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Bayesian Estimation of Parameters of Augmenting Gamma Strength Reliability for Symmetric and Asymmetric Loss Functions

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Abstract: This paper deals with the Bayesian and maximum likelihood estimation of augmented strength reliability $R_k(k = 1,2,3)$ under Augmentation Strategy Plan (ASP). In Bayesian context we consider gamma prior for unknown parameters of augmented strength reliability model under squared error loss function (SELF) and linex loss function (LLF) for the generalized case of ASP. A Monte-Carlo importance sampling procedure has been implemented to approximate the Bayes and quasi-Bayes estimators of R_k. The performances of Bayes and quasi-Bayes estimators of augmented strength reliability under both the loss functions are compared with that of maximum likelihood estimators on the basis of their mean square errors and absolute biases. We analyze simulated and real data sets for illustrative purpose for validation of proposed estimators.

Keywords: Stress Strength Reliability; Augmentation; MLE; Bayesian Estimators; Monte Carlo Importance Sampling.

1 Introduction

The Gamma distribution has widely been used to model the time to event analysis in survival and reliability theory. This distribution is been also used in many other areas e.g. life insurance claims and credit risks, climatology, meteorology, telecommunication etc., the reader is suggested to refer [1] and [2]. for further applications and discussions of gamma distribution. There exist comparatively less attempts on gamma model, may be because of non-availability of closed form for cumulative distribution function, survival function and hazard rate etc. Moreover, this distribution consists its reproductive property, which leads us to choose this model for the proposed ASP introduced by [28].

The strength reliability is defined as the probability that the equipment will survive its usual life if its random strength (X) is higher than the random stress(Y) imposed on it, which is expressed as $R = P(Y < X)$. In reliability engineering R is often called as measure of system performance. The stress strength model was first considered by [3] using the non-parametric approach. Thereafter the problem of system reliability under the stress strength set up have been attracted to the researchers due to its applicability in various real life situations. A plenty of works on system reliability and its inferences have widely been attempted by several authors, some of the pioneer contributions are, [4], [5], [6], [7] and [8] and references therein.

In literature, a number of works on estimation of system reliability parameters are cited particularly for gamma life distribution. The problem of estimation of system reliability for gamma distribution was firstly considered by [10]. They extended the work of [33] and attempted to find out the different representations of $R = P(Y < X)$ for real and integer valued shape parameters. A comparison between ML and uniformly minimum variance unbiased estimators of the stress-strength reliability were presented for known integer-valued shape parameters. The similar work was followed by series of work of [11] and [12], in which bootstrap and different non-parametric confidence intervals of R are presented. In the similar manner, [9] attempted the estimation of R and compared the parametric and non-parametric methods of estimation.

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Recently, [13] have studied the MLE and UMVUE of parameters of gamma stress and strength reliability with assumption that the two shape parameters are known arbitrary real numbers. Confidence interval estimation of strength reliability have discussed by [14] for generalized gamma family. [15] proposed normal approximation due to [16] for two parameter gamma distribution and they rectified that the proposed approximation was suitable and useful for the calculation of prediction and tolerance intervals and for the estimation of system reliability parameters and also given number of real life examples on two parameter gamma distribution. [17] derived the mathematical expressions for strength reliability for several life distributions named as in gamma, compound gamma, log gamma and generalized gamma models. A non-parametric approach is also considered in estimating the strength reliabilities for life distributions, namely, normal, exponential, gamma and beta distributions by [18].

In Bayesian paradigm, the choice of appropriate prior distribution is most essential. The gamma distribution is frequently chosen as a prior distribution over the decades. [19] and [20] attempted ML and Bayes estimation of R under gamma prior distribution, by considering both stress and strength are independently distributed as scaled Burr Type X distribution and generalized exponential distributions respectively. [21] considered estimation of modified ML and Bayes estimation of R by assuming that X and Y are distributed as two independent 3-parameter generalized exponential random variables having different shape but same location and scale parameters for gamma prior distribution. The importance sampling procedure was employed for Bayes computations. A numerical comparison between Bayes and modified ML estimators of $R = P(Y < X)$ through importance sampling procedure was carried out by [22] by assuming that the random variables X and Y are distributed as two independent four-parameter generalized gamma distribution with same location and scale parameters. Some of the recent attempts on Bayesian estimation of R under the assumption of gamma prior may be referred from [23], [24], [25] and [26] and references therein.

In this paper, we propose Bayesian and classical approach for drawing inferences on augmenting gamma strength reliability for the generalized case of ASP. In fact, every new brand of existing system has two obvious characteristics (i) reliable and (ii) unreliable. For life time data analysis of reliable equipments, several ideas including accelerated life testing method are considered by researchers and the references are available in literature. There is a great difficulty in assessing and obtaining the failure time observations of unreliable equipment. We therefore recommend ASP to overcome the situation when equipment has an impression of early failure of new system and frequent failures occur in used systems due to poor quality of component. The ASP is useful in enhancing the strength reliability and protect from unwanted such failures and sustain to survive its usual life.

ASP comprises three possible situations for enhancing the strength of an equipment to face the common stress. The cases under ASP are stated as: In the first case, the strength of equipment, having initially Gamma strength, is increased by m times of its initial stress. For second case, a suggestion is made to add n independent components, each having Gamma initial strength with the equipment to face the stress. Finally, in third case, the strength of the equipment is increased by adding independent components, each having m times of initial Gamma stress. It is to be noticed that case-I and case-II of ASP are special cases of case-III, which we call it as generalized case of ASP.

Initially, the augmenting strength reliability problem under exponential stress strength set-up was considered by [27] for three different possible cases. After one decade, [28] discussed applicability of augmenting strength reliability of an equipment under these three cases which is named as augmentation strategy plan (ASP). They derived gamma strength reliability models under ASP and numerically observed that augmentation is fruitful. [29] attempted for augmenting Inverse Gaussian strength reliability under ASP and numerically it is verified that ASP performs quite effectively. Recently, [30] and [31] have attempted the augmentation of strength reliability of a coherent system, when its components are connected in series and parallel set-up for exponential and gamma life time models respectively.

Here, we consider that the stress (Y) and strength (X) are independently and identically gamma distributed random variables with scale and shape parameters respectively. The probability density function (pdf) of X (or Y) is given by

$$f_X(x/\alpha,\lambda) = \frac{\alpha^\lambda}{\Gamma(\lambda)}\exp(-\alpha x)x^{\lambda-1};\;\; x>0,\alpha,\lambda>0. \tag{1}$$

Under the generalized case of ASP (i.e. case-III), the pdf of enhanced gamma strength $Z_k (k = 1,2,3)$, where k stands for case-I,II and III respectively, is given by

$$f_{Z_k}(z_k/\alpha,\lambda) = \frac{\alpha^{n\lambda}}{m^{n\lambda}\Gamma(n\lambda)}\;\exp(\frac{-\alpha z_k}{m})\;z_k^{n\lambda-1};\;\; z_k>0,\alpha,\lambda>0 \tag{2}$$

where,$'m'$ is a positive real number and $'n'$ is positive integer. From equation 2, the density functions of cases-I and II of ASP can directly be obtained by substituting $k = n = 1$ and $k = 2, m = 1$ respectively.

Recently, [34] have attempted Bayes estimation of augmenting gamma strength reliability of a System under non-informative prior distributions. In similar manner, here we attempted ML, Bayes and quasi-Bayes estimation of augmented strength reliability for the generalized case of ASP by assuming that both scale (α) and shape (λ) parameters

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having gamma informative priors.

The rest of article is organized as follows. In section 2, a generalized form of augmented strength reliability models under ASP are introduced. A generalized form of ML estimators of augmented strength reliability under ASP is presented in section 3. The asymptotic distributions as well as asymptotic confidence intervals for α, λ and $R_k(k=1,2,3)$ of ASP are discussed in section 4. In section 5, we propose a generalized form of Bayes and quasi-Bayes estimators of augmented strength reliability parameters using importance sampling under SELF and LLF. The methods proposed are illustrated by analyzing simulated and real data sets in section 6. A simulation study and its discussions based on findings of the generalized case of ASP are reported in section 7. Finally, the concluding remarks are given in section 8.

2 Generalized Augmented Strength Reliability Models

In this section, a generalized form of augmented strength reliability model under ASP is presented, which is suitable to handle the situation of early stage failures of sophisticated new equipments as well as frequent failures of used equipments due to its poor or weaker strength. To overcome such circumstances, the ASP is recommended to make the system failure free by boosting the existing strength to survive its usual life. In this view augmenting gamma strength reliability models for three different possible cases under ASP are validated and developed by [28]. A more generalized form of augmenting gamma strength reliability is given by

$$R_k = P(Z_k > Y) = \frac{m^{\lambda}}{\Gamma(n\lambda)(1+m)^{n\lambda+\lambda}} \sum_{j=0}^{\infty} \frac{\Gamma(n\lambda+\lambda+j)}{\Gamma(\lambda+1+j)} (\frac{m}{m+1})^{j};\ k=1,2,3. \tag{3}$$

The augmenting strength reliability expressions for case I and II are particular cases of R_k can be obtained by substituting $k=n=1$ and $k=2, m=1$ respectively in equation 3. One can find the strength reliability expression for case-III by substituting $k=3$ in equation 3.

3 Maximum Likelihood Estimation of Generalized Augmented Strength Reliability

In this section, we present the estimation of parameters of augmented strength reliability R_k. Suppose $Z_k = (z_{k1}, z_{k2} \ldots z_{kn_1})$ and $Y = (y_1, y_2 \ldots y_{n_2})$ be the two independent random samples of sizes n_1 and n_2 drawn from the augmented gamma strength and gamma stress distributions respectively. Then the likelihood function is given as follows

$$L_k(\alpha,\lambda/\mathbf{data}) = \frac{\alpha^{\lambda(nn_1+n_2)}}{m^{\lambda nn_1}\Gamma(n\lambda)^{n_1}\Gamma(\lambda)^{n_2}} \exp -\alpha(\frac{n_1\bar{z}_k}{m}+n_2\bar{y}) \prod_{i=1}^{n_1} z_{ki}^{(n\lambda-1)} \prod_{j=1}^{n_2} y_j^{(\lambda-1)}. \tag{4}$$

The likelihood equations with respect to α and λ are given by

$$\frac{\partial logL_k(\mathbf{data}/\alpha,\lambda)}{\partial\alpha} = 0 \text{ and } \frac{\partial logL_k(\mathbf{data}/\alpha,\lambda)}{\partial\lambda} = 0. \tag{5}$$

The maximum likelihood equations based on random samples are obtained by partial derivatives with respect to α and λ and equating to zero, which are given by

$$\frac{\partial logL_k(\mathbf{data}/\alpha,\lambda)}{\partial\alpha} = \frac{\lambda(nn_1+n_2)}{\alpha} - (\frac{n_1\bar{z}_k}{m}+n_2\bar{y}) = 0 \tag{6}$$

$$\frac{\partial logL_k(\mathbf{data}/\alpha,\lambda)}{\partial\lambda} = (nn_1+n_2)log(\alpha) - nn_1 log(m) - n_1\psi(n\lambda) - n_2\psi(\lambda) + n\sum_{i=1}^{n_1} log z_{ki} + \sum_{i=1}^{n_2} log y_j = 0 \tag{7}$$

where, $\psi(\lambda)$ is the digamma function, defined as $\psi(\lambda) = \frac{\partial ln\Gamma(\lambda)}{\partial\lambda}$. The maximum likelihood estimators $\hat{\alpha}$ and $\hat{\lambda}$ are obtained as the simultaneous solution of equations 6 and 7. As the closed form solution is not possible to evaluate the

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above equations, thus any one of numerical iterative technique may be used. The MLE of augmented strength reliability (R_k) for generalized case of ASP can be obtained through invariance property as follow

$$\hat{R}_k = \frac{m^{\hat{\lambda}}}{\Gamma(n\hat{\lambda})(1+m)^{n\hat{\lambda}+\hat{\lambda}}} \sum_{j=0}^{\infty} \frac{\Gamma(n\hat{\lambda}+\hat{\lambda}+j)}{\Gamma(\hat{\lambda}+1+j)} (\frac{m}{m+1})^j;\ k=1,2,3. \tag{8}$$

Remarks: (1) The ML estimators $\hat{\alpha}_k$ and $\hat{\lambda}_k$ of augmented strength reliability parameters α and λ, respectively can be obtained for each of respective cases I, II and III by substituting $k=n=1$; $k=2, m=1$ and $k=3$ separately in the solution of equations 6 and 7.
(2) The MLE of augmented strength reliability (R_k) for Cases-I, II and III under ASP can be obtained directly by substituting $k=n=1$; $k=2, m=1$ and $k=3$ respectively in equation 8.

4 Asymptotic Distributions and Confidence Intervals of R_k

In this section, the asymptotic distributions and asymptotic confidence intervals (C.I.) for α, λ and $R_k(k=1,2,3)$ for each of the generalized case of ASP are derived. The asymptotic distributions of α and λ for large samples are given as

$$\sqrt{n}(\hat{\alpha}-\alpha) \to N(0,\ I_{11}^{-1}(\alpha))\ and\ \sqrt{n}(\hat{\lambda}-\lambda) \to N(0,\ I_{22}^{-1}(\lambda))$$

where, $I_k(\Theta)$ is the Fisher information matrix of $\Theta=(\alpha,\lambda)$, defined as

$$I_k(\Theta) = -E \begin{pmatrix} \frac{\partial^2 l_k}{\partial \alpha^2} & \frac{\partial^2 l_k}{\partial \alpha \partial \lambda} \\ \frac{\partial^2 l_k}{\partial \lambda \partial \alpha} & \frac{\partial^2 l_k}{\partial \lambda^2} \end{pmatrix} = \begin{pmatrix} I_{k,11} & I_{k,12} \\ I_{k,21} & I_{k,22} \end{pmatrix}$$

where, $l_k = logL_k$. The $100(1-p)\%$ confidence interval of α and λ are given by

$$\{\hat{\alpha} \mp z_{p/2}\sqrt{v(\hat{\alpha})}\}\ and\ \{\hat{\lambda} \mp z_{p/2}\sqrt{v(\hat{\lambda})}\}\ respectively,$$

where, $z_{p/2}$ is the upper $100(p/2)^{th}$ percentile of a standard normal random variable. The asymptotic distribution of augmented strength reliability as $n_1 \to \infty$ and $n_2 \to \infty$ is given by

$$\frac{\hat{R}_k - R_k}{\sqrt{\frac{R_{k1}^2}{n_1 I_{11}} + \frac{R_{k2}^2}{n_2 I_{22}}}} \to N(0,1)$$

where, $R_{k1} = \frac{\partial R_k}{\partial \alpha}$ and $R_{k2} = \frac{\partial R_k}{\partial \lambda}$. Here, MLE of R_k, $\hat{R}_k$ is not in explicit form, therefore, it is difficult to find out the exact distribution of $\hat{R}_k$. We, therefore, construct the $100(1-p)\%$ asymptotic confidence interval of $\hat{R}_k$, is given by

$$\left[\hat{R}_k \mp z_{p/2}\sqrt{\frac{\hat{R}_{k1}^2}{n_1 I_{k,11}(\hat{\alpha})} + \frac{\hat{R}_{k2}^2}{n_2 I_{k,22}(\hat{\lambda})}}\right]$$

where, $I_{k,11}(\hat{\alpha})$, $I_{k,22}(\hat{\lambda})$, $\hat{R}_{k1}^2$ and $\hat{R}_{k2}^2$ are the MLEs of $I_{k,11}(\alpha)$, $I_{k,22}(\lambda)$, R_{k1}^2 and R_{k2}^2 respectively and the Fisher information matrix is given as

$$I_k(\Theta) = \begin{pmatrix} \frac{\lambda(nn_1+n_2)}{\alpha^2} & \frac{-(nn_1+n_2)}{\alpha} \\ \frac{-(nn_1+n_2)}{\alpha} & n_1\psi'(n\lambda)+n_2\psi'(\lambda) \end{pmatrix}$$

where, $\psi'(.)$ is the tri-gamma function, which is defined as follows

$$\psi'(n\lambda) = \frac{\partial^2 log\Gamma(n\lambda)}{\partial \lambda^2} = \frac{\Gamma''(n\lambda)\Gamma(n\lambda)-(\Gamma'(n\lambda))^2}{(\Gamma(n\lambda))^2}$$
$$\psi'(\lambda) = \frac{\partial^2 log\Gamma(\lambda)}{\partial \lambda^2} = \frac{\Gamma''(\lambda)\Gamma(\lambda)-(\Gamma'(\lambda))^2}{(\Gamma(\lambda))^2}.$$

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Thus the $100(1-p)\%$ confidence intervals of α and λ for the generalized case of ASP is given by

$$\left[\hat{\alpha} \mp z_{p/2}\sqrt{\frac{\hat{\alpha}^2}{\hat{\lambda}(nn_1+n_2)}}\right] \text{ and } \left[\hat{\lambda} \mp z_{p/2}\sqrt{(n_1\psi'(n\hat{\lambda})+n_2\psi'(\hat{\lambda}))^{-1}}\right]$$

respectively. One can also find the asymptotic distribution of $\hat{R}_k$, which is asymptotically normally distributed with mean R_k and variance $\sigma^2_{R_k} = \left(\frac{R^2_{k1}}{n_1 I_{k,11}} + \frac{R^2_{k2}}{n_2 I_{k,22}}\right)$, where R_{k1} and R_{k2} are defined as

$$R_{k1} = \frac{\partial R_k}{\partial \alpha} = 0$$

$$R_{k2} = \frac{\partial R_k}{\partial \lambda} = \frac{m^\lambda}{\Gamma(n\lambda)}\sum_{j=0}^{\infty}\frac{\Gamma(n\lambda+\lambda+j)}{\Gamma(\lambda+1+j)}\left(\frac{m}{m+1}\right)^{(n\lambda+\lambda+j)}$$
$$\{(n+1)PG(0,n\lambda+\lambda+j) - nlogm - PG(0,\lambda+j+1) + nPG(0,n\lambda)\}$$

Remark 3: $PG(0,z)$ defines the first derivative of logarithmic of gamma function, which is defined by $PG(0,z) = \frac{\partial log(\Gamma(z))}{\partial z}$.

5 Bayesian Estimation of Generalized Augmenting Strength Reliability Models

5.1 Prior and Posterior

This subsection deals with the Bayes estimation of $R_k (k = 1,2,3)$ and its parameters α and λ for generalized case under ASP. In Bayesian paradigm, the choice of appropriate prior is most essential and is also challenging task. The general ideology behind the choosing of such prior is depends on personal belief and subjective knowledge. If one has adequate information about the parameter(s), one should use informative prior(s), which are combined with the likelihood function to update the information about a particular characteristic of the known data. In this study, we consider α and λ are independent random variables having conjugate (informative) gamma prior, i.e., $\alpha \sim G(a,b)$ and $\lambda \sim G(c,d)$. The joint prior probability density function of α and λ is given by

$$g(\alpha,\lambda) \propto \alpha^{a-1}\lambda^{c-1}exp\{-(b\alpha+d\lambda)\}\ ;\ \alpha,\lambda>0;\ a,b,c,d>0. \tag{9}$$

The hyper-parameters a,b,c and d of prior density function are assumed to be known and are chosen in such a way to reflect the prior belief about the unknown parameters. The joint posterior probability distribution of α and λ is given as

$$\Pi_k(\alpha,\lambda) = Kg(\alpha,\lambda)L_k(\mathbf{data}/\alpha,\lambda) \tag{10}$$

where, $L_k(data/\alpha,\lambda)$ is the likelihood function and K is normalizing constant which is defined as

$$K^{-1} = \int_0^\infty\int_0^\infty g(\alpha,\lambda)L_k(\mathbf{data}/\alpha,\lambda)\partial\alpha\partial\lambda. \tag{11}$$

The marginal posteriors densities of α and λ respectively can be obtained from equation (10) as

$$\pi_{k1}(\alpha/\mathbf{data},\lambda) \propto \int_0^\infty \Pi_k(\alpha,\lambda/\mathbf{data})\partial\lambda \tag{12}$$

$$\pi_{k2}(\lambda/\mathbf{data},\alpha) \propto \int_0^\infty \Pi_k(\alpha,\lambda/\mathbf{data})\partial\alpha. \tag{13}$$

Here, we consider two different loss functions for better comprehension of Bayesian analysis, first one is squared error loss function (SELF) which is symmetric and other is linex loss function (LLF) which is asymmetric. The Bayes estimator of any parametric function, say $\phi(\alpha,\lambda)$ under SELF as well as LLF are respectively defined by

$$\hat{\phi}(\alpha,\lambda)^{self} = \int_{(\alpha,\lambda)}\phi(\alpha,\lambda)\Pi(\alpha,\lambda/\mathbf{data})\partial\alpha\partial\lambda \tag{14}$$

$$\hat{\phi}(\alpha,\lambda)^{llf} = \frac{-1}{p}log\left[\int_{(\alpha,\lambda)}exp\{-p\phi(\alpha,\lambda)\}\Pi(\alpha,\lambda/\mathbf{data})\partial\alpha\partial\lambda\right]. \tag{15}$$

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5.2 Bayes Estimation of R_k under squared error loss function (SELF)

In this subsection, we propose the Bayes estimators of augmenting strength reliability $(R_k; k = 1, 2, 3)$ under squared error loss function for the general case of ASP. The joint posterior probability distribution of random variables α and λ for general case of ASP is obtained by combining likelihood function $L_k(\alpha, \lambda/\textbf{data})$ and joint prior probability density $g(\alpha, \lambda)$ given by

$$\Pi_k(\alpha, \lambda/\textbf{data}) \propto \frac{exp\{s_1(n\lambda - 1) - s_2\}}{m^{nn_1\lambda}\Gamma(n\lambda)^{n_1}\Gamma(\lambda)^{n_2}}\lambda^{c-1}exp\{-\lambda(d - s_2)\}$$
$$\alpha^{\lambda(nn_1+n_2)+a-1}exp\left[-\alpha(\frac{n_1\bar{z}_k}{m} + n_2\bar{y} + b)\right]. \tag{16}$$

The equation 16 can also have a form as

$$\Pi_k(\alpha, \lambda/\textbf{data}) \propto \pi_{k1}(\alpha/\textbf{data}, \lambda)\pi_{k2}(\lambda/\textbf{data}, \alpha)W_k(\alpha, \lambda) \tag{17}$$

where, $s_1 = \sum_{i=1}^{n_1} logz_{ki}$, $s_2 = \sum_{j=1}^{n_2} logy_j$ and the adjustment factor $W_k(\alpha, \lambda)$ is defined as

$$W_k(\alpha, \lambda) = \frac{exp\{-(s_1 + s_2)\}\alpha^{\lambda(nn_1+n_2)}}{m^{nn_1\lambda}\Gamma(n\lambda)^{n_1}\Gamma(\lambda)^{n_2}} \tag{18}$$

and the marginal posterior density functions of α and λ are respectively given by

$$\pi_{k1}(\alpha/\textbf{data}, \lambda) \propto G\left(a,\ (\frac{n_1\bar{z}k}{m} + n_2\bar{y} + b)\right) \tag{19}$$

$$\pi_{k2}(\lambda/\textbf{data}, \alpha) \propto G_\lambda(c,\ d - ns_1 - s_2). \tag{20}$$

Therefore, the Bayes estimator of augmented strength reliability under SELF for a generalized case of ASP is given as

$$\hat{R}_k^{self} = \int_{(\alpha,\lambda)} R_k\Pi_k(\alpha, \lambda/\textbf{data})\partial\alpha\partial\lambda \tag{21}$$

where, R_k is the augmented strength reliability under ASP. The expression of Bayes estimator in equation 21 does not have closed form solution and it cannot be solved analytically, therefore numerical method is used for solution of the proposed estimator. We therefore suggested Monte-Carlo importance sampling procedure to evaluate equation 21. Hence, Bayes estimator $\hat{R}_{kIS}^{self}$ of R_k under this sampling procedure for generalized case of ASP is given by

$$\hat{R}_{kIS}^{self} = \frac{1}{N}\sum_{i=1}^{N}\left[R_k\right]_{\alpha=\alpha_i;\lambda=\lambda_i}$$
$$= \frac{1}{N}\sum_{i=1}^{N} W_k(\alpha_i, \lambda_i)\frac{m_i^\lambda}{\Gamma(n\lambda_i)(1+m)^{n\lambda_i+\lambda_i}}\sum_{j=0}^{\infty}\frac{\Gamma(n\lambda_i + \lambda_i + j)}{\Gamma(\lambda_i + 1 + j)}(\frac{m}{m+1})^j. \tag{22}$$

As an alternative, we obtain the quasi-Bayes estimator $\hat{R}_{kQB}^{self}$ of augmented strength reliability R_k by substituting the Bayes estimators $\hat{\lambda}_k^{self}$ and $\hat{\alpha}_k^{self}$ in the place of λ and α respectively in the augmented strength reliability expression of R_k given in equation 3. Under SELF, the Bayes estimators $\hat{\lambda}_k^{self}$ and $\hat{\alpha}_k^{self}$ are obtained by its posterior means of λ and α, respectively given by

$$\hat{\alpha}_k^{self} = \int_0^\infty \alpha\pi_{k1}(\alpha/\textbf{data}, \lambda) \tag{23}$$

$$\hat{\lambda}_k^{self} = \int_0^\infty \lambda\pi_{k2}(\lambda/\textbf{data}, \alpha). \tag{24}$$

It may be notice that the expressions of augmented strength reliability R_k given in equation 3 is free from α, therefore, only Bayes estimator of λ is required in order to obtain quasi-Bayes estimator. Thus, the required quasi-Bayes estimator $(\hat{R}_{kQB}^{self})$ of augmented strength reliability R_k for general case of ASP is given by

$$\hat{R}_{kQB}^{self} = \frac{m^{\hat{\lambda}_k^{self}}}{\Gamma(n\hat{\lambda}_k^{self})(1+m)^{n\hat{\lambda}_k^{self}+\hat{\lambda}_k^{self}}}\sum_{j=0}^{\infty}\frac{\Gamma(n\hat{\lambda}_k^{self} + \hat{\lambda}_k^{self} + j)}{\Gamma(\hat{\lambda}_k^{self} + 1 + j)}(\frac{m}{m+1})^j. \tag{25}$$

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Remark 4: The expressions for Bayes estimators (and quasi-Bayes estimators) of augmented strength reliability (R_k) for Case-I, II and III under ASP can be obtained directly by substituting $k=n=1$, $k=2m=1$ and $k=3$ separately in equations 22 and 25.

5.3 Bayes Estimation of R_k under Linex loss function (LLF)

Under LLF, The Bayes estimator $(\hat{R}_k^{llf})$ of augmented strength reliability (R_k) for generalized case of ASP is given as

$$\hat{R}_k^{llf} = \frac{-1}{p} ln\big[E(e^{-pR_k}/data)\big] = \frac{-1}{p} ln\big[\int_{(\alpha,\lambda)} e^{-pR_k} \Pi_k(\alpha,\lambda/\mathbf{data})\partial\alpha\partial\lambda\big] \tag{26}$$

where, $\Pi_k(\alpha,\lambda/\mathbf{data})$ being the joint posterior density of α and λ for general case of ASP, which is defined in the equation 16. The expression of Bayes estimator $\hat{R}_k^{llf}$ of augmenting strength reliability for general case of ASP is given in equation 26, have not closed form solution and therefore analytically cannot be solved. Only numerical approximation methods can be used for solution. We therefore importance sampling approximation is proposed. As an alternatively, we obtain the quasi-Bayes estimator $\hat{R}_{kQB}^{llf}$ of augmented strength reliability for general case of ASP under LLF by substituting the Bayes estimators $\hat{\alpha}_k^{llf}$ and $\hat{\lambda}_k^{llf}$ in place of α and λ respectively in R_k defined in equation 3. The Bayes estimators of α and λ under linex loss function for generalized case of ASP are denoted by $\hat{\alpha}_k^{llf}$ and $\hat{\lambda}_k^{llf}$ are respectively given by

$$\hat{\alpha}_k^{llf} = \frac{-1}{p} ln\big[E(e^{-p\alpha_k}/data)\big] = \frac{-1}{p} ln\left[\int_{(\alpha)} e^{-p\alpha_k}\pi_{k1}(\alpha/\mathbf{data},\lambda)\partial\alpha\right] \tag{27}$$

$$\hat{\lambda}_k^{llf} = \frac{-1}{p} ln\big[E(e^{-p\lambda_k}/data)\big] = \frac{-1}{p} ln\left[\int_{(\lambda)} e^{-p\lambda_k}\pi_{k1}(\lambda/\mathbf{data},\lambda)\partial\lambda\right] \tag{28}$$

where, $\pi_{k1}(\alpha/\mathbf{data},\lambda)$ and $\pi_{k2}(\lambda/\mathbf{data},\alpha)$ are defined in equations 19 and 20 respectively. The quasi-Bayes estimator of augmented strength reliability $(\hat{R}_{kQB}^{llf})$ for the generalized case of ASP is given by

$$\hat{R}_{kQB}^{llf} = \frac{m^{\hat{\lambda}_k^{llf}}}{\Gamma(n\hat{\lambda}_k^{llf})(1+m)^{n\hat{\lambda}_k^{llf}+\hat{\lambda}_k^{llf}}} \sum_{j=0}^{\infty} \frac{\Gamma(n\hat{\lambda}_k^{llf}+\hat{\lambda}_k^{llf}+j)}{\Gamma(\hat{\lambda}_k^{llf}+1+j)} \left(\frac{m}{m+1}\right)^j. \tag{29}$$

Remark 5: Under LLF, **(a)** the expressions of Bayes estimators of augmented strength reliability (R_k) for all three Cases of ASP can be obtained under importance sampling in similar manner of Bayes estimator under SELF (see, eq. 22).

(b) The quasi-Bayes estimators of augmented strength reliability (R_k) for Case-I, II and III of ASP can directly be obtained by substituting $k=n=1$, $k=2\ m=1$ and $k=3$ in equation 29.

6 Data analysis

In this section we illustrate the proposed ML and Bayesian procedures to analyze the augmented strength reliabilities for the generalized case (i.e., Case-III) of ASP by considering the simulated and real data sets.

Example 1: We generate the strength and stress data sets of 30 observations each from $G(\alpha/m, n\lambda)$ and $G(\alpha,\lambda)$ respectively with $\alpha=2.5, \lambda=0.5, m=2$ and $n=2$. The true value for augmented strength reliability for case-III is given as $R_3=0.81650$. Thus the maximum likelihood estimates of unknown stress-strength parameters $(\alpha \ and \ \lambda)$ are given as $\alpha_3 = \hat{0}.16654$ and $\hat{\lambda}_3 = 0.341475$. Hence, the maximum likelihood estimate of augmented strength reliability (R_3) is $\hat{R}_3 = 0.78162$. To find out the Bayes estimate of augmented strength reliability under square error and Linex loss functions separately, we fixed the hyper-parameters $a=1.25, b=0.15, c=2.5, d=0.65$.The Bayes estimate of augmented strength reliability and its parameters have been numerically approximated through the general importance sampling of fifty thousand random generations of α and λ. The Bayes estimate of augmented strength reliability under SELF (LLF) is $0.74288(0.74287)$. Similarly, the quasi-Bayes estimate of augmented strength reliability under SELF (LLF) is $0.758475(0.758209)$. To test which set of parameter estimates give better fit to the data sets, the

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Kolmogorov-Smirnov (K-S) distance between the empirical and fitted distribution based on MLEs and Bayes estimators have been calculated. The test was carried out with 5% level of significance. For data set 1, we get the p-value for MLEs (Bayes estimators) as $0.5161(0.6452)$ and for data set 2, the p-value for MLEs (Bayes estimators) is $0.06452(0.06455)$. From the K-S test it is noticed that for both data sets, Bayes estimates gives the better fit than MLEs.

Example2 : We analyze the strength data sets reported by [32] using the two parameters gamma distribution. The ML and Bayes estimates of augmented strength reliability and its parameters are obtained. For the same data sets, [22] observed that 4-parameter generalized gamma distribution works quite well.
The data sets were initially reported by [32], represent the strength of single carbon fiber and impregnated 1000-carbon fiber tows which were measured in GPa. Single fibers were tested under tension at gauge lengths of 1, 10, 20, and 50mm. Impregnated tows of 1000 fibers were tested at gauge lengths of 20, 50, 150 and 300 mm. We analyze here the transformed strength data sets were considered by [19]. The data sets are consider the single fibers of 20 mm (Data Set I) and 10 mm (Data Set II) in gauge length, with sample sizes n = 69 and m = 63, respectively. The data sets are presented in Tables 1 and 2.

Assuming Data set 1 (x) as strength and Data set 2 (y) as stress, the generalized case (i.e., case-III) of ASP was applied to the strength data to augment the strength of the carbon-fiber. In order to obtain the enhanced strength data sets, we added two units each having strength 0.004 (two times of initial stress (0.002)) to the existing strength of carbon fiber. The maximum likelihood estimates of augmented strength reliability (R_3) based on augmented strength data set is $\hat{R}_3 = 0.94122$ and ML estimates of parameters $(\alpha\ and\ \lambda)$ are given as $\hat{\alpha}_3 = 2.6333$ and $\hat{\lambda}_3 = 1.6987$.

Table 1: Data set 1 (x)

0.312	0.314	0.479	0.552	0.700	0.803	0.861	0.865	0.944	0.958
0.966	0.997	1.006	1.021	1.027	1.055	1.063	1.098	1.140	1.179
1.224	1.240	1.253	1.270	1.272	1.274	1.301	1.301	1.359	1.382
1.382	1.426	1.434	1.435	1.478	1.490	1.511	1.514	1.535	1.554
1.566	1.570	1.586	1.629	1.633	1.642	1.648	1.684	1.697	1.726
1.770	1.773	1.800	1.809	1.818	1.821	1.848	1.880	1.954	2.012
2.067	2.084	2.090	2.096	2.128	2.233	2.433	2.585	2.585	

Table 2: Data set 2 (y)

0.101	0.332	0.403	0.428	0.457	0.550	0.561	0.596	0.597	0.645
0.654	0.674	0.718	0.722	0.725	0.732	0.775	0.814	0.816	0.818
0.824	0.859	0.857	0.938	0.940	1.056	1.117	1.128	1.137	1.137
1.177	1.196	1.230	1.325	1.339	1.345	1.420	1.423	1.435	1.443
1.464	1.472	1.494	1.532	1.546	1.577	1.608	1.635	1.693	1.701
1.737	1.754	1.762	1.828	2.052	2.071	2.086	2.171	2.224	2.227
2.425	2.595	3.220							

To find out the Bayes estimate of augmented strength reliability under square error and Linex loss functions separately, we fixed the hyper-parameters $a = 1.25, b = 0.15, c = 2.5, d = 0.65$. The Bayes estimate of augmented strength reliability and its parameters have been obtained through the general importance sampling of fifty thousand random generations of α and λ. The Bayes estimates of augmented strength reliability under SELF (LLF) is $0.745028(0.737579)$. Similarly, the Bayes estimates of augmented strength reliability parameters α and λ under SELF (LLF) respectively are $0.445426(0.445226)$ and $0.208503(0.208506)$, thus the quasi-Bayes estimate of augmented strength reliability under SELF (LLF) is $745028(745027)$.
In order to test which of the estimation method better fits the given data sets, we compute the Kolmogorov-Smirnov (K-S) distance between empirical and fitted distributions based on ML and Bayes methods for different loss functions (SELF and LLF) and tested at 5% level of significance. For the data set 1, the K-S distance based on MLEs (Bayes estimates) is 0.9275(0.913) with corresponding p-values 0.3646(0.3838). Similarly, for data set 2, the K-S distance based on MLEs (Bayes estimates) is 0.7302(0.6349) and the corresponding p-values are 0.6703(0.8223).Thus, for data sets 1 and 2 it is observed that the Bayes estimates give better fit than that of MLEs.

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7 Simulation Study and Discussion

This section presents the behavior of augmented strength reliability parameters under the proposed augmentation strategy plan through simulated samples with different combinations of sample sizes and the stress-strength reliability parameters. For this purpose, the random samples of different sizes (n_1,n_2) from the distributions of stress and augmented strength random variables were drawn. The performance of proposed maximum likelihood estimators of strength reliability parameters have been compared with that of Bayes and Quasi-Bayes estimators under SELF and LLF. The comparison among the different proposed estimators of augmented strength reliabilities have been done on the basis of their mean square errors (MSEs) and absolute biases for different combinations of fixed strength reliability parameters as well as hyper parameters with different sample sizes. In order to evaluate MSEs and absolute biases of the ML, Bayes and quasi-Bayes estimators of augmented strength reliability, the whole procedure was randomly replicated 1000 times.
The derived expressions of Bayes and quasi-Bayes estimators of augmented strength reliability models under ASP are not in explicit form and involve the ratios of implicit integrals. We, therefore used Monte-Carlo importance sampling method to evaluate the integrals involved in the equations of posterior expectations of augmented strength reliabilities and its parameters. The Monte-Carlo importance sampling was carried out with 5000 of intermediate iterations. The importance sampling is a well-established method to approximate the integrals. To carry out the importance sampling procedure numerically the following steps were taken as:

(i)Set trail densities $g_1(\alpha)$ and $g_2(\lambda)$ whose support is same as that of corresponding joint posterior density $\Pi(\alpha,\lambda/\textbf{data})$.
(ii)Generate random samples α_i and $\lambda_i; (i=1,2,...,N)$ of size N from the trail densities $g_1(\alpha)$ and $g_2(\lambda)$.
(iii)Find the product of $R_k(\alpha_i,\lambda_i)$ and $W_k(\alpha_i,\lambda_i)$ at each values of α_i and λ_i drawn from the corresponding marginal posterior densities $g_1(\alpha)$ and $g_2(\lambda)$ respectively.
(iv)The importance sampling estimate $ER_k(\alpha_i,\lambda_i)$ can be found by evaluating the following equation

$$\left[\hat{E}R_k(\alpha_i,\lambda_i)\right]_{IS} = \frac{(1/N)\sum_{i=1}^{N} R_k(\alpha_i,\lambda_i)W_k(\alpha_i,\lambda_i)}{(1/N)\sum_{i=1}^{N} W_k(\alpha_i,\lambda_i)}$$

The comparison among the proposed estimators of augmented strength reliabilities, for the generalized of ASP are presented for varying values of stress-strength parameters λ, m, n and α while keeping all the hyper-parameters fixed $(a=0.5, b=0.75, c=0.25, d=0.75, p=1.5)$ for different combinations of sample sizes (n_1,n_2). Similarly the effect of hyper-parameters (a,b,c,d) in Bayesian estimation of augmented strength reliability models have also been observed for the generalized case and compared by that of MLE. The results obtained through simulation for the generalized case (i.e. case-III) of ASP are presented in the tables $(3-6)$. In these tables the average estimates, mean square errors (MSEs) and absolute biases are tabulated for MLE as well as for Bayes and quasi-Bayes estimates under SELF and LLF. The following observations are made based on the results reported in the tables.

- In Table 3 the effect of different values of $\lambda(0.5,1.5,3)$ are presented by fixing rest of the other parameters and hyper-parameters and it is observed that the MSEs of all the estimators decrease for increasing sample sizes. For smaller value of $\lambda(0.5)$ the Bayes estimators are dominated by ML estimators but for higher values of $\lambda(1.5,3)$ the Bayes estimators dominate the ML estimator. It is also to be noticed that the quasi-Bayes estimators perform quite effective as an alternative to Bayes estimators.
- The results for variation in $n(3,5)$ are presented in Table 4 and it is noticed from the table that the quasi-Bayes estimators under SELF perform well with minimum MSEs and absolute biases as compared to ML and Bayes estimators. It is also observed that the larger sample sizes reduce the MSEs.
- Table 5 presents the variation in $m(2.5,5)$ and it is noticed that the MSEs and absolute biases are in decreasing in nature with increasing values of sample sizes (n_1,n_2). For $m=2.5$ the ML estimators dominate the Bayesian estimators. Moreover the quasi-Bayes estimators of augmented strength reliability gives more accurate result than Bayes estimator. Similarly the Bayes estimators dominate that of ML estimators for $m=5$.
- In Table 6, three different choices of prior variances (small, moderate and large) are chosen to observe the effect of variability in the considered prior distributions, it is noticed that the mean square errors for small variance prior are lesser than that of the moderate and large variance priors. Thus, the choice of minimum variance prior gives the better precision in the Bayes estimates of the parameters.

8 Concluding remarks

In this paper, we have attempted the estimation of augmenting strength reliability under ASP by adopting ML and Bayes methods. Bayes and quasi-Bayes methods of estimation with importance sampling for two different loss functions (SELF

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and LLF) have been employed. We also derived the asymptotic distribution of MLE of augmented strength reliability and its parameters to construct the associated confidence intervals. The comparison among the different estimates of augmented strength reliability models of ASP have been carried out on the basis of mean square errors (MSEs) and absolute biases with 1000 replications of Monte-Carlo simulation. The estimates of augmented strength reliabilities and mean square errors and absolute biases for the generalized case of ASP are tabulated in the corresponding tables. From the given tables, it may be notice that the mean square errors (MSEs) and absolute biases gradually decrease for increasing values of sample sizes. It may be noticed that, the Bayes estimates and quasi-Bayes estimates for SELF and LLF performs quite effectively than that of ML estimates. It is seen from the tables that, there are not much differences among the different estimates with respect to loss functions, i.e., the Bayes and quasi-Bayes estimates give almost similar results for SELF and LLF. The choice of priors with minimum variability are suggestive. To validate both the methods of estimation, the data analysis was carried out with simulated as well as real data sets and it is observed that the Bayes and quasi-Bayes estimates gives the better fit in compare to ML method.
This present problem remains some open problems to the researchers for future attempt, the problem of Bayesian and ML estimation of augmenting strength reliability models for (i) some other life time distributions; (ii) different censoring schemes of life time experiments (iii) cost aspects of augmenting strength by adding new components and its cost estimation.

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Table 3: Average estimates, MSE and Absolute bias of the estimators of augmented strength reliability (R_3) under SELF and LLF with variation in λ and samples sizes (n_1,n_2) when $\alpha = 0.75, n = m = 2; a = 0.5, b = 0.75, c = 0.25, d = 0.75, p = 1.5$

(n_1,n_2)		MLE	Bayes		Quasi-Bayes	
			SELF	LLF	SELF	LLF
$\lambda = 0.5$; $R_3 = 0.816497$						
(10, 20)	Estimate	0.823853	0.777285	0.777179	0.77930	0.778898
	MSE	0.000663	0.001542	0.00155	0.001397	0.001427
	Abs. bias	0.007356	0.039212	0.039318	0.037197	0.037599
(20, 10)	Estimate	0.823541	0.777740	0.777584	0.774337	0.773809
	MSE	0.000663	0.001510	0.001522	0.001808	0.001852
	Abs. bias	0.007045	0.038757	0.038913	0.042160	0.042687
(20, 30)	Estimate	0.820333	0.799454	0.799349	0.791741	0.791325
	MSE	0.000329	0.000293	0.000296	0.000634	0.000655
	Abs. bias	0.003836	0.017043	0.017148	0.024755	0.025172
(30, 50)	Estimate	0.819213	0.799393	0.799325	0.795842	0.79543
	MSE	0.000211	0.000294	0.000296	0.000435	0.000452
	Abs. bias	0.002717	0.017103	0.017172	0.020655	0.021067
(50, 50)	Estimate	0.818179	0.789268	0.789210	0.789124	0.788731
	MSE	0.000145	0.000744	0.000747	0.000757	0.000779
	Abs. bias	0.001683	0.027229	0.027287	0.027373	0.027766
$\lambda = 1.5$; $R_3 = 0.929899$						
(10, 20)	Estimate	0.934318	0.898406	0.898244	0.907882	0.904306
	MSE	0.000540	0.000999	0.001009	0.000512	0.000681
	Abs. bias	0.004419	0.031493	0.031655	0.022017	0.025592
(20, 10)	Estimate	0.934490	0.887998	0.887562	0.907100	0.901410
	MSE	0.000559	0.001785	0.001821	0.000558	0.000850
	Abs. bias	0.004592	0.041901	0.042337	0.022799	0.028489

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(20, 30)	Estimate	0.932995	0.901384	0.901249	0.910285	0.908019
	MSE	0.000304	0.000822	0.000829	0.000401	0.000495
	Abs. bias	0.003096	0.028515	0.02865	0.019614	0.021880
(30, 50)	Estimate	0.931218	0.902236	0.902141	0.909598	0.908319
	MSE	0.000208	0.000772	0.000777	0.000421	0.000475
	Abs. bias	0.001319	0.027663	0.027757	0.020301	0.021579
(50, 50)	Estimate	0.931271	0.898933	0.898850	0.905873	0.904713
	MSE	0.000163	0.000964	0.000969	0.000585	0.000642
	Abs. bias	0.001372	0.030965	0.031049	0.024026	0.025186
$\lambda = 3$; $R_3 = 0.980338$						
(10, 20)	Estimate	0.981264	0.955612	0.95556	0.961330	0.958668
	MSE	0.000144	0.000614	0.000616	0.000370	0.000478
	Abs. bias	0.000926	0.024726	0.024778	0.019008	0.021670
(20, 10)	Estimate	0.980702	0.974492	0.974443	0.980357	0.977567
	MSE	0.000142	3.8e-050	3.9e-050	3.0e-060	1.2e-050
	Abs. bias	0.000363	0.005846	0.005896	1.9e-050	0.002771
(20, 30)	Estimate	0.980851	0.961341	0.961302	0.966366	0.964763
	MSE	9.5e-050	0.000364	0.000365	0.000199	0.000246
	Abs. bias	0.000513	0.018997	0.019037	0.013972	0.015576
(30, 50)	Estimate	0.980894	0.965043	0.965022	0.968324	0.967396
	MSE	5.5e-050	0.000236	0.000236	0.000146	0.000169
	Abs. bias	0.000556	0.015295	0.015316	0.012014	0.012942
(50, 50)	Estimate	0.980388	0.975704	0.975693	0.978172	0.977499
	MSE	4e-050	2.2e-050	2.2e-050	6e-060	9e-060
	Abs. bias	5e-050	0.004634	0.004646	0.002167	0.00284

Table 4: Average estimates, MSE and Absolute bias of the estimators of augmented strength reliability (R_3) under SELF and LLF with variation in **n** and samples sizes (n_1, n_2) when $\alpha = 0.75, \lambda = 1.5 n = 2; a = 0.5, b = 0.75, c = 0.25, d = 0.75, p = 1.5$

(n_1, n_2)		MLE	Bayes		Quasi-Bayes	
			SELF	LLF	SELF	LLF
$n = 3$; $R_3 = 0.984284$						
(10, 20)	Estimate	0.985109	0.968781	0.968691	0.977894	0.975662
	MSE	8.7e-05	0.000247	0.00025	4.7e-05	8.2e-05
	Abs. bias	0.000825	0.015504	0.015594	0.00639	0.008622
(20, 10)	Estimate	0.984311	0.967389	0.967225	0.979911	0.976371
	MSE	0.000101	0.000298	0.000304	2.7e-05	7.3e-05
	Abs. bias	2.7e-05	0.016896	0.01706	0.004373	0.007913
(20, 30)	Estimate	0.984711	0.968854	0.968799	0.975238	0.973851
	MSE	5.2e-05	0.000243	0.000245	8.6e-05	0.000114
	Abs. bias	0.000427	0.015431	0.015486	0.009046	0.010433
(30, 50)	Estimate	0.984742	0.962976	0.962929	0.968082	0.967224
	MSE	3.7e-05	0.000458	0.00046	0.000267	0.000295
	Abs. bias	0.000458	0.021309	0.021355	0.016203	0.01706
(50, 50)	Estimate	0.984466	0.977063	0.977043	0.980529	0.979884
	MSE	2.6e-05	5.4e-05	5.4e-05	1.6e-05	2.1e-05
	Abs. bias	0.000182	0.007221	0.007241	0.003755	0.004401
$n = 5$; $R_3 = 0.999281$						
(10, 20)	Estimate	0.999051	0.987265	0.987247	0.985257	0.984875
	MSE	2e-06	0.000145	0.000145	0.000199	0.00021
	Abs. bias	0.00023	0.012017	0.012035	0.014024	0.014407

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(20, 10)	Estimate	0.999065	0.978481	0.978442	0.977449	0.976853
	MSE	1e-06	0.000434	0.000435	0.000481	0.000507
	Abs. bias	0.000216	0.020801	0.020839	0.021833	0.022428
(20, 30)	Estimate	0.99915	0.998608	0.998608	0.999319	0.999207
	MSE	1e-06	4.8e-06	4.9e-07	1.9e-08	2.9e-08
	Abs. bias	0.000131	0.000673	0.000674	3.8e-05	7.4e-05
(30, 50)	Estimate	0.999208	0.99822	0.99822	0.99883	0.998736
	MSE	4.2e-07	1e-06	1e-06	2.3e-07	2.2e-07
	Abs. bias	7.3e-05	0.001061	0.001061	0.000451	0.000545
(50, 50)	Estimate	0.999253	0.991546	0.991540	0.993802	0.993531
	MSE	2.4e-07	6e-05	6e-05	3e-05	3.4e-05
	Abs. bias	2.8e-05	0.007735	0.007742	0.005479	0.005750

Table 5: Average estimates, MSE and Absolute bias of the estimators of augmented strength reliability (R_3) under SELF and LLF with variation in **m** and samples sizes (n_1, n_2) when $\alpha = 0.75, \lambda = 1.5 m = 2; a = 0.5, b = 0.75, c = 0.25, d = 0.75, p = 1.5$

(n_1, n_2)		MLE	Bayes		Quasi-Bayes	
			SELF	LLF	SELF	LLF
$m = 2.5;\ R_3 = 0.954803$						
(10, 20)	Estimate	0.958217	0.911241	0.911099	0.905190	0.904043
	MSE	0.000373	0.001900	0.001912	0.002485	0.002600
	Abs. bias	0.003414	0.043562	0.043703	0.049613	0.050759
(20, 10)	Estimate	0.956846	0.909120	0.908657	0.928975	0.923411
	MSE	0.000356	0.002119	0.002161	0.000703	0.001025
	Abs. bias	0.002043	0.045683	0.046146	0.025828	0.031392
(20, 30)	Estimate	0.956263	0.924730	0.924599	0.933589	0.931464
	MSE	0.000212	0.000913	0.000920	0.000464	0.000559
	Abs. bias	0.001461	0.030072	0.030204	0.021214	0.023339
(30, 50)	Estimate	0.955973	0.943939	0.943876	0.950188	0.949047
	MSE	0.000134	0.000122	0.000124	2.6e-05	3.8e-05
	Abs. bias	0.001171	0.010863	0.010926	0.004615	0.005755
(50, 50)	Estimate	0.955646	0.937522	0.937452	0.943672	0.942635
	MSE	1e-04	0.000303	0.000306	0.000129	0.000153
	Abs. bias	0.000843	0.017281	0.01735	0.011131	0.012168
$m = 5;\ R_3 = 0.990527$						
(10, 20)	Estimate	0.990475	0.963454	0.963324	0.974657	0.972343
	MSE	5.3e-05	0.000743	0.000751	0.000262	0.000341
	Abs. bias	5.2e-05	0.027073	0.027203	0.015871	0.018184
(20, 10)	Estimate	0.990123	0.970524	0.97038	0.982685	0.979879
	MSE	5.3e-05	0.000414	0.00042	6.9e-05	0.000123
	Abs. bias	0.000404	0.020003	0.020147	0.007842	0.010648
(20, 30)	Estimate	0.990433	0.97536	0.975305	0.981563	0.980388
	MSE	3.4e-05	0.000234	0.000236	8.4e-05	0.000107
	Abs. bias	9.4e-05	0.015167	0.015222	0.008964	0.010139
(30, 50)	Estimate	0.990662	0.99338	0.993377	0.994878	0.994586
	MSE	1.9e-05	8e-06	8e-06	1.9e-05	1.7e-05
	Abs. bias	0.000135	0.002853	0.00285	0.004351	0.004059
(50, 50)	Estimate	0.990429	0.986802	0.986792	0.989444	0.989031
	MSE	1.6e-05	1.5e-05	1.5e-05	2e-06	3e-06
	Abs. bias	9.8e-05	0.003726	0.003735	0.001083	0.001496

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Table 6: Average estimates, MSE and Absolute bias of the estimators of augmented strength reliability (R_3) under SELF and LLF with variation in hyper-parameters (a,b,c,d) and samples sizes (n_1,n_2) when $\alpha = 0.75, \lambda = 1.5 m = n = 2; R_3 = 0.929899$

(n_1,n_2)		MLE	Bayes		Quasi-Bayes	
			SELF	LLF	SELF	LLF
{(Small variance=0.25)} $a = 1.5, b = 0.56, c = 1.75, d = 0.75, p = 1.5$						
(10, 20)	Estimate	0.934724	0.88793	0.88777	0.883442	0.881445
	MSE	0.000533	0.001765	0.001778	0.002175	0.002366
	Abs. bias	0.004825	0.041969	0.042129	0.046457	0.048454
(20, 10)	Estimate	0.93437	0.873376	0.872905	0.891077	0.885499
	MSE	0.000514	0.003228	0.003281	0.001554	0.002019
	Abs. bias	0.004471	0.056523	0.056993	0.038822	0.044400
(20, 30)	Estimate	0.933653	0.881582	0.881471	0.885748	0.88395
	MSE	0.000345	0.002338	0.002349	0.001964	0.002125
	Abs. bias	0.003754	0.048317	0.048428	0.044151	0.045949
(30, 50)	Estimate	0.931871	0.907022	0.906937	0.913673	0.912357
	MSE	0.000188	0.000528	0.000532	0.000272	0.000316
	Abs. bias	0.001972	0.022877	0.022962	0.016225	0.017542
(50, 50)	Estimate	0.93095	0.913402	0.913326	0.919503	0.918333
	MSE	0.000151	0.000278	0.00028	0.000115	0.00014
	Abs. bias	0.001051	0.016497	0.016573	0.010396	0.011566
{(Moderate variance=5)} $a = 0.25, b = 0.30, c = 0.35, d = 0.60, p = 1.5$						
(10, 20)	Estimate	0.933952	0.876329	0.876188	0.876813	0.875695
	MSE	0.000517	0.002876	0.002891	0.002835	0.002954
	Abs. bias	0.004053	0.053569	0.053711	0.053086	0.054203
(20, 10)	Estimate	0.934842	0.958691	0.95858	0.967972	0.964274
	MSE	0.000556	0.000836	0.00083	0.001456	0.001189
	Abs. bias	0.004943	0.028792	0.028681	0.038073	0.034375
(20, 30)	Estimate	0.93285	0.886513	0.886415	0.890372	0.888597
	MSE	0.000316	0.001886	0.001895	0.001577	0.00172
	Abs. bias	0.002951	0.043386	0.043484	0.039527	0.041302
(30, 50)	Estimate	0.931498	0.894544	0.894486	0.898174	0.897045
	MSE	0.000206	0.001253	0.001257	0.001015	0.001088
	Abs. bias	0.001599	0.035355	0.035413	0.031725	0.032854
(50, 50)	Estimate	0.930934	0.914588	0.914514	0.920782	0.919605
	MSE	0.000157	0.00024	0.000242	8.9e-05	0.000112
	Abs. bias	0.001036	0.015311	0.015385	0.009117	0.010293
{(Large variance=10)} $a = 0.20, b = 0.40, c = 0.50, d = 2.50, p = 1.5$						
(10, 20)	Estimate	0.934376	0.914562	0.914416	0.924093	0.920474
	MSE	0.000521	0.000242	0.000246	5.5e-05	0.000109
	Abs. bias	0.004477	0.015337	0.015483	0.005806	0.009425
(20, 10)	Estimate	0.933585	0.872962	0.872535	0.888538	0.883028
	MSE	0.000533	0.003271	0.003319	0.001759	0.002244
	Abs. bias	0.003687	0.056937	0.057364	0.041361	0.046871
(20, 30)	Estimate	0.932557	0.920857	0.92072	0.929528	0.927202
	MSE	0.000324	9.2e-05	9.4e-05	1.1e-05	1.8e-05
	Abs. bias	0.002658	0.009042	0.009179	0.000371	0.002697
(30, 50)	Estimate	0.932105	0.909400	0.909316	0.915809	0.914499
	MSE	0.000203	0.000427	0.000430	0.000207	0.000246
	Abs. bias	0.002206	0.020499	0.020583	0.014090	0.015400
(50, 50)	Estimate	0.930578	0.889414	0.889323	0.896728	0.895601
	MSE	0.000153	0.001645	0.001652	0.001109	0.001185
	Abs. bias	0.000679	0.040484	0.040576	0.033171	0.034298

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