Quantum Physics Letters
An International Journal

http://dx.doi.org/10.18576/qpl/050101

Physical Consistency of Theories with Fermions in the Division Algebra Modules

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Received: 7 Nov. 2015, Revised: 21 Mar. 2016, Accepted: 23 Mar. 2016
Published online: 1 Apr. 2016

Abstract: The spinor space of the standard model can be cast in the form of a direct sum of a tensor product of division algebra modules. The consistency of interactions of fermions belonging to these modules is verified.

Keywords: division algebras, string theory, associativity anomaly, vertex operator algebras

1 Introduction

The condition of a nonvanishing interaction vertex between spinors in each irreducible product of fermion modules in the spinor space of the standard model is satisfied by composition algebras. The classification of alternative finite-dimensional composition algebras, necessary for the consistent evaluation of the diagrammatic expansion of the scattering matrix, restricts the dimensions of the algebras [1].

The analytic methods for the evaluation of diagrams in the expansion of the S-matrix can be extended to octonionic analysis. Then it may be determined whether there is any possibility of interactions of particles corresponding to elements of the octonion module in the spinor space $\oplus_{i=1}^{3}(\mathbb{C}\otimes\mathbb{H}\otimes\mathbb{O})$ [2]. There is a class of diagrams representing four-fermion interactions, for example, can occur with the spinors belonging to any division algebra [1]. The general three-point vertex, however, may not have a unique interpretation, because $(o_1o_2)o_3 \neq o_1(o_2o_3)$. With point particle interactions, the order in which the vertex wavefunctions are multiplied would not affect physical results. A string diagram, however, reveals a natural temporal order given the location of the interaction loops in space-time. The introduction of nonassociativity in string theory, therefore, is feasible.

There is an associativity anomaly that results from the interaction of three strings [3]. It may be removed by replacing the initial open string interaction by an overlap along the edges [4] which may be moved to the ends. The interactions at the ends have been verified by the range of the scalar fifth force [1].

The three-fermion vertex cannot exist in the diagrammatic expansion because the spin angular momentum is not conserved. Since $\frac{1}{2}\otimes\frac{1}{2} = 1\oplus 0$, the tensor product is not be matched with a spin-$\frac{1}{2}$ state. Instead the three-fermion vertex is replaced by a vertex with one boson and two fermions. Given the vector and two spinor representations of $SO(8)$, together with a restriction to the octonion module, the existence of this vertex is related to the triality automorphism of the octonions [5,6]. Furthermore, it is necessary for the formulation of a renormalizable gauge theory in place of the nonrenormalizable four-fermion theory in four dimensions [7,8].

2 The Twistor Formulation of Heterotic String Theory

The role of octonionic vertex interactions in string theory is elucidated by considering the twistor form of the ten-dimensional heterotic string action. The space-time vector X^μ and the 16-component anticommuting spinor θ^α in the Green-Schwarz theory satisfy the twistor equation $-\lambda^\alpha \Gamma^\mu{}_{\alpha\beta}\lambda^\beta = \partial_- X^\mu - i\theta^\alpha\Gamma^\mu{}_{\alpha\beta}\partial_-\theta^\beta$, where $\partial_- = \frac{1}{2}(\partial_\tau - \partial_\sigma)$ and $\partial_- \frac{1}{2}\left(\frac{\partial}{\partial\tau} - \frac{\partial}{\partial\sigma}\right)$, by a gamma matrix identity, may be written in terms of a set of commuting spinors λ^A, restricted to the sphere S^7,

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anticommuting spinors θ^A and X^{AB}, where $A,\ B = \pm$ and

$$\lambda^+ = \sum_{a=0}^{7} \lambda_a e^a, \lambda^- = \sum_{a=0}^{7} \lambda^{a+8} e_a, \theta^+ = \sum_{a=0}^{7} \theta^a e_a$$

$$\theta^- = \sum_{a=0}^{7} \theta^{a+8} e_a, X^{++} = \frac{1}{2}(X^0 + X^9), X^{--} = \frac{1}{2}(X^0 - X^9),$$

$$X^{+-} = \sum_{a=0}^{7} X^a e_a \text{ and } X^{-+} = \sum_{a=0}^{7} X^a \overline{e}_a$$

with $\{e_a\}$ being the basis for the octonions, including the bilinear: $2\lambda^A \overline{\lambda}^B = 2\partial_- X^{AB} + i\theta^A \overline{\partial_- \theta^B} - i\partial_- \theta^A \overline{\theta}^B$, which is invariant under the transformations $\delta\lambda^+ = \lambda^+ \left(\sum_{i=17}^{7} c^i e_i\right)$ and $\delta\lambda^- = \frac{(\lambda^- \overline{\lambda^+}}{|\lambda^+|^2} \left(\lambda^+ (\sum_{i=1}^{7} c^i e_i)\right)$ with $\partial_+ c^i = 0$ [9]. Given that the octonion commutators are $[e_i, e_j] = f^k{}_{ij} e_k$, where $f^k{}_{ij}$ are the structure constants,

$$[\delta_1, \delta_2]\lambda^+ = \delta_1 \left(\lambda^+ \left(\sum_{i=1}^{7} c_2^i e_i\right)\right) - \delta_2 \left(\lambda^+ \left(\sum_{i=1}^{7} c_1^i e_i\right)\right) \tag{2.1}$$
$$= \left(\sum_{a=0}^{7} \lambda^a e_a \sum_{j=1}^{7} c_1^j e_j\right) \left(\sum_{i=1}^{7} c_2^i e_i\right)$$
$$- \left(\sum_{a=0}^{7} \lambda^a e_a \sum_{j=1}^{7} c_2^j e_j\right) \sum_{i=1}^{7} c_1^i e_i$$
$$= 2\lambda^0 \sum_{k=1}^{7} c_1^i c_2^j f^k{}_{ij} e_k + \sum_{i,k=1}^{7} \lambda^k c_{2k} c_1^i e_i - \sum_{i,k=1}^{7} \lambda^k c_{1k} c_2^i e_i$$
$$- \sum_{i,j,k,\ell,m=1}^{7} (\lambda^k c_1^j f^\ell{}_{kj} c_2^j - \lambda^k c_2^j f^\ell{}_{kj} c_1^j)(f^m{}_{\ell i} e_m - \delta_{\ell i} e_0)$$

while

$$\lambda^+ \left(\sum_{i=1}^{7} c_3^i e_i\right) = \lambda_0 \sum_{i=1}^{7} c_3^i e_i + \sum_{i,j,k=1}^{7} \lambda^j c_3^i f^k{}_{ij} e_k - \sum_{k=1}^{7} \lambda^k c_{3k} e_0. \tag{2.2}$$

If $c_3 = 2\sum_{i,j=1}^{7} f^k{}_{ij} c_1^i c_2^j$,

$$2\lambda^0 \sum_{i,j,k=1}^{7} c_1^i c_2^j f^k{}_{ij} e_k = \lambda^0 \sum_{k=1}^{7} c_3^k e_k \sum_{i,j,k,\ell=1}^{7} (\lambda^k c_2^j f^\ell{}_{kj} c_1^i) \tag{2.3}$$
$$- \lambda^k c_1^j f^\ell{}_{kj} c_2^i \delta_{\ell i} e_0$$
$$= \sum_{i,j,k=1}^{7} (\lambda^k c_2^j f_{ikj} c_1^i - \lambda^k c_1^j f_{ikj} c_2^i) e_0$$
$$= -2 \sum_{i,j,k=1}^{7} f_{ijk} c_1^i c_2^j \lambda^k e_0 = -\sum_{k=1}^{7} \lambda^k c_{3k} e_0.$$

The remaining terms do not match because

$$- \sum_{i,j,k,\ell,m=1}^{7} (\lambda^k c_2^j f^\ell{}_{kj} c_1^i - \lambda^k c_1^j f^\ell{}_{kj} c_2^i) f^m{}_{\ell i} e_m \tag{2.4}$$
$$= - \sum_{i,j,k,\ell,m=1}^{7} \lambda^k (f^\ell{}_{jk} f^m{}_{i\ell} + f^\ell{}_{ki} f^m{}_{j\ell}) c_1^i c_2^j e_m \sum_{i,j,k=1}^{7} \lambda^j c_3^i f^k{}_{ji} e_k$$
$$= 2 \sum_{i,j,k,\ell,m=1}^{7} \lambda^k f^\ell{}_{ij} f^m{}_{k\ell} c_1^i c_2^j e_m.$$

From the identity $6[x,y,z] = [x,[y,z]] + [y,[z,x]] + [z,[x,y]]$ and $[e_i, e_j, e_k] = 2^m{}_{ijk} e_m$ [10], it follows that

$$\sum_{\ell=1}^{7} (f^\ell{}_{ij} f^m{}_{k\ell} + f^\ell{}_{ki} f^m{}_{j\ell} + f^\ell{}_{jk} f^m{}_{i\ell}) = 3\phi^m{}_{ijk}, \tag{2.5}$$

which does not yield the necessary cancellation and the commutator cannot be expressed as $\delta_3 \lambda^+$. The commutator also equals

$$\lambda^+ \left(\left(\sum_{j=1}^{7} c_1^j e_j\right) \left(\sum_{i=1}^{7} c_2^i e_i\right) \right) \tag{2.6}$$
$$- \lambda^+ \left(\left(\sum_{j=1}^{7} c_2^j e_j\right) \left(\sum_{i=1}^{7} c_i e_i\right) \right)$$
$$= \lambda^+ \sum_{i,j=1}^{7} c_1^i c_2^j (e_j e_i - e_i e_j) + [\lambda^+, c_1, c_2] - [\lambda^+, c_2, c_1]$$

and the associators equal

$$[\lambda^+, c_1, c_2] = \left[\lambda^0 e_0 + \sum_{i=1}^{7} \lambda^i e_i, \sum_{j=1}^{7} c_1^j e_j, \sum_{k=1}^{7} c_2^k e_k\right], \tag{2.7}$$
$$= \left[\lambda^0 e_0, \sum_{j=1}^{7} c_1^j e_j, \sum_{k=1}^{7} c_2^k e_k\right] + \left[\sum_{i=1}^{7} \lambda^i e_i, \sum_{j=1}^{7} c_1^j e_j \sum_{k=1}^{7} c_2^k e_k\right],$$
$$= \sum_{i,j,k=1}^{7} \lambda^i c_1^j c_2^k [e_i, e_j, e_k] = 2 \sum_{i,j,k=1}^{7} \phi^m ijk \lambda^i c_1^j c_2^k [\lambda^i, c_2, c_1]$$
$$= \sum_{i,j,k=1}^{7} \lambda^i c_2^j c_1^k [e_i, e_j, e_k] = 2 \sum_{i,j,k=1}^{7} \phi^m{}_{ikj} \lambda^i c_1^i c_2^k e_m$$

It follows from Eqs.(2.1), (2.4), (2.6) and (2.7) that

$$-2(\phi^m{}_{kij} - \phi^m{}_{kji}) = \sum_{\ell=1}^{7} (\delta_{ik} \delta_j{}^m - \delta_{jk} \delta^{im} + f^\ell f^m{}_{i\ell} + f^\ell{}_{ki} f^m{}_{j\ell} + 2 f^\ell{}_{ij} f^m{}_{k\ell}) \tag{2.8}$$

Similarly,

$$[\delta_1, \delta_2]\lambda^- = \delta_1 \left\{ \frac{(\lambda^- \overline{\lambda}^+)}{|\lambda^+|^2} \left(\lambda^+ \sum_{i=1}^{7} c_2^i e_i\right) \right\} - \delta_2 \left\{ \frac{(\lambda^- \overline{\lambda}^+)}{|\lambda^+|^2} \left(\lambda^+ \sum_{i=1}^{7} c_1^i e_i\right) \right\} \tag{(2.9)}$$
$$= \left(\frac{(\lambda^- \overline{\lambda}^+)}{|\lambda^+|^2} \left(\lambda^+ \sum_{j=1}^{7} c_1^j e_j\right) \frac{\overline{\lambda}^+}{|\lambda^+|^2} \right) \left(\lambda^+ \sum_{i=1}^{7} c_2^i e_i\right)$$
$$+ \left(\frac{\lambda^-}{|\lambda^+|^2} \left(\left(\sum_{j=1}^{7} c_1^j \overline{e}_j\right) \overline{\lambda}^+ \right) \right) \left(\lambda^+ \sum_{i=1}^{7} c_2^i e_i\right)$$
$$- \left(\lambda^- \overline{\lambda}^+ \left(\frac{1}{|\lambda^+|^4} (\delta_1 \lambda^+ \overline{\lambda}^+ + \lambda^+ \delta_1 \overline{\lambda}^+) \right) \right) \left(\lambda^+ \sum_{i=1}^{7} c_2^i e_i\right)$$
$$+ \frac{\lambda^- \overline{\lambda}^+}{|\lambda^+|^2} \left(\left(\lambda^+ \left(\sum_{j=1}^{7} c_1^j e_j\right) \right) \left(\sum_{i=1}^{7} c_2^i e_i\right) \right)$$
$$- \left(\frac{(\lambda^- \overline{\lambda}^+)}{|\lambda^+|^2} \left(\lambda^+ \sum_{j=1}^{7} c_2^j e_j\right) \frac{\overline{\lambda}^+}{|\lambda^+|^2} \right) \left(\lambda^+ \sum_{i=1}^{7} c_1^i e_i\right)$$
$$- \left(\frac{\lambda^-}{|\lambda^+|^2} \left(\left(\lambda^+ \sum_{j=1}^{7} c_2^j \overline{e}_j\right) \overline{\lambda}^+ \right) \right) \left(\lambda^+ \sum_{i=1}^{7} c_1^i e_i\right)$$
$$+ \left(\lambda^- \overline{\lambda}^+ \left(\frac{1}{|\lambda^+|^4} (\delta_2 \lambda^+ \overline{\lambda}^+ + \lambda^+ \delta_2 \overline{\lambda}^+) \right) \right) \left(\lambda^+ \sum_{i=1}^{7} c_1^i e_i\right)$$
$$- \frac{\lambda^- \overline{\lambda}^+}{|\lambda^+|^2} \left(\lambda^+ \left(\sum_{j=1}^{7} c_2^j e_j\right) \right) \left(\sum_{i=1}^{7} c_1^i e_i\right)$$

with $\overline{z_1 z_2} = \bar{z}_2 \bar{z}_1$, is not equal to $\delta_3 \lambda^-$ for another choice of c_3^k. Other symmetries such as κ-invariance generate a Kac-Moody-Malcev algebra with field-dependent commutation relations [11,12]. The current algebra of the superstring theory has been found to have the form $\hat{S}^7$, which may be demonstrated to represent an invariance of twistor-string theory [13].

The algebra $\hat{S}$ includes S^7, which is known to act on Hopf bundles $S^{15} \to S^8$ and the space of lightlike lines in ten dimensions [14]. Since the commutator algebra of the octonions does not satisfy a Jacobi identity, it cannot be integrated to a finite associative action. It follows that, an intermediate vector boson, if it is represented by vector fields on S^7, cannot be transported along a lightlike line in ten dimensions and form a state in a physical quantum theory with real observables unless it is combined with other vector bosons in a state transforming under a trivial representation.

The vertex operator algebra, defined by the associative operator product expansion of combinations of fermion fields for elements of a lattice and a new product which includes a phase and cocycle factor, has been demonstrated to isomorphic to the division algebras over the real numbers with orbits of order 2, 4 and 8 for A_n, C_{n+1} and F_4 [15]. These coefficients represent two-cocyles, and it has been noted that Dirac charge quantization is equivalent for the two-cycle and three-cocycle conditions through Poincare duality [16]. In ten dimensions, however, the three-cocycles would be distiguished from the two-cocycles, and there would be an effect on the nonassocativity of the product of three vertex operators.

The BRST operator for the Kac-Moody algebra $\hat{S}^7$ is $Q = c^i \mathscr{J}_i - T_{ij}{}^k c^i c^j b_k$. The nilpotency of the BRST charge $Q^2 = 0$ requires

$$\underline{\mathscr{J}_i \mathscr{J}_j} = -\frac{24[ij]}{(z-\zeta)^2} + \frac{1}{(z-\zeta)}(\mathscr{J} + 8X^*\partial X)_{[i,j]_X} \tag{2.10}$$

where $[ij]$ is the real component of the product [13]. The Sugawara construction relates the current algebra to the energy momentum tensor and the central charge [17]. When

$$\underline{\mathscr{J}_i(z)\mathscr{J}_j(\zeta)} = \frac{k\delta^{ij}}{(z-\zeta)^2} + \frac{2}{z-\zeta} : T_{ijk}(X)(\mathscr{J}_k - 8[X^*\partial X e_k])(\zeta) :, \tag{2.11}$$

the central charge is $\frac{7k}{k-12}$ [18]. Since $[ij] = -\delta^{ij}$ and $[i,j]_X = 2T_{ijk}(X)k$, $(X^*\partial X)_{[i,j]_X} = 2T_{ijk}(X)(X^*\partial X)_k$ and $[X^*\partial X e_k] = (X^*\partial X)_i [e_i e_k] = -(X^*\partial X)_i \delta^{ik} = -(X^*\partial X)_k$. Then

$$\underline{\mathscr{J}_i \mathscr{J}_j} = \frac{24\delta^{ij}}{(z-\zeta)^2} + \frac{2}{z-\zeta} : T_{ijk}(X)(\mathscr{J}_k + 8(X^*\partial X)_k)(\zeta) : \tag{2.12}$$

$$= \frac{k\delta^{ij}}{(z-\zeta)^2} + \frac{2}{z-\zeta} : T_{ijk}(X)(\mathscr{J}_k + 8(X^*\partial X)_k)(\zeta) :$$

if $k = 24$. Then the central charge is $c = 14$ and equal to the dimension of $S^7 \times S^7$. It may be noted that the contraction of $\hat{S}^7$ requires generators which belong to the Lie algebra of a group. There exists a matrix representation of the left and right multiplication by octonions with commutators that close in $so(7)$.

3 Three-Cocycles and Nonassociativity in String Theory

The phase factors in a $U(1)$ gauge theory, which represents the minimal set of variables that describes completely a field configuration, can be extended to arbitrary compact Lie groups through $exp(\oint \mathbf{A} \cdot d\mathbf{s})$. While the covariance of this operator under nonabelian gauge transformations can be verified for contours of infinitesimal size, and if the commutator of the covariant derivative vanishes, which requires a vanishing field strength, or valid to second order for infinitesimal gauge transformations when a scalar field is included [19], it will provide a model for determining the deformation of the translation group on a curved manifold and the cocycles which define the change in the tangent bundle to various submanifolds.

When the potential is Lie-algebra valued, the product of exponential factors again yields an infinite series that belongs to the Lie group. This correspondence is known not to be precise for vector field on parallelizable manifolds that are not Lie groups and infinite-dimensional algebras.

Suppose that $[A_1, A_2] \neq 0$ and $[A_1, [A_1, A_2]] + [[A_1, A_2], A_2] = 0$. Then $[A_1, ..., [A_1, [A_1, A_2]], ...] + ... + [..., [[A_1, A_2], A_2], ..., A_2] = 0$ and $e^{A_1} e^{A_2} = e^{A_1 + A_2 + \frac{1}{2}[A_1, A_2]}$, which does not equal $e^{\frac{1}{2}[A_1, A_2]} e^{A_1 + A_2}$ generally because that equality would require $\frac{1}{2}[A_1 + A_2, [A_1, A_2]] = \frac{1}{2}([A_1, [A_1, A_2]] - [[A_1, A_2], A_2])$ to vanish, which is not equivalent to the above condition unless $[A_1, [A_1, A_2]] = [[A_1, A_2], A_2] = 0$. Therefore, it will be required further that all higher-order commutators of the two elements of the algebra equal zero For three matrices, the conditions would be $[A_i, A_j] \neq 0$ and $[A_i, [A_i, A_j]] = [[A_i, A_j], A_j] = 0$ for $i \neq j$. and

$$(e^{A_1} e^{A_2}) e^{A_3} \tag{3.1}$$
$$= e^{A_1 + A_2 + \frac{1}{2}[A_1, A_2]} e^{A_3} = e^{A_1 + A_2 + A_3 + \frac{1}{2}[A_1, A_2] + \frac{1}{2}[A_1 + A_2 + \frac{1}{2}[[A_1, A_2], A_3]}$$
$$= e^{A_1 + A_2 + A_3 + \frac{1}{2}[A_1, A_2] + \frac{1}{2}[A_1, A_3] + \frac{1}{2}[A_2, A_3] + \frac{1}{4}[[A_1, A_2], A_3]} e^{A_1} (e^{A_2} e^{A_3})$$
$$= e^{A_1} (e^{A_2 + A_3 + \frac{1}{2}[A_2, A_3]}) = e^{A_1 + A_2 + A_3 + \frac{1}{2}[A_2, A_3] + \frac{1}{2}[A_1, A_2 + A_3 + \frac{1}{2}[A_2, A_3]]}$$
$$= e^{A_1 + A_2 + A_3 + \frac{1}{2}[A_1, A_2] + \frac{1}{2}[A_1, A_3] + \frac{1}{2}[A_2, A_3] + \frac{1}{4}[A_1, [A_2, A_3]]}$$

The difference between the exponents is

$$\frac{1}{4}([[A_1, A_2], A_3] - [A_1, [A_2, A_3]]) \tag{3.3}$$

The elements of a Lie algebra would satisfy

$$[[A_1, A_2], A_3] + [[A_3, A_1], A_2] + [[A_2, A_3], A_1] \tag{3.4}$$
$$= [[A_1, A_2], A_3] - [A_1, [A_2, A_3]] - [A_2, [A_3, A_1]] = 0.$$

yielding the difference of $\frac{1}{4}[A_2,[A_3,A_1]$. However, this formula cannot be taken to be valid until the two products in Eq.(3.1) are proven to be equal. This condition also may be satisfied if $[A_i,A_j]$ belong to the center of the algebra [20]. Therefore, the commutators $[[A_1,A_2],A_3]$ and $[A_1,[A_2,A_3]]$ must coincide for the mutliplication of the exponentials to be associative. This condition is separate and cannot be deduced from the commutation relations

$$[A_1,[A_1,A_2]] = [[A_1,A_2],A_2] = 0 \tag{3.5}$$
$$[A_1,[A_1,A_3]] = [[A_1,A_3],A_3]$$
$$= 0[A_2,[A_2,A_3]] = [[A_2,A_3,A_3] = 0.$$

Therefore, $[[A_1,A_2],A_3] - [A_1,[A_2,A_3]]$ measures nonassociativity in the multiplication rule of the exponential operators. Furthermore, the exponential factor $e^{\frac{1}{4}([[A_1,A_2],A_3]-[A_1,[A_2,A_3]]}$ may be identified with a three-cocycle.

It may be noted that a variety of nonassociative algebras can satisfy this property with the difference between the commutators replaced by an associator. Consider the basis $e_a = \{e_0, e_i\}$ of the octonions. Since the algebra is alternative, $e_i(e_ie_j) = (e_ie_i)e_j$ and $[e_i,e_i,e_j] = 0$. Since $[e_i,e_j,e_k] \neq 0$ for unequal i, j and k, it parallels the nonvanishing of the difference between the commutators.

The translation operator on a general curved manifold can be defined to be $e^{it\nabla}$ representing the coordinate transformation $e^{it\nabla}\psi(x) = e^{i\int_C \mathbf{A}\cdot d\mathbf{s}}\psi(exp_t(V)(\xi))$ where the covariant derivative represents parallel transport along the integral curve of the vector field V by an affine parameter t. Similar considerations to the exponential matrices yield measures of nonassociativity in the product of the translation operators. The interaction of strings introduces the equivalent of a gravitational field, and therefore, curvature in the manifold. Consequently, the associativity anomaly in the translation operator of open string field theory can be regarded as a indication of the a non-zero measure of nonassociativity in the products of the exponentials under conditions on the commutators of the covariant derivatives.

The conditions on the covariant derivatives of the manifold would be

$$[\nabla_{X_i},[\nabla_{X_i},\nabla_{X_j}]] - [[\nabla_{X_i},\nabla_{X_i}],\nabla_{X_j}] = 0. \tag{3.6}$$

Since $[\nabla_{X_i},\nabla_{X_i}] = 0$, it would follow that

$$[\nabla_{X_i},[\nabla_{X_i},\nabla_{X_j}]] = 0. \tag{3.7}$$

In a two-dimensional space spanned by the X_i and X_j, a conformal transformation exists such that the metric on the surface is flat. In string theory, the physical processes would be independent of the conformal transformation and consequently, $[\nabla_{X_i},[\nabla_{X_i},\nabla_{X_j}]] = 0$. It would not be possible to set $[\nabla_{X_i},[\nabla_{X_j},\nabla_{X_k}]]$ equal to zero. There exists a parallelizing torsion and a connection on S^7 such that the curvature may be set equal to zero. That allows the condition (3.7) to consistently imposed on the motion of the string while preserving the symmetries of a fermion vertex operator algebra. The nonvanishing of $[[\nabla_{X_i},\nabla_{X_j}],\nabla_{X_k}] - [\nabla_{X_i},[\nabla_{X_j},\nabla_{X_k}]]$ then would be a measure of nonassociativity in the products of the translation operators.

The absence of a correspondence of the exponential infinite-dimensional Lie algebras and Lie groups provides another method for establishing the nonassociativity of the operators products in open string field theory. The infinite-dimensional matrix products in the commutator of momentum and coordinate operators, $X(\frac{\pi}{2})$ and P^R, do not commute as a result of the order of the summation [2]. When the interaction occurs at the ends of the string, P^R can be removed together with the nonassociativity of these two operators with respect to a four-string state.

4 Conclusion

The restriction of finite-dimensional composition algebras over the real numbers allows a class of algebras that could serve as a basis of a theory of elementary particle interactions. The condition of alternativity is required for uniqueness of the evaluation of a four-point diagram [1]. Then, the algebras are restricted to be isomorphic to the real division algebras. It follows that the known subatomic forces can be described with fermion modules in these division algebras. The necessity for a formalism of this kind, however, would not yet be proven. Specifically, it may be established if the octonion algebra is required in a theory of the strong interactions.

String theory has been found to resolve the problem of divergences in the perturbation expansion of scattering elements in the quantum theory of the Einstein-Hilbert action. It is not known whether the one-dimensional nature of the fundamental objects is preserved at macroscopic scales. Nevertheless, the amplitudes are well described by a sum over the metrics of surfaces which is given by a moduli space integral at each genus. There is a potential problem with the nonassociativity in the evaluation of expectation values of star products of string fields. This nonassociativity has been observed in the translation generator which moves the string and must annihilate the state representing the three-string vertex while preserving the midpoint of the open string. This nonassociativity has been traced to the midpoint defining the region of interaction of the strings. The contour integrals defining the momentum operators is defined over a specified domain adjoining this region. The contour integral would be affected by the temporal order of the folding of the string field states. This result also can be derived from a measure of the nonassociativity of an exponential of a generator of translations that does not affect the fermion vertex operator algebra. Consequently, the nonassocativity can be deduced from algebraic invariances as well as summation ordering.

The removal of the associativity anomaly also reflects a formulation of the theory without the octonions. In addition to the methods suggested for eliminating this anomaly in string field theory, the transition from the vertex operator algebra with octonionic structure constants to fermion operator product expansions that yield an associative algebra provides a formalism which consists of an entirely consistent description of physical observables.

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