}^{t} H(t,\sigma(s))a(s)\Delta s-\int_{t_1}^{t}[H(t,\sigma(s))a(s)\int_{t_1}^{s} g(u)x(\tau(u))\Delta u]\Delta s\\ & \geq B(t,t_1)+k_2\int_{t_1}^{t} H(t,\sigma(s))a(s)\Delta s\\ & -\int_{t_1}^{t}[H(t,\sigma(s))a(s)\int_{t_1}^{s} g(u)x(\tau(u))\Delta u]\Delta s\\ & +\int_{t_1}^{\tau(t)}[(H^{\Delta s}(t,\sigma^*(s))r(\sigma^*(s)))^{\Delta s}(\sigma^*(s))^{\Delta}x(s)\\ & -H(t,\sigma(\tau^*(s)))p^*(\tau^*(s))(\tau^*(s))^{\Delta}x^{\gamma}(s)]\Delta s, \end{aligned} \tag{4.15}$$

where $B(t,t_1) = A(t,t_1)+\int_{\sigma(t_1)}^{t_1}(H^{\Delta_s}(t,\sigma^*(s))r(\sigma^*(s)))^{\Delta_s}x(s)(\sigma^*(s))^{\Delta}\Delta s$ $-\int_{\tau(t_1)}^{t_1} H(t,\sigma(\tau^*(s)))p^*(\tau^*(s))x^{\gamma}(s)(\tau^*(s))^{\Delta}\Delta s.$
Applying Lemma 3.1, we get

$$\begin{aligned}&(H^{\Delta s}(t,\sigma^*(s))r(\sigma^*(s)))^{\Delta s}(\sigma^*(s))^{\Delta}x(s)-H(t,\sigma(\tau^*(s)))p^*(\tau^*(s))(\tau^*(s))^{\Delta}x^{\gamma}(s)\\ &\geq(\gamma-1)\gamma^{\gamma/(1-\gamma)}[(H^{\Delta s}(t,\sigma^*(s))r(\sigma^*(s)))^{\Delta s}(\sigma^*(s))^{\Delta}]^{\gamma/(\gamma-1)}\\ &(H(t,\sigma(\tau^*(s)))p^*(\tau^*(s))(\tau^*(s))^{\Delta})^{1/(1-\gamma)}\\ &=G(t,s)\end{aligned}$$

Hence, (4.15) becomes

$$\begin{aligned}\int_{t_1}^{t} H(t,\sigma(s))A(s)\Delta s \geq & B(t,t_1)+\int_{t_1}^{\tau(t)} G(t,s)\Delta s+k_2\int_{t_1}^{t} H(t,\sigma(s))a(s)\Delta s\\ & -\int_{t_1}^{t}[H(t,\sigma(s))a(s)\int_{t_1}^{s}\tau(u)g(u)\frac{x(\tau(u))}{\tau(u)}\Delta u]\Delta s.\end{aligned}$$

$$\int_{t_1}^{t} H(t,\sigma(s))[A(s)-k_2a(s)]\Delta s-\int_{t_1}^{\tau(t)} G(t,s)\Delta s \tag{4.16}$$

$$\geq B(t,t_1)-\int_{t_1}^{t}[H(t,\sigma(s))a(s)\int_{t_1}^{s}\tau(u)g(u)\frac{x(\tau(u))}{\tau(u)}\Delta u]\Delta s. \tag{4.17}$$

Multiplying (4.16) by $H^{-1}(t,t_1)$, using (4.4), and taking the lower limit of (4.16), we get a contradiction with(4.6). This completes the proof.

Theorem 4.2*Assume that $\tau(t)\geq t$, H_1-H_4 hold and there exists a kernel function $H(t,s)$ such that*

$$(H^{\Delta_s}(t,s)r(s))^{\Delta_s}\geq 0, \tag{4.18}$$

$$\limsup_{t\to\infty}\frac{1}{H(t,t_1)}\int_{t_1}^{t}[H(t,\sigma(s))a(s)\int_{t_1}^{s}\tau(u)g(u)\Delta u]\Delta<\infty, \tag{4.19}$$

$$\limsup_{t\to\infty}\frac{1}{H(t,t_1)}\int_{t}^{\tau(t)} H(t,\sigma(\tau^*(s)))s^{\gamma}p^*(\tau^*(s))(\tau^*(s))^{\Delta}\Delta s<\infty \tag{4.20}$$

$$\limsup_{t\to\infty}\frac{1}{H(t,t_1)}[\int_{t_1}^{t} H(t,\sigma(s))[A(s)-k_2a(s)]\Delta s+\int_{\tau(t_1)}^{t} G(t,s)\Delta s]=\infty, \tag{4.21}$$

$$\liminf_{t\to\infty}\frac{1}{H(t,t_1)}[\int_{t_1}^{t} H(t,\sigma(s))[A(s)-k_2a(s)]\Delta s-\int_{\tau(t_1)}^{t} G(t,s)\Delta s]=-\infty, \tag{4.22}$$

where,
$G(t,s)=[(\gamma-1)\gamma^{\gamma/(1-\gamma)}[(H^{\Delta s}(t,\sigma^*(s))r(\sigma^*(s)))^{\Delta s}(\sigma^*(s))^{\Delta}]^{\gamma/(\gamma-1)}$
$(H(t,\sigma(\tau^*(s)))p^*(\tau^*(s))(\tau^*(s))^{\Delta})^{1/(1-\gamma)}]$, $\sigma^*(s)$ *and* $\tau^*(s)$ *are the inverse functions of* $\sigma(s)$ *and* $\tau(s)$ *respectively, and* $p^*(t)=\max\{\pm p(t),0\}$. *Then every solution* $x(t)=O(t)$ *of Eq. (1.1) is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1). We may assume that $x(t)>0$, $x(\tau(t))>0$ for all $t\geq t_1$¿0. Proceeding as in the proof of Theorem 4.1 to get (4.14)

$$\begin{aligned}\int_{t_1}^{t} H(t,\sigma(s))A(s)\Delta s \geq & A(t,t_1)+\int_{\sigma(t_1)}^{\sigma(t)}(H^{\Delta s}(t,\sigma^*(s))r(\sigma^*(s)))^{\Delta s}x(s)(\sigma^*(s))^{\Delta}\Delta s\\ & -\int_{\tau(t_1)}^{\tau(t)} H(t,\sigma(\tau^*(s)))p^*(\tau^*(s))x^{\gamma}(s)(\tau^*(s))^{\Delta}\Delta s\\ & +k_2\int_{t_1}^{t} H(t,\sigma(s))a(s)\Delta s\\ & -\int_{t_1}^{t}[H(t,\sigma(s))a(s)\int_{t_1}^{s} g(u)x(\tau(u))\Delta u]\Delta s.\end{aligned}$$

since $\tau(t)\geq t$, then

$$\begin{aligned}\int_{t_1}^{t} H(t,\sigma(s))A(s)\Delta s \geq & A(t,t_1)+\int_{\sigma(t_1)}^{t}(H^{\Delta s}(t,\sigma^*(s))r(\sigma^*(s)))^{\Delta s}x(s)(\sigma^*(s))^{\Delta}\Delta s\\ & -\int_{\tau(t_1)}^{t} H(t,\sigma(\tau^*(s)))p^*(\tau^*(s))x^{\gamma}(s)(\tau^*(s))^{\Delta}\Delta s\\ & -\int_{t}^{\tau(t)} H(t,\sigma(\tau^*(s)))p^*(\tau^*(s))x^{\gamma}(s)(\tau^*(s))^{\Delta}\Delta s\\ & +k_2\int_{t_1}^{t} H(t,\sigma(s))a(s)\Delta s-\int_{t_1}^{t}[H(t,\sigma(s))a(s)\int_{t_1}^{s} g(u)x(\tau(u))\Delta u]\Delta s.\end{aligned}$$

Hence,

$$\begin{aligned}&\int_{t_1}^{t} H(t,\sigma(s))A(s)\Delta s\geq B(t,t_1)-\int_{t}^{\tau(t)} H(t,\sigma(\tau^*(s)))p^*(\tau^*(s))x^{\gamma}(s)(\tau^*(s))^{\Delta}\Delta s\\ &+\int_{\tau(t_1)}^{t}[(H^{\Delta_s}(t,\sigma^*(s))r(\sigma^*(s)))^{\Delta_s}x(s)(\sigma^*(s))^{\Delta}\\ &-H(t,\sigma(\tau^*(s)))p^*(\tau^*(s))x^{\gamma}(s)(\tau^*(s))^{\Delta}]\Delta s\\ &+k_2\int_{t_1}^{t} H(t,\sigma(s))a(s)\Delta s-\int_{t_1}^{t}[H(t,\sigma(s))a(s)\int_{t_1}^{s} g(u)x(\tau(u))\Delta u]\Delta s,\end{aligned}$$

where,

$B(t,t_1) = A(t,t_1)+\int_{\sigma(t_1)}^{\tau(t_1)}(H^{\Delta_s}(t,\sigma^*(s))r(\sigma^*(s)))^{\Delta_s}x(s)(\sigma^*(s))^{\Delta}\Delta s.$
Therefore,

$$\begin{aligned}&\int_{t_1}^{t} H(t,\sigma(s))[A(s)-k_2a(s)]\Delta s-\int_{\tau(t_1)}^{t} G(t,s)\Delta s\geq B(t,t_1)\\ &-\int_{t_1}^{t}[H(t,\sigma(s))a(s)\int_{t_1}^{s}\tau(u)g(u)\frac{x(\tau(u))}{\tau(u)}\Delta u]\Delta s\\ &-\int_{t}^{\tau(t)} H(t,\sigma(\tau^*(s)))s^{\gamma}p^*(\tau^*(s))(\frac{x(s)}{s})^{\gamma}(\tau^*(s))^{\Delta}\Delta s,\end{aligned} \tag{4.23}$$

Multiplying (4.23) by $H^{-1}(t,t_1)$, using (4.19), (4.20) and taking the lower limit of (4.23), we get a contradiction with(4.22). This completes the proof.

Example 41*Consider the integro-differential equation ($\mathbb{T} = \mathbb{R}$)*

$$x''(t) = t^3 \pm t^2 \sin t x^\gamma(t) + \int_0^t \frac{x^\beta(s)}{(t^4+1)(s^8+1)} \Delta s, \qquad t \geq 0, \tag{4.24}$$

where, $0 < \gamma, \beta < 1$. Here, $r(t) = 1$, $e(t) = t^3$, $p(t) = t^2 \sin t$, $k(t,s) = \frac{1}{(t^4+1)(s^8+1)}$, $a(t) = \frac{1}{t^4}$, $m(s) = \frac{1}{s^8}$, $f(x) = x^\beta$, $\tau(t) = t$. *To apply Theorem 4.1, let* $g(t) = m(t)$ *and* $H(t,s) = t - s$. *Therefore*

$$(H'(t,s)r(s))' = 0,$$

and

$$\begin{aligned}\int_{t_1}^t H(t,s)a(s)[\int_{t_1}^s \tau(u)g(u)du]ds &= \int_{t_1}^t H(t,s)a(s)[\int_{t_1}^s \frac{1}{u^7}du]ds \\ &= \int_{t_1}^t H(t,s)a(s)[\frac{-1}{6}[\frac{1}{s^5} - \frac{1}{t_1^5}]ds \\ &= \frac{-1}{6}\int_{t_1}^t (t-s)\frac{1}{s^4}[\frac{1}{s^5} - \frac{1}{t_1^5}]ds.\end{aligned}$$

Hence,

$$\limsup_{t\to\infty} \frac{1}{H(t,t_1)} \int_{t_1}^t H(t,s)a(s)[\int_{t_1}^s \tau(u)g(u)du]ds < \infty,$$

$$\int_{t_1}^{\tau(t)} G(t,s)ds = 0,$$

and

$$\begin{aligned}&\int_{t_1}^t H(t,s)[A(s) - k_2 a(s)]ds \\ &= \int_{t_1}^t (t-s)[s^3 - \frac{1}{7}(1-\beta)^{\beta/(1-\beta)}[\frac{1}{s^{11}} - \frac{1}{t_1^7 s^4}] - k_2\frac{1}{s^4}]ds \\ &\to \infty \qquad as\ t \to \infty.\end{aligned}$$

Therefore, Eq. (4.24) is oscillatory.

4.2 *Oscillation criteria for Eq. (1.2):*

Here, we establish oscillation criteria for Eq. (1.2) subject to the following conditions:

$(M_1) r, e, p : \mathbb{T} \to \mathbb{R}$ and $k : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ are rd-continuous, $r(t) > 0$, $k(t,s) \geq 0$ for $t > s$ and there exist rd-continuous functions $b, n : \mathbb{T} \to (0,\infty)$ such that

$$k(t,s) \geq b(t)n(s) \text{ for all } t \geq s.$$

$(M_2) f : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is continuous and there exist rd-continuous function $u : \mathbb{T} \to (0,\infty)$ and a real number λ with $\lambda > 1$ such that

$$xf(t,x) \geq u(t)|x|^{\lambda+1} \text{ for } x \neq 0,\ t \geq 0. \tag{4.25}$$

$(M_3) \tau : \mathbb{T} \to \mathbb{T}$ with $\tau(t) \leq t$ and $\lim_{t\to\infty} \tau(t) = \infty$.

In the following, we denote

$$h_\pm(t) = e(t) \pm (\lambda-1)\lambda^{\lambda/(1-\lambda)} b(t) \int_0^t v^{\lambda/(\lambda-1)}(s) n^{1/(1-\lambda)}(s) u^{1/(1-\lambda)}(s) \Delta s, \tag{4.26}$$

where $v \in C_{rd}(\mathbb{T}, (o,\infty))$.

Now, we give sufficient conditions under which a nonoscillatory solutions $x(t)$ of (1.2) satisfying

$$x(t) = O(1) \qquad t \to \infty.$$

Theorem 4.3*Let $\lambda > 1$, $M_1 - M_3$ hold for all $t_0 > 0$ such that*

$$\int_{t_0}^\infty \frac{1}{r(s)} \Delta s < \infty. \tag{4.27}$$

$$\int_{t_0}^\infty [\frac{1}{r(s)} \int_{t_0}^s b(\xi)\Delta\xi]\Delta s < \infty, \tag{4.28}$$

$$\int_{t_0}^\infty \frac{1}{r(s)} [\int_{\tau(t_1)}^{\tau(s)} p^*(\tau^*(\xi))(\tau^*(\xi))^\Delta \Delta\xi]\Delta s < \infty. \tag{4.29}$$

If,

$$\begin{aligned}&\limsup_{t\to\infty} \int_{t_0}^t \frac{1}{r(s)}[\int_{t_0}^s h_+(\xi)\Delta\xi]\Delta s < \infty \\ &\liminf_{t\to\infty} \int_{t_0}^t \frac{1}{r(s)}[\int_{t_0}^s h_-(\xi)\Delta\xi]\Delta s > -\infty,\end{aligned} \tag{4.30}$$

then every nonoscillatory solution $x(t)$ of (1.2) satisfies

$$\limsup_{t\to\infty} x(t) < \infty. \tag{4.31}$$

Proof. Let $x(t)$ be a nonoscillatory solution of (1.2). We may assume that $x(t) > 0$, $x(\tau(t)) > 0$ for all $t \geq t_1$, for some $t_1 > 0$. Using M_1 and M_2 in (1.2), we have

$$\begin{aligned}(r(t)x^\Delta(t))^\Delta &= e(t) + p(t)x(\tau(t)) \\ &\quad - \int_0^{t_1} k(t,s)f(s,x(\tau(s)))\Delta s \\ &\quad - \int_{t_1}^t k(t,s)f(s,x(\tau(s)))\Delta s \\ &\leq e(t) + p(t)x(\tau(t)) - b(t)c_1\int_0^{t_1} n(s)\Delta s \\ &\quad - b(t)\int_{t_1}^t n(s)u(s)x^\lambda(\tau(s))\Delta s \\ &\leq e(t) + p^*(t)x(\tau(t)) + c_2 b(t) \\ &\quad - b(t)\int_{t_1}^t n(s)u(s)x^\lambda(\tau(s))\Delta s,\end{aligned} \tag{4.32}$$

where

$$c_1 := \min\{f(t,x(\tau(t))) : t \in [0,t_1]_\mathbb{T}\} \leq 0,$$

$$c_2 = -c_1\int_0^{t_1} n(s)\Delta s \geq 0 \text{ and } p^*(t) = \max\{0, p(t)\}.$$

Hence,

$$
\begin{aligned}
(r(t)x^{\Delta}(t))^{\Delta} &\leq e(t) + p^{*}(t)x(\tau(t)) + c_2 b(t) \\
&\quad - b(t)\int_{t_1}^{t} n(s)u(s)x^{\lambda}(\tau(s))\Delta s \\
&\leq e(t) + p^{(}*t)x(\tau(t)) + c_2 b(t) \\
&\quad + b(t)\int_{t_1}^{t} [v(s)x(\tau(s)) - n(s)u(s)x^{\lambda}(\tau(s))]\Delta s,
\end{aligned} \tag{4.33}
$$

Applying Lemma 3.1 to $v(s)x(\tau(s)) - n(s)u(s)x^{\lambda}(\tau(s))$ with

$$X = (nu)^{\frac{1}{\lambda}}x, \quad \text{and} \quad Y = (\tfrac{1}{\lambda}v(nu)^{-\frac{1}{\lambda}})^{\frac{1}{\lambda-1}},$$

we obtain

$$
\begin{aligned}
& v(s)x(\tau(s)) - n(s)u(s)x^{\lambda}(\tau(s)) \leq \\
& (\lambda-1)\lambda^{\lambda/(1-\lambda)}v^{\lambda/(\lambda-1)}(s)n^{1/(1-\lambda)}(s)u^{1/(1-\lambda)}(s).
\end{aligned}
$$

Therefore,

$$(r(t)x^{\Delta}(t))^{\Delta} \leq h_{+}(t) + p^{*}(t)x(\tau(t)) + c_2 b(t). \tag{4.34}$$

Integrating (4.34) from t_1 to t, we have

$$
\begin{aligned}
r(t)x^{\Delta}(t) &\leq r(t_1)x^{\Delta}(t_1) + \int_{t_1}^{t} h_{+}(s)\Delta s \\
&\quad + \int_{t_1}^{t} p^{*}(s)x(\tau(s))\Delta s + c_2\int_{t_1}^{t} b(s)\Delta s.
\end{aligned} \tag{4.35}
$$

Therefore,

$$
\begin{aligned}
x^{\Delta}(t) &\leq \frac{r(t_1)x^{\Delta}(t_1)}{r(t)} + \frac{1}{r(t)}\int_{t_1}^{t} h_{+}(s)\Delta s \\
&\quad + \frac{1}{r(t)}\int_{t_1}^{t} p^{*}(s)x(\tau(s))\Delta s + \frac{c_2}{r(t)}\int_{t_1}^{t} b(s)\Delta s \\
&= \frac{r(t_1)x^{\Delta}(t_1)}{r(t)} + \frac{1}{r(t)}\int_{t_1}^{t} h_{+}(s)\Delta s \\
&\quad + \frac{1}{r(t)}\int_{\tau(t_1)}^{\tau(t)} p^{*}(\tau^{*}(s))x(s)(\tau^{*}(s))^{\Delta}\Delta s + \frac{c_2}{r(t)}\int_{t_1}^{t} b(s)\Delta s.
\end{aligned} \tag{4.36}
$$

Integrating from t_1 to t, we get

$$
\begin{aligned}
x(t) &\leq x(t_1) + r(t_1)x^{\Delta}(t_1)\int_{t_1}^{t}\frac{\Delta s}{r(s)} \\
&\quad + \int_{t_1}^{t}[\frac{1}{r(s)}\int_{t_1}^{s} h_{+}(\xi)\Delta\xi]\Delta s + c_2\int_{t_1}^{t}[\frac{1}{r(s)}\int_{t_1}^{s} b(\xi)\Delta\xi]\Delta s \\
&\quad + \int_{t_1}^{t}\frac{1}{r(s)}[\int_{\tau(t_1)}^{\tau(s)} p^{*}(\tau^{*}(\xi))x(\xi)(\tau^{*}(\xi))^{\Delta}\Delta\xi]\Delta s.
\end{aligned} \tag{4.37}
$$

Hence,

$$x(t) \leq K + \int_{t_1}^{t}\frac{1}{r(s)}[\int_{\tau(t_1)}^{\tau(s)} p^{*}(\tau^{*}(\xi))x(\xi)(\tau^{*}(\xi))^{\Delta}\Delta\xi]\Delta s, \tag{4.38}$$

where K is an upper bound for the expression

$$
\begin{aligned}
& x(t_1) + r(t_1)x^{\Delta}(t_1)\int_{t_1}^{t}\frac{\Delta s}{r(s)} + \int_{t_1}^{t}\frac{1}{r(s)}\int_{t_1}^{s} h_{+}(\xi)\Delta\xi\Delta s + \\
& c_2\int_{t_1}^{t}[\frac{1}{r(s)}\int_{t_1}^{s} b(\xi)\Delta\xi]\Delta s,
\end{aligned}
$$

for $t \geq t_1$. Applying Lemma 3.2 to inequality (4.38) and then using condition (4.29), we get

$$\limsup_{t\to\infty} x(t) < \infty. \tag{4.39}$$

If $x(t)$ is eventually negative, we can set $y = -x$. Hence, y satisfies Eq. (1.2) with $e(t)$ replaced by $-e(t)$ and $f(t,x)$ by $-f(t,-y)$. In a similar way, we get

$$\limsup_{t\to\infty}(-x(t)) < \infty. \tag{4.40}$$

From (4.39) and (4.40), we conclude that (4.31) holds.

Theorem 4.4*Let $\lambda > 1$, $M_1 - M_3$, (4.27), (4.28), (4.29) and (4.30) hold. If*

$$
\begin{aligned}
\limsup_{t\to\infty}\int_{t_0}^{t}\frac{1}{r(s)}[\int_{t_0}^{s} h_{-}(\xi)\Delta\xi]\Delta s &= \infty, \\
\liminf_{t\to\infty}\int_{t_0}^{t}\frac{1}{r(s)}[\int_{t_0}^{s} h_{+}(\xi)\Delta\xi]\Delta s &= -\infty,
\end{aligned}
$$

for all $t_0 \geq 0$, then every solution of (1.2) is oscillatory.

Proof. Assume that (1.2) is nonoscillatory on $[t_0,\infty)_{\mathbb{T}}$. Then there is a solution x of (1.2) and a point $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t)$ and $x(\tau(t))$ are of the same sign on $[t_1,\infty)_{\mathbb{T}}$. Consider the case $x(t)$ and $x(\tau(t))$ are positive on $[t_1,\infty)_{\mathbb{T}}$. The proof when x is eventually negative is similar. Proceeding as in the proof of Theorem 4.3, we get

$$
\begin{aligned}
x(t) &\leq x(t_1) + r(t_1)x^{\Delta}(t_1)\int_{t_1}^{t}\frac{\Delta s}{r(s)} \\
&\quad + \int_{t_1}^{t}\frac{1}{r(s)}\int_{t_1}^{s} h_{+}(\xi)\Delta\xi\Delta s + c_2\int_{t_1}^{t}[\frac{1}{r(s)}\int_{t_1}^{s} b(\xi)\Delta\xi]\Delta s \\
&\quad + \int_{t_1}^{t}\frac{1}{r(s)}[\int_{\tau(t_1)}^{\tau(s)} p^{*}(\tau^{*}(\xi))x(\xi)(\tau^{*}(\xi))^{\Delta}\Delta\xi]\Delta s.
\end{aligned}
$$

Clearly, the conclusion of Theorem 4.3 holds. Hence, the second and the last two integrals in the above inequality are bounded. Finally, taking liminf as $t \to \infty$ and using (**??**), we get a contradiction with the fact that $x(t)$ is eventually positive. This contradiction completes the proof.

Theorem 4.5*Let $\lambda = 1$, $M_1 - M_3$, (4.27), (4.28), and (4.29) hold. If*

$$\int_{t_0}^{\infty}\frac{1}{r(s)}[\int_{t_0}^{s} b(u)[\int_{\tau(t_0)}^{\tau(u)} n(\tau^{*}(\xi))u(\tau^{*}(\xi))(\tau^{*}(\xi))^{\Delta}\Delta\xi]\Delta\xi]\Delta s < \infty, \tag{4.41}$$

and,

$$
\begin{aligned}
\limsup_{t\to\infty}\int_{t_0}^{t}\frac{1}{r(s)}[\int_{t_0}^{s} e(\xi)\Delta\xi]\Delta s &= \infty, \\
\liminf_{t\to\infty}\int_{t_0}^{t}\frac{1}{r(s)}[\int_{t_0}^{s} e(\xi)\Delta\xi]\Delta s &= -\infty,
\end{aligned}
$$

for all $t_0 \geq 0$, then every solution of (1.2) is oscillatory.

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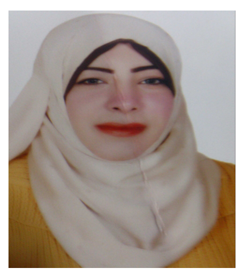

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