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Dynamics of a Plant-Herbivore Model with Fractional Order

Moustafa El-Shahed[1,*], *A. M. Ahmed*[2] *and Ibrahim M. E. Abdelstar*[2,3]

[1] Department of Mathematics, Faculty of Arts and Sciences Qassim University, P.O. Box 3771, Qassim, Unizah 51911, Saudi Arabia.
[2] Department of Mathematics, Faculty of Science, Al-Azhar University, Nasr City, P.O.Box: 11884, Cairo, Egypt.
[3] Quantitative methods unit, Faculty of Business & Economics Qassim University, P.O.Box: 6633, Qassim, Buridah 51452, Saudi Arabia.

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Abstract: This paper studies the fractional order model of a plant-herbivore. For this model, the stability of three fixed points are analyzed. To solve and simulate the system of differential equations we utilized the Adams-Bashforth-Moulton algorithm.

Keywords: Fractional derivative, plant-herbivore, stability, Adams-Bashforth-Moulton method.

1 Introduction

During the past decades, several mathematical models have been investigated to model plant-herbivore interactions [1,2,3,4,5,6,7,8]. These models are based on prey-predator system. The understanding of the relationships between herbivores and plants are extremely important for land management. Fractional differential equations has been an active field of research currently due to their applications in many areas of life [9,10,11,12,13,14,15]. In the present paper, we consider fractional order to model plant-herbivore interactions. The paper is organized in the following manner. Section 2 introduces a plant-herbivore model with fractional order and discusses the boundeness of the solutions of the plant-herbivore fractional order model. Sections 3 discuss the stability of the equilibrium points of the model. Section 4 simulates the dynamics of the system of plant-herbivore fractional order model using generalized predictor corrector algorithm. Section 5 summarizes the results obtained in this paper.

2 Model Formulation

The plant-herbivore model can be written as follows [15]:

$$\begin{aligned} \frac{dx}{dt} &= x(q-x) - \frac{\beta x^2 y}{1+x^2}, \\ \frac{dy}{dt} &= \frac{\beta_1 x^2 y}{1+x^2} - \gamma y, \end{aligned} \tag{1}$$

where α, β, β_1 and γ are real positive constants. Recently, mathematical models with fractional order are become suitable than models with integer order as fractional order models allow more degrees of freedom and due to the existence of memory effects, see [9,10]. The plant-herbivore interactions are described by the following system of nonlinear fractional ordinary differential equations:

$$\begin{aligned} D_t^{\alpha} x &= x(q-x) - \frac{\beta x^2 y}{1+x^2}, \\ D_t^{\alpha} y &= \frac{\beta_1 x^2 y}{1+x^2} - \gamma y, \end{aligned} \tag{2}$$

where D_t^{α} is the Caputo fractional derivative.

* Corresponding author e-mail: elshahedm@yahoo.com

Lemma 1. *The solutions of the plant-herbivore model are uniformly bounded* .

Let $(x(t), y(t))$ be the solution of the plant-herbivore fractional order model (2). Since

$$D_t^\alpha x = x(q-x) - \frac{\beta x^2 y}{1+x^2} \leq x(q-x), \tag{3}$$

then

$$x \leq q \text{ for } t \to \infty. \tag{4}$$

Let

$$W = x + \frac{\beta}{\beta_1} y, \tag{5}$$

then

$$D_t^\alpha W = D_t^\alpha x + \frac{\beta}{\beta_1} D_t^\alpha y = x(q-x) - \frac{\beta\gamma}{\beta_1} y = x(q-x) + \gamma x - \gamma\left(x + \frac{\beta}{\beta_1} y\right). \tag{6}$$

Now, the maximum value of $x(q-x)$ is $\frac{q^2}{4}$ since $0 \leq x \leq q$, then

$$D_t^\alpha W \leq \gamma q + \frac{q^2}{4} - \gamma W = L - \gamma W \tag{7}$$

where $L = \gamma q + \frac{q^2}{4}$. By Lemma 9 [16] , we have

$$0 \leq W(x,y) \leq W(x(0), y(0)) E_\alpha(-\gamma t^\alpha) + L t^\alpha E_{\alpha,\alpha+1}(-\gamma t^\alpha) = W_1, \tag{8}$$

where E_α and $E_{\alpha,\alpha+1}$ are the one-parameter and two-parameter Mittag-Leffler function respectively. Then the solutions of the plant-herbivore fractional order model (2) with non negative initial conditions in the region Ω, s.t

$$\Omega = \{(X,Y,Z) \in W : 0 \leq W \leq W_1\} \tag{9}$$

remain in the region Ω . Thus, the region Ω is positively invariant with respect to the of the plant-herbivore fractional order model (2). ■

In the next section, we will study the dynamical analysis of the fractional order plant-herbivore model (2).

3 Stability of Equilibrium Points and Hopf Bifurcation

Let $\alpha \in (0,1]$ and consider the following fractional order commensurate dynamical system:

$$D_t^\alpha x_i = f_i(x_1, x_2), \qquad i = 1, 2. \tag{10}$$

Let $E = (x_1^*, x_2^*)$ be an equilibrium point for the fractional order system (10) and $x_i = x_i^* + \eta_i$, where η_i is a small disturbance from the equilibrium point. This implies that

$$\begin{aligned} D_t^\alpha \eta_i &= D_t^\alpha x_i \\ &= f_i(x_1^* + \eta_1, x_2^* + \eta_2) \\ &\approx \eta_1 \frac{\partial f_i(E)}{\partial x_1} + \eta_2 \frac{\partial f_i(E)}{\partial x_2}. \end{aligned} \tag{11}$$

The system (11) can be written as:

$$D_t^\alpha \eta = J\eta, \tag{12}$$

where $\eta = (\eta_1, \eta_2)^T$ and J is the variational matrix evaluated at the point $E = (x_1^*, x_2^*)$. Following Matignon's theorems [17], the fractional order linear system (12) is asymptotically stable if for all eigenvalues of the Jacobian matrix J at the fixed points, the condition $|arg(\lambda)| > \frac{\alpha\pi}{2}$ is satisfied.

Theorem 1.[18,19,20] *Consider the following commensurate nonlinear fractional order system:*

$$D_t^{\alpha} x = g(x), \qquad x(0) = x_0, \qquad \alpha \in (0,1). \tag{13}$$

an equilibrium point of system (13) is locally asymptotically stable if all the eigenvalues of the Jacobian matrix satisfy $|arg(\lambda)| > \frac{\alpha\pi}{2}$.

If $\Phi(x) = x^2 + a_1 x + a_2$, then the discriminant $D(\Phi)$ of a polynomial Φ is given by

$$D(\Phi) = -\begin{vmatrix} 1 & a_1 & a_2 \\ 2 & a_1 & 0 \\ 0 & 2 & a_1 \end{vmatrix} = a_1^2 - 4a_2. \tag{14}$$

The generalized Routh-Hurwitz stability conditions are given by the following proposition[18,19,20].

Proposition

One assumes that E exists in R_+^2.

1. If $D(\Phi) \geq 0$, $a_1 > 0$ and $a_2 > 0$, then the equilibrium point is locally asymptotically stable.

2. If $D(\Phi) < 0$ and $\left|\tan^{-1}\left(\frac{\sqrt{4a_2 - a_1^2}}{a_1}\right)\right| > \frac{\alpha\pi}{2}$, $\alpha \in [0,1)$ then the equilibrium point is locally asymptotically stable.

In the following we evaluate the equilibrium points. Let

$$D_t^{\alpha} x = 0, \qquad D_t^{\alpha} y = 0.$$

Then we obtain:

1. The first trivial equilibrium point is $E_0 = (0,0)$ .The point E_0 always exists.

The Jacobian matrix J_0 for the plant-herbivore fractional order model (2) evaluated at the equilibrium point E_0 is:

$$J_0 = \begin{pmatrix} q & 0 \\ 0 & -\gamma \end{pmatrix}.$$

Theorem 2. *The trivial equilibrium point E_0 of system (2) is a saddle point.*

The trivial equilibrium point E_0 is locally asymptotically stable if all the eigenvalues λ_{0i}, $i = 1,2$ of J_0 satisfy Matignon's conditions. The eigenvalues corresponding to the equilibrium E_0 are $\lambda_{01} = q$ and $\lambda_{02} = -\gamma$.

Then we have $\lambda_{01} > 0$ and $\lambda_{02} < 0$. It follows that the node equilibrium point of system (2) is a saddle point, non-empty stable manifolds and an unstable manifold. ■

2. The second free herbivore fixed point is $E_1 = (x_1, y_1) = (q, 0)$ when the herbivore is absent in the plant, in this case $(y = 0)$, therefore the plant is fully susceptible. The point E_1 always exists.

Theorem 3. *For the fractional order plant-herbivore model (2), the basic reproduction number is*

$$R_0 = \frac{\beta_1 q^2}{\gamma(1+q^2)}.$$

Rewrite the equations by which classes of the herbivore population y first and then the plant population x secondly, we have

$$D_t^{\alpha} y = \frac{\beta_1 x^2 y}{1+x^2} - \gamma y, \tag{15}$$
$$D_t^{\alpha} x = x(q-x) - \frac{\beta x^2 y}{1+x^2}.$$

one can write the system (14) in the form

$$\frac{d^{\alpha} X}{dt^{\alpha}} = f(X) - v(X),$$

where

$$f(X) = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} \frac{\beta_1 x^2 y}{1+x^2} \\ \frac{-\beta x^2 y}{1+x^2} \end{bmatrix}, \; v(X) = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \gamma y \\ -x(q-x) \end{bmatrix}.$$

Next, we define the matrices $F(X)$ and $V(x)$, such that

$$F(X)=\begin{bmatrix}\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y}\\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y}\end{bmatrix},V(X)=\begin{bmatrix}\frac{\partial v_1}{\partial x} & \frac{\partial v_1}{\partial y}\\ \frac{\partial v_2}{\partial x} & \frac{\partial v_2}{\partial y}\end{bmatrix},$$

then

$$F(X)=\begin{bmatrix}\frac{2\beta_1 xy}{(1+x^2)^2} & \frac{\beta_1 x^2}{1+x^2}\\ \frac{-2\beta xy}{(1+x^2)^2} & \frac{-\beta x^2}{1+x^2}\end{bmatrix},V(X)=\begin{bmatrix}0 & \gamma\\ -q+2x & 0\end{bmatrix},$$

at the free herbivore point $E_1=(q,0)$, we have

$$F(X)=\begin{bmatrix}0 & \frac{\beta_1 q^2}{1+q^2}\\ 0 & \frac{-\beta q^2}{1+q^2}\end{bmatrix},\ V(X)=\begin{bmatrix}0 & \gamma\\ q & 0\end{bmatrix}.$$

Using the equation $\left|F\cdot V^{-1}-\lambda I\right|=0$, one obtain

$$\begin{vmatrix}\frac{\beta_1 q^2}{\gamma(1+q^2)}-\lambda & 0\\ \frac{-\beta q^2}{\gamma(1+q^2)} & -\lambda\end{vmatrix}=0,$$

then $\lambda_1=\frac{\beta_1 q^2}{\gamma(1+q^2)}$, $\lambda_2=0$.

Therefore, the spectral radius is $\rho\left(F\cdot V^{-1}\right)=\max(\lambda_i),\ i=1,2$. Then $R_0=\frac{\beta_1 q^2}{\gamma(1+q^2)}$. ■

3. By (2), The third point is a positive equilibrium point $E_2=(x_2,y_*)=\left(\sqrt{\frac{\gamma}{\beta_1-\gamma}},\frac{(q-x_2^2)(1+x_2^2)}{\beta x_2}\right)$.

Remark 1.

1) The free plant whose population density is denoted by x ($x=0$ and $y=+ve$), does not exist, because herbivore depends on the existence of the plant, so if $y=0$ then it should be that, $x=0$ is the free equilibrium point E_0 again.

2) E_2 must be have non negative component, then we have the condition $\beta_1>\gamma$ and $x_2<\sqrt{q}$ for E_2.

The variational matrix J_1 for the plant-herbivore fractional order system given in (2) evaluated at the free herbivore equilibrium point E_1 is given by:

$$J_1=\begin{bmatrix}-q & \frac{\beta q^2}{1+q^2}\\ 0 & \gamma(R_0-1)\end{bmatrix}.$$

Theorem 4. *The free herbivore equilibrium point E_1 is a saddle unstable point.*

The Jacobian matrix J_1 has the following eigenvalues $\lambda_{11}=-q$ and $\lambda_{12}=\gamma(R_0-1)>0$. Hence, the free herbivore equilibrium point E_1 is not locally asymptotically stable.■ In the next section, we will discuss the asymptotic stability of the positive equilibrium point E_2 of the plant herbivore fractional order model (2). The Jacobian matrix J_2 of the positive equilibrium point $E_2=(x_2,y_*)$ is given as:

$$J_2=\begin{pmatrix}q-2x_2-\frac{2\beta x_2 y_2}{(1+x_2^2)^2} & \frac{-\beta x_2}{1+x_2^2}\\ \frac{2\beta_1 x_2 y_2}{(1+x_2^2)^2} & 0\end{pmatrix}.$$

The characteristic equation of J_2 is

$$\lambda^2-\mathrm{Tr}(J_2)\lambda+\det(J_2)=0, \tag{16}$$

where

$$\mathrm{Tr}(J_2)=q-2x_2-\frac{2\beta x_2 y_2}{(1+x_2^2)^2}\ \text{and}\ \det(J_2)=\frac{2\beta\beta_1 x_2^2 y_2}{(1+x_2^2)^3}>0.$$

The characteristic equation (16) have the roots $\lambda_{21},\lambda_{22}=\frac{1}{2}\left[\mathrm{Tr}(J_2)\pm\sqrt{\mathrm{Tr}^2(J_2)-4\det(J_2)}\right]$.

Theorem 5. *The equilibrium point E_2 of the plant herbivore fractional order system (2) is locally asymptotically stable if the following conditions are satisfied:*
i) $q < \frac{2x_2\gamma}{2\gamma-\beta_1}$,
ii) $2\gamma > \beta_1$ *and*
iii) $\mathrm{Tr}(J_2) < 2\sqrt{\det(J_2)}\cos(\frac{\alpha\pi}{2})$.

It is clear that $\mathrm{Tr}(J_2) < 0$ if and only if $q < \frac{2x_2\gamma}{2\gamma-\beta_1}$ implies that $2\gamma > \beta_1$, then $\left|\arg\left(\lambda_{2j}\right)\right| > \frac{\alpha\pi}{2}$, $j = 1,2$, if and only if the three conditions hold. ■

Theorem 6. *For the plant herbivore fractional order model (2), the following statements can be obtained.*
(a) If $q \le \frac{2x_2\gamma}{2\gamma-\beta_1}$, *then for* $0 < \alpha < 1$, *the equilibrium point* E_2 *is locally asymptotically stable,*
(b) $2\gamma > \beta_1$,
(c) If $0 < \mathrm{Tr}(J_2) < 2\sqrt{\det(J_2)}$, *then for any* $\alpha \in (0,\alpha^*)$, *the equilibrium point* E_2 *is locally asymptotically stable, where* $\alpha^* = \frac{2}{\pi}\left|\cos^{-1}\left(\frac{\mathrm{Tr}(J_2)}{2\sqrt{\det(J_2)}}\right)\right|$ *and*
(d) If $\mathrm{Tr}(J_2) \ge 2\sqrt{\det(J_2)}$, *the equilibrium* E_2 *is unstable for any* $\alpha \in (0,1)$.

The conclusions (a), (b) and (d) are obvious. For the statement (c), due to $0 < \mathrm{Tr}(J_2) < 2\sqrt{\det(J_2)}$, the equation (16) has two complex roots $\lambda_{21}, \lambda_{22}$, and their real part is $\frac{\mathrm{Tr}(J_2)}{2} > 0$. Then $\left|\arg(\lambda_{2j})\right| = \cos^{-1}\left(\frac{\mathrm{Tr}(J_2)}{2\sqrt{\det(J_2)}}\right)$, $j = 1,2$. Besides, according to the condition $\cos^{-1}\left(\frac{\mathrm{Tr}(J_2)}{2\sqrt{\det(J_2)}}\right) = \frac{\alpha^*\pi}{2}$, $\alpha \in (0,\alpha^*)$ if and only if $\left|\arg(\lambda_{2j})\right| > \frac{\alpha\pi}{2}$, $j = 1,2$ [18,19,20,2,4,21], it is concluded that Theorem 3.6 is true. ■

According to the statement of Theorem 4 and Theorem 5, it can be concluded that the positive equilibrium is locally asymptotically stable if and only if $\alpha \in (0,\alpha^*)$. At $\alpha = \alpha^*$ the Hopf bifurcation is expected to take place. As increases above the critical value α^* the positive equilibrium is unstable and a limit cycle is expected to appear in the proximity of E_2 due to the Hopf bifurcation phenomenon.

4 Numerical Methods and Simulations

We applying the generalized predictor corrector algorithm to find the numerical solution of the plant herbivore fractional-order model (2). By setting $h = \frac{T}{M}$, $t_m = mh$, $m = 0,1,2,,\ldots,M \in Z^+$, then Eq. (2) can be discretized as follows:

$$x_{m+1} = x_0 + \frac{h^\alpha}{\Gamma(\alpha+2)}\left[x^p_{m+1}(q - x^p_{m+1}) - \frac{\beta\left(x^p_{m+1}\right)^2 y^p_{m+1}}{1+\left(x^p_{m+1}\right)^2}\right] + \frac{h^\alpha}{\Gamma(\alpha+2)}\sum_{j=1}^{m} a_{j,m+1}\left[x_j(q-x_j) - \frac{\beta x_j^2 y_j}{1+x_j^2}\right],$$

$$y_{m+1} = y_0 + \frac{h^\alpha}{\Gamma(\alpha+2)}\left[\frac{\beta_1\left(x^p_{m+1}\right)^2 y^p_{m+1}}{1+\left(x^p_{m+1}\right)^2} - \gamma y^p_{m+1}\right] + \frac{h^\alpha}{\Gamma(\alpha+2)}\sum_{j=1}^{m} a_{j,m+1}\left[\frac{\beta_1 x_j^2 y_j}{1+x_j^2} - \gamma y_j\right],$$

where

$$x^p_{m+1} = x_0 + \frac{1}{\Gamma(\alpha)}\sum_{j=0}^{m} b_{j,m+1}\left[x_j(q-x_j) - \frac{\beta x_j^2 y_j}{1+x_j^2}\right],$$

$$y^p_{m+1} = y_0 + \frac{1}{\Gamma(\alpha)}\sum_{j=0}^{m} b_{j,m+1}\left[\frac{\beta_1 x_j^2 y_j}{1+x_j^2} - \gamma y_j\right],$$

$$a_{j,m+1} = \begin{cases} m^{\alpha-1} - (m-\alpha)(m+1), & j = 0,\\ (m-j-2)^{\alpha+1} + (m-j)^{\alpha+1} - 2(m-j+1)^{\alpha+1} & 1 \le j \le m,\\ 1 & j = m+1,\end{cases}$$

$$b_{j,m+1} = \frac{h^\alpha}{\alpha}\left[(m-j+1)^\alpha - (m-j)^\alpha\right], \qquad 0 \le j \le m.$$

Numerical results of the fractional order plant-herbivore model (2) are presented in Figs. 1-4, it is clear that the numerical solutions of the fractional order plant-herbivore model (2) depends on the fractional order α. We use some parameters like $q = 8$, $\beta = 1.25$, $\beta_1 = 1.2$, $\gamma = 1$ and $(x_0, y_0) = (2.2, 12)$.The approximate solutions $x(t)$ and $y(t)$ by the generalized predictor corrector algorithm are displayed in Figs. 1-4, with different values of α. Where $\mathrm{Tr}(J_2) = 1.606553371$, $\det(J_2) = 1.921310675$ the values of the basic reproductive number $R_0 = 1.181538462$, the equilibrium point $E_2 = (x_2, y_2) = (2.236067977, 12.37300207)$ and $\alpha^* = 0.606482273$. When $\alpha < \alpha^*$ the trajectory of fractional order plant-herbivore system (2) converges to the equilibrium E_2 as shown in figure 1 for $\alpha = 0.59$ and figure 2 for $\alpha = 0.6$. When $\alpha > \alpha^*$ the trajectory of fractional order plant-herbivore system (2) converges to converges to an asymptotically stable limit cycle as shown in figure 3 for $\alpha = 0.61$ and figure 4 for $\alpha = 0.62$.

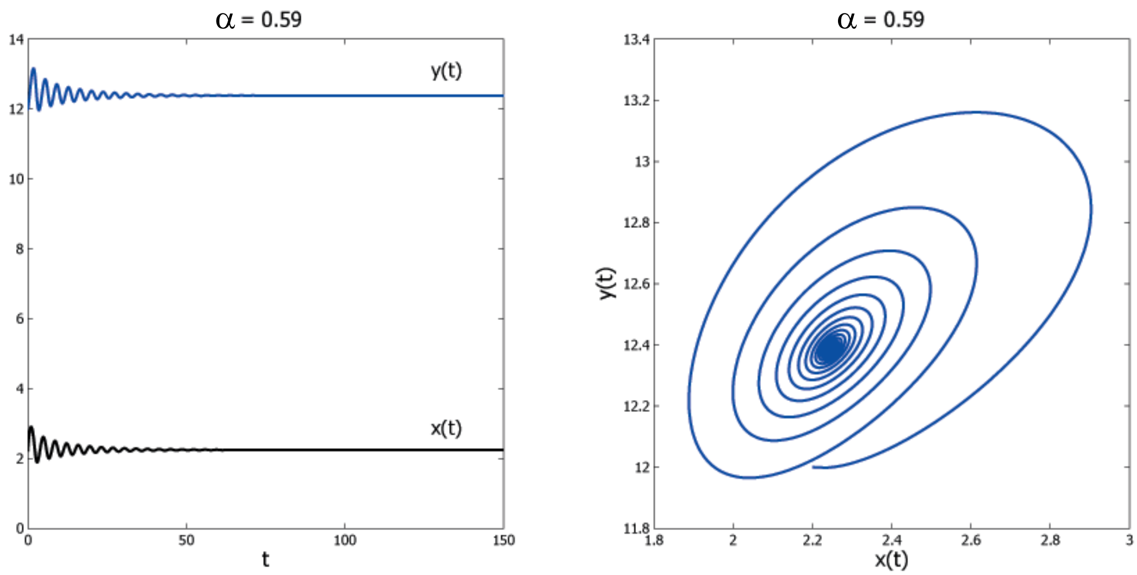

Fig.1. The phase portraits of plant herbivore model (2) converges to the equilibrium E_2 for $\alpha = 0.59$.

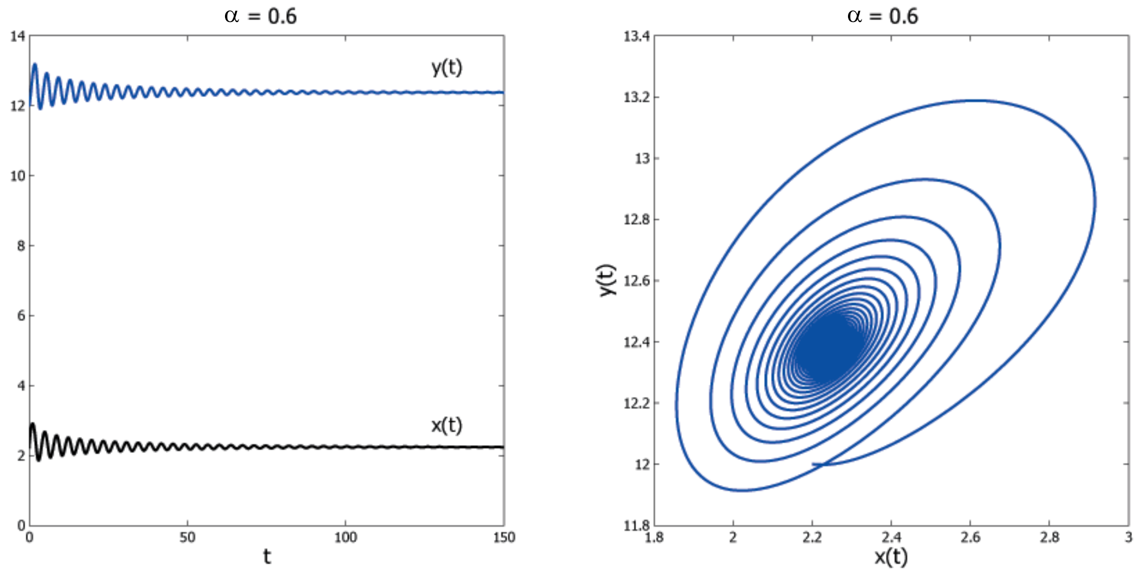

Fig.2. The phase portraits of plant herbivore model (2) converges to the equilibrium E_2 for $\alpha = 0.6$.

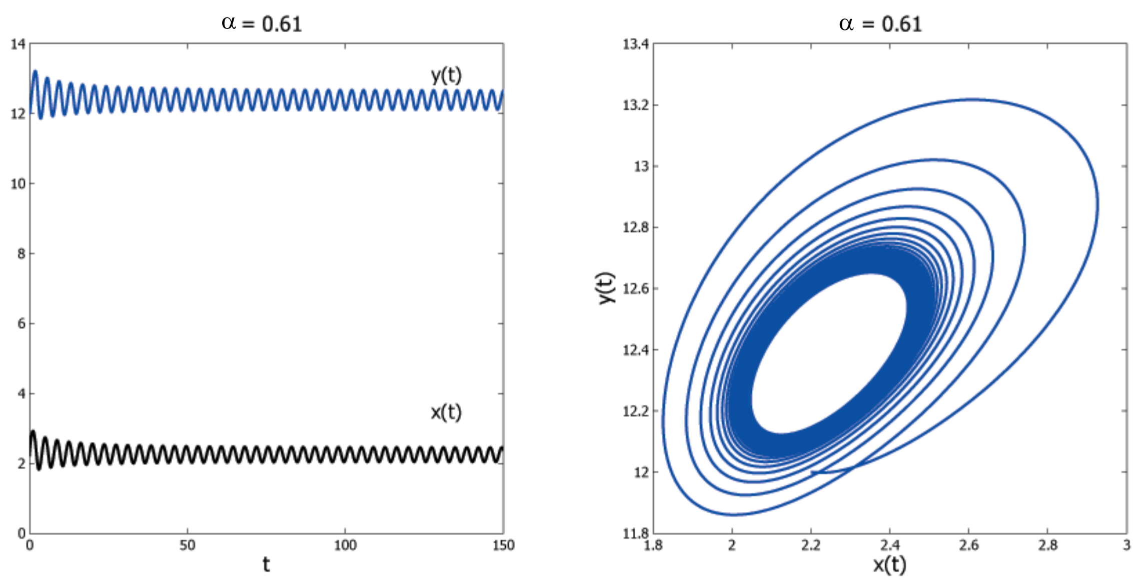

Fig.3. The trajectory of plant herbivore model (2) converges to an asymptotically stable limit cycle for $\alpha = 0.61$.

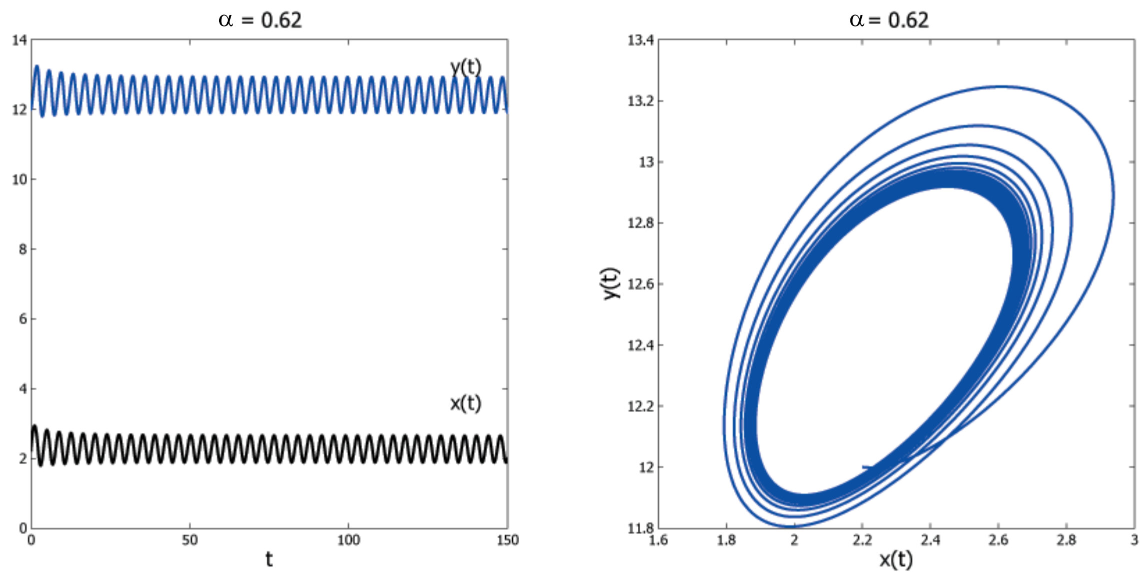

Fig.4. When $\alpha = 0.62$ the trajectory of system (2) converges to an asymptotically stable limit cycle.

5 Conclusions

In this paper, We have proposed a fractional order model for the interaction between plant and herbivore. We analyze the fractional order model with regard to stability of the equilibrium points. We have established the condition for uniform boundeness of the model. We have also given a numerical results using Adams-Bashforth-Moulton algorithm. The theoretical and numerical results for the fractional dynamical system model presented in the paper show that the plant-herbivore model may exhibit rich dynamical behavior.

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