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Natural Transform Method to Solve Nonhomogeneous Fractional Ordinary Differential Equations

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Abstract: This paper is committed to solve nonhomogeneous fractional ordinary differential equations both of single and multiple fractional orders where in most cases, the celebrated Mittag-Leffler function is involved in the nonhomogeneous part. A vital result is established and the solution of forced nearly simple harmonic vibration equation is graphically represented amongst others.

Keywords: Natural transform, multiple order fractional differential equations, fractional calculus.

1 Introduction

Integral transform methods are amongst the most effective methods of solving both differential and integral equations of integer and fractional order. According to Belgacem and Silambarasan [1], integral transform method is most gifted technique in the mathematics world for solving differential equations.
The Natural transform evolved from the Fourier integral and it converges into Laplace transform [2,3,4] and Sumudu transform [5,6] given a unit value to each transform parameters respectively. The definition of the Natural transform and the study of their properties and applications were first done by Khan and Khan [7]. Several research works in connection with Natural transforms properties and applications are published in [1,7]. More about Laplace and Sumudu transforms relating to definition and properties, list of transform of several functions and applications are available in [2,3,4] and [8,9,10,11] respectively.
In this paper, we solve non-homogeneous fractional ordinary differential equations where the non-homogeneous part is a product of polynomial of fractional order and Mittag-Leffler functions using the Natural transform which makes it unique from the previous articles such as [12]. Indeed, the Natural transform is the generalization of the Laplace and Sumudu transforms hence, the technique has combined advantages associated with Laplace and Sumudu methods. An important theorem is proved and some basic definitions and concepts used in this paper are also presented. In Section 2 we review basic definitions and results. In Section 3 the Natural transform will be used to solve certain non-homogeneous fractional differential equations. In the last section we state our conclusions.

2 Basic Definitions

Fractional calculus: Given a function $v(t)$, the Caputo derivative [13] is defined as

$$
{}_0^C D_x^{\alpha}(v(x)) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} \frac{d^n v(t)}{dt^n} dt, & n-1<\alpha<n \\ \frac{d^n}{dt^n} v(t), & \alpha = n \end{cases} \tag{1}
$$

while Riemann-Liouville derivative [14] is

$$
D_x^{\alpha}(v(x)) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x (x-t)^{n-\alpha-1} v(t) dt, & n-1<\alpha<n \\ \frac{d^n}{dt^n} v(t), & \alpha = n \end{cases}, \tag{2}
$$

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where $n \in \mathbb{Z}^+$ $\alpha \in \mathbb{R}^+$.
The Riemann-Liouville fractional integral operator [14] of order $\alpha \geq 0$, of a function $v(t)$, is defined as

$$J^{\alpha} v(x) = \frac{1}{\Gamma\alpha} \int_0^x (x-t)^{\alpha-1} v(t) dt, \qquad \alpha > 0,\ x > 0. \tag{3}$$

The Natural transform of a real valued, sectionwise continuous and exponential order function $v(t) > 0$ and $v(t) = 0$ for $t < 0$ defined in the set

$$A = \left\{ v(t) \,|\exists M,\ \tau_1,\ \tau_2 > 0,\ |v(t)| < Me^{\frac{t}{\tau_j}},\ \text{if } t \in (-1)^j \times [0\infty) \right\}, \tag{4}$$

is given by [1]

$$\mathbb{N}^+ [v(t)] = R(s,u) = \int_0^{\infty} e^{-st} v(ut)\, dt;\ s > 0,\ u > 0, \tag{5}$$

where s and u are the transform variables. The Natural transform converges to Laplace transform when $u \equiv 1$ [2,3,4] while it converges to Sumudu transform when $s \equiv 1$ [5,6].

The inverse Natural transform is defined in [15]

$$\mathbb{N}^- [R(s,u)] = v(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\frac{st}{u}} R(s,u) ds. \tag{6}$$

The duality relation between Natural-Laplace and Natural –Sumudu transforms [1] are given by

$$R(s,u) = \frac{1}{u} F\left(\frac{s}{u}\right); R(s,u) = \frac{1}{s} G\left(\frac{u}{s}\right). \tag{7}$$

where $R(s,u)$ is the Natural transform, $F(s)$ is the Laplace transform and $G(u)$ is the Sumudu transform. The following standard results on Natural transform of derivatives, the Natural transform of fractional derivative and integral of order α, multiple shift and convolution theorem were found to be useful in this paper and are stated inform of lemmas but proof with other details are referred to [16,12].

Lemma 1.*If $f(t)$ is a function defined on the set A as in equation (4), and its nth order derivative of $f(t)$ with respect to t exists then, the Natural transform of the nth derivative is given by*

$$\mathbb{N}^+ \left[f'(t)\right] = \frac{s}{u} R(s,u) - \frac{f(0)}{u}, \tag{8}$$

$$\mathbb{N}^+ \left[f''(t)\right] = \frac{s^2}{u^2} \left[R(s,u) - s f(0)\right] - \frac{f'(0)}{u}, \tag{9}$$

and generally,

$$\mathbb{N}^+ \left[f^{(n)}(t)\right] = \frac{s^n}{u^n} R(s,u) - \sum_{k=0}^{n} \frac{s^{n-(k+1)}}{u^{n-k}} f^{(k)}(0). \tag{10}$$

where $R(s,u)$ denotes the Natural transform of the function $f(t)$. Furthermore, the Natural transforms of the fractional derivative and fractional integral both of order α of the function $f(t)$ are defined respectively by

$$\mathbb{N}^+ \left[D^{\alpha} \left[f(t)\right]\right] = \frac{s^{\alpha}}{u^{\alpha}} R(s,u) - \sum_{k=0}^{n} \frac{s^{\alpha-(k+1)}}{u^{\alpha-k}} f^{(k)}(0), \tag{11}$$

and

$$\mathbb{N}^+ \left[D^{-\alpha} \left[f(t)\right]\right] = \frac{u^{\alpha}}{s^{\alpha}} R(s,u). \tag{12}$$

Lemma 2.*If $F_1(s,u)$ and $F_2(s,u)$ are the Natural transforms of the functions $f_1(x)$ and $f_2(x)$ respectively, defined in set A then the convolution is given by*

$$\mathbb{N}^- \left\{uF_1(s,u) F_2(s,u)\right\} = \int_0^t f_1(x) f_2(t-x) dx. \tag{13}$$

Mittag-Leffler function: The special function

$$E_{\alpha}(x)=\sum_{k=0}^{\infty}\frac{x^{k}}{\Gamma(\alpha k+1)},\ \ \alpha>0, \tag{14}$$

and its general form

$$E_{\alpha,\beta}(x)=\sum_{k=0}^{\infty}\frac{x^{k}}{\Gamma(\alpha k+\beta)},\ \ \alpha>0,\ \ \beta>0, \tag{15}$$

are called Mittag-Leffler functions [17,18].

Theorem 1. *For $\alpha,\beta\in\mathbb{R}$ and if $f(t)$ is defined on A then,*

$$\mathbb{N}^{+}\left[t^{\beta-1}E_{\alpha,\beta}(\lambda t^{\alpha})\right]=\frac{1}{u}\left[\frac{\left(\frac{s}{u}\right)^{\alpha-\beta}}{\left(\frac{s}{u}\right)^{\alpha}-\lambda}\right],\ \text{for all }\alpha>0,\ \beta>0. \tag{16}$$

Proof: *By definition of the Natural transform*

$$\mathbb{N}^{+}[v(t)]=R(s,u)=\frac{1}{u}\int_{0}^{\infty}e^{-st}v(ut)\,dt\quad\text{or}\quad\mathbb{N}^{+}[v(t)]=R(s,u)=\frac{1}{u}\int_{0}^{\infty}e^{-\frac{st}{u}}v(t)\,dt,$$

so that

$$\mathbb{N}^{+}\left[t^{\beta-1}E_{\alpha,\beta}(\lambda t^{\alpha})\right]=\frac{1}{u}\int_{0}^{\infty}e^{-\frac{st}{u}}t^{\beta-1}E_{\alpha,\beta}(\lambda t^{\alpha})\,dt=\frac{1}{u}\int_{0}^{\infty}e^{-\frac{st}{u}}t^{\beta-1}\sum_{k=0}^{\infty}\frac{(\lambda t^{\alpha})^{k}}{\Gamma(\alpha k+\beta)}dt,$$

$$=\frac{1}{u}\sum_{k=0}^{\infty}\frac{\lambda^{k}}{\Gamma(\alpha k+\beta)}\int_{0}^{\infty}e^{-\frac{st}{u}}t^{\beta+\alpha-1}dt=\frac{1}{u}\sum_{k=0}^{\infty}\frac{\lambda^{k}}{\Gamma(\alpha k+\beta)}\int_{0}^{\infty}e^{-\frac{st}{u}}t^{\beta+\alpha k-1}dt.$$

Putting $x=\frac{st}{u}$,

$$\frac{1}{u}\sum_{k=0}^{\infty}\frac{\lambda^{k}}{\Gamma(\alpha k+\beta)}\int_{0}^{\infty}e^{-\frac{st}{u}}t^{\beta+\alpha k-1}dt=\frac{1}{u}\sum_{k=0}^{\infty}\frac{\lambda^{k}}{\Gamma(\alpha k+\beta)\left(\frac{s}{u}\right)^{\beta+\alpha k}}\int_{0}^{\infty}e^{-x}x^{\beta+\alpha k-1}dx,$$

$$=\frac{1}{u}\sum_{k=0}^{\infty}\frac{\lambda^{k}}{\Gamma(\alpha k+\beta)\left(\frac{s}{u}\right)^{\beta+\alpha k}}\Gamma(\alpha k+\beta)=\frac{1}{u}\frac{1}{\left(\frac{s}{u}\right)^{\beta}}\sum_{k=0}^{\infty}\frac{\lambda^{k}}{\left(\frac{s}{u}\right)^{\alpha k}},$$

$$=\frac{1}{u}\frac{1}{\left(\frac{s}{u}\right)^{\beta}}\sum_{k=0}^{\infty}\left(\frac{\lambda}{\left(\frac{s}{u}\right)^{\alpha}}\right)^{k}=\frac{1}{u}\frac{1}{\left(\frac{s}{u}\right)^{\beta}}\left[\frac{\left(\frac{s}{u}\right)^{\alpha}}{\left(\frac{s}{u}\right)^{\alpha}-\lambda}\right]=\frac{1}{u}\left[\frac{\left(\frac{s}{u}\right)^{\alpha-\beta}}{\left(\frac{s}{u}\right)^{\alpha}-\lambda}\right],$$

as required.

Remarks: The following results are worth noting as special cases of the theorem above:

1.

$$\text{If }\alpha>0,\ \beta=1\text{ then, }\ \mathbb{N}^{+}[E_{\alpha,1}(\lambda t^{\alpha})]=\frac{1}{u}\frac{1}{\left(\frac{s}{u}\right)}\left[\frac{\left(\frac{s}{u}\right)^{\alpha}}{\left(\frac{s}{u}\right)^{\alpha}-\lambda}\right]. \tag{17}$$

2.

$$\text{If }\alpha>0,\ \beta=\alpha\text{ then, }\ \mathbb{N}^{+}\left[t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^{\alpha})\right]=\frac{1}{u}\left[\frac{1}{\left(\frac{s}{u}\right)^{\alpha}-\lambda}\right]. \tag{18}$$

3.

$$\text{If }\alpha>0,\ \beta=1\text{ then, }\mathbb{N}^{+}[1-E_{\alpha}(-\lambda t^{\alpha})]=\frac{1}{u}\frac{1}{\left(\frac{s}{u}\right)}\left[\frac{\lambda}{\left(\frac{s}{u}\right)^{\alpha}+\lambda}\right]. \tag{19}$$

3 Natural Transform Application

In this section, the natural transform will be used to solve certain non-homogenous fractional differential equations.
Example 1. Solve a forced nearly simple harmonic vibration equation using the Natural transform,

$$D^{\alpha}[V(t)]+\omega^{2}V(t)=k, \qquad 1\leq\alpha\leq 2 \tag{20}$$

with the initial conditions

$$V(0)=c_0,\ \ V'(0)=c_1.$$

Solution: By taking the Natural transform of equation (20) and applying the initial conditions, using equation (11) and together with relevant standard results (16) and (19), it gives

$$\mathbb{N}^{+}[D^{\alpha}[V(t)]]+\omega^{2}\mathbb{N}^{+}[V(t)]=k\mathbb{N}^{+}[1]$$

$$\left(\frac{s}{u}\right)^{\alpha}R(s,u)-\sum_{k=0}^{n}\frac{s^{\alpha-(k+1)}}{u^{\alpha-k}}f^{(k)}(0)+\omega^{2}R(s,u)=\frac{1}{u}\frac{k}{\left(\frac{s}{u}\right)},$$

i.e.

$$\left(\frac{s}{u}\right)^{\alpha}R(s,u)-\frac{s^{\alpha-1}}{u^{\alpha}}V(0)-\frac{s^{\alpha-2}}{u^{\alpha-1}}V'(0)+\omega^{2}R(s,u)=\frac{1}{u}\frac{k}{\left(\frac{s}{u}\right)},$$

or

$$\left(\frac{s}{u}\right)^{\alpha}R(s,u)+\omega^{2}R(s,u)=\frac{1}{u}\left[\frac{s^{\alpha-1}}{u^{\alpha-1}}c_0+\frac{s^{\alpha-2}}{u^{\alpha-2}}c_1+\frac{k}{\left(\frac{s}{u}\right)}\right],$$

or

$$\left[\left(\frac{s}{u}\right)^{\alpha}+\omega^{2}\right]R(s,u)=\frac{1}{u}\left[\frac{s^{\alpha-1}}{u^{\alpha-1}}c_0+\frac{s^{\alpha-2}}{u^{\alpha-2}}c_1+\frac{k}{\left(\frac{s}{u}\right)},\right]$$

$$R(s,u)=\frac{1}{u}\left[\frac{\left(\frac{s}{u}\right)^{\alpha-1}}{\left[\left(\frac{s}{u}\right)^{\alpha}+\omega^{2}\right]}c_0+\frac{\left(\frac{s}{u}\right)^{\alpha-2}}{\left[\left(\frac{s}{u}\right)^{\alpha}+\omega^{2}\right]}c_1+\frac{k}{\omega^{2}}\frac{\omega^{2}}{\left(\frac{s}{u}\right)\left[\left(\frac{s}{u}\right)^{\alpha}+\omega^{2}\right]}\right].$$

Taking the inverse Natural transform, it gives

$$V(t)=c_0E_{\alpha,1}\left(-\omega^{2}t^{\alpha}\right)+c_1tE_{\alpha,2}\left(-\omega^{2}t^{\alpha}\right)+\frac{k}{\omega^{2}}\left[1-E_{\alpha}(-\omega^{2}t^{\alpha})\right]. \tag{21}$$

The solution is plotted below in Figure 1a and 1b at different values of ω for $c_0=c_1=k=1$:

Example 2. Solve the non-homogeneous fractional ordinary differential equation using Natural transform method

$$D^{\alpha}[V(t)]+V(t)=t^{\beta-1}E_{\alpha,\beta}(t^{\alpha}),\ \ 0<\alpha\leq 1,\ \ \beta>0,\ \ t>0, \tag{22}$$

subject to initial condition $V(0)=0$.
Solution: By invoking the Natural transform on equation (18) and applying the initial conditions, using equations (11) and together with standard results from Theorem 1, so that

$$\mathbb{N}^{+}[D^{\alpha}[V(t)]]+\mathbb{N}^{+}[V(t)]=\mathbb{N}^{+}\left[t^{\beta-1}E_{\alpha,\beta}(t^{\alpha})\right],$$

$$\frac{s^{\alpha}}{u^{\alpha}}R(s,u)-\sum_{k=0}^{n}\frac{s^{\alpha-(k+1)}}{u^{\alpha-k}}f^{(k)}(0)+R(s,u)=\frac{1}{u}\left[\frac{\left(\frac{s}{u}\right)^{\alpha-\beta}}{\left(\frac{s}{u}\right)^{\alpha}-1}\right],$$

$$\left[\left(\frac{s}{u}\right)^{\alpha}+1\right]R(s,u)=\frac{1}{u}\left[\frac{\left(\frac{s}{u}\right)^{\alpha-\beta}}{\left(\frac{s}{u}\right)^{\alpha}-1}\right],$$

$$R(s,u)=\frac{1}{u}\frac{1}{\left[\left(\frac{s}{u}\right)^{\alpha}+1\right]}\left[\frac{\left(\frac{s}{u}\right)^{\alpha-\beta}}{\left(\frac{s}{u}\right)^{\alpha}-1}\right],$$

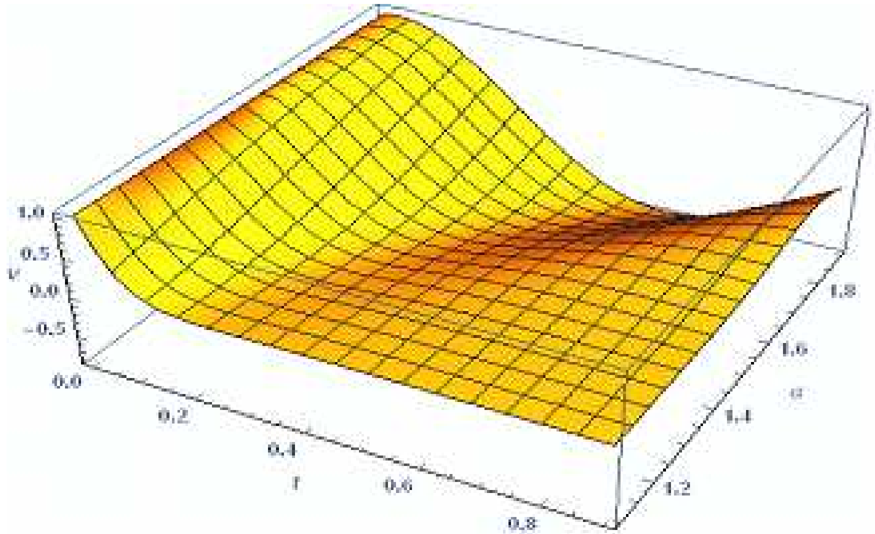

Figure 1a: A 3D plot of solution (21) as a function of t and α for $\omega = 5$.

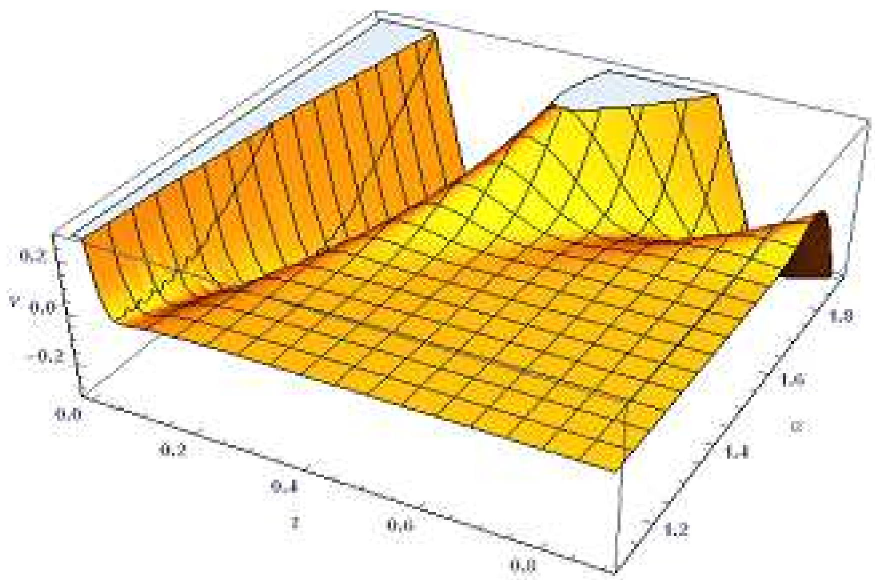

Figure 1b: A 3D plot of solution (21) as a function of t and α for $\omega = 10$.

$$V(t) = \mathbb{N}^{-}\left\{\frac{1}{u}\frac{1}{\left[\left(\frac{s}{u}\right)^{\alpha}+1\right]}\left[\frac{\left(\frac{s}{u}\right)^{\alpha-\beta}}{\left(\frac{s}{u}\right)^{\alpha}-1}\right]\right\}.$$

Now, the convolution theorem is easily applied to evaluate the inverse Natural transform thus:

$$\mathbb{N}^{-}\{uF_1(s,u).F_2(s,u)\} = \int_0^t f_1(x)\, f_2(t-x)dx,$$

$$\mathbb{N}^{-}\left\{\frac{1}{\left[\left(\frac{s}{u}\right)^{\alpha}+1\right]}\right\} = t^{\alpha-1}E_{\alpha,\alpha}(t^{\alpha}),$$

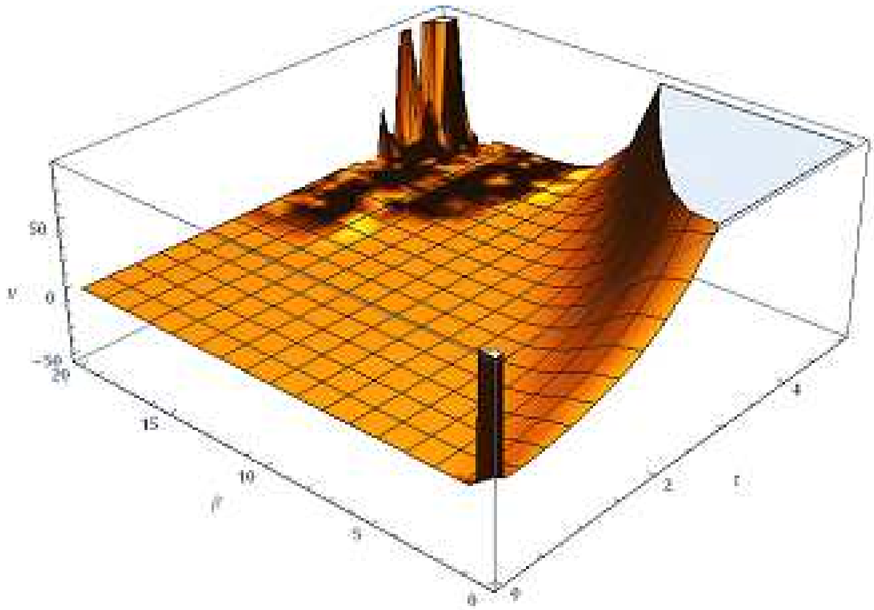

Figure 2a: A 3D plot of solution (23) as a function of t and β for $\alpha = 0.1$.

$$\mathbb{N}^{-}\left[\frac{\left(\frac{s}{u}\right)^{\alpha-\beta}}{\left(\frac{s}{u}\right)^{\alpha}-1}\right] = t^{\beta-1}E_{\alpha,\beta}\left(-t^{\alpha}\right),$$

$$\mathbb{N}^{-}\left\{\frac{1}{u}\frac{1}{\left[\left(\frac{s}{u}\right)^{\alpha}+1\right]}\left[\frac{\left(\frac{s}{u}\right)^{\alpha-\beta}}{\left(\frac{s}{u}\right)^{\alpha}-1}\right]\right\} = \left[t^{\alpha-1}E_{\alpha,\alpha}\left(t^{\alpha}\right)\right]*\left[t^{\beta-1}E_{\alpha,\beta}\left(-t^{\alpha}\right)\right] = \int_0^t x^{\alpha-1}E_{\alpha,\alpha}\left(x^{\alpha}\right)(t-x)^{\beta-1}E_{\alpha,\beta}\left[-(t-x)^{\alpha}\right]dx,$$

Comparing with the standard result below as in [16]

$$\int_0^x u^{\gamma-1}E_{\alpha,\gamma}\left(yu^{\alpha}\right)(x-u)^{\beta-1}E_{\alpha,\beta}\left[z(x-u)^{\alpha}\right]du = \frac{yE_{\alpha,\beta+\gamma}(yx^{\alpha}) - zE_{\alpha,\beta+\gamma}(zx^{\alpha})}{y-z}x^{\beta+\gamma-1},$$

where $y,z \in C; y \neq z,\ \ \beta > 0,\ \gamma > 0$, it implies that

$$\int_0^t x^{\alpha-1}E_{\alpha,\alpha}\left(x^{\alpha}\right)(t-x)^{\beta-1}E_{\alpha,\beta}\left[-(t-x)^{\alpha}\right]dx = \frac{E_{\alpha,\beta+\alpha}\left(t^{\alpha}\right)+E_{\alpha,\beta+\alpha}(-t^{\alpha})}{2}t^{\beta+\alpha-1},$$

$$\therefore\ V(t) = \frac{E_{\alpha,\beta+\alpha}\left(t^{\alpha}\right)+E_{\alpha,\beta+\alpha}(-t^{\alpha})}{2}t^{\beta+\alpha-1}. \tag{23}$$

The solution (23) is also plotted at four different values of α against the t and β in Figure 2a-2c.

Example 3. Solve the multi-order non-homogeneous fractional ordinary differential equation using Natural transform method.

$$D^{\alpha}\left[V(t)\right] + D^{\beta}\left[V(t)\right] = -V(t) + t^{-\alpha}E_{1,1-\alpha}(t) + t^{-\beta}E_{1,1-\beta}(t) + e^{t} \tag{24}$$

subject to initial condition $V(0) = 0$.

Solution: By invoking the Natural transform on equation (24) and applying the initial conditions, using equations (11) and standard results from Theorem 1,

$$\mathbb{N}^{+}\left[D^{\alpha}\left[V(t)\right]+D^{\beta}\left[V(t)\right]\right] = -\mathbb{N}^{+}\left[V(t)\right] + \mathbb{N}^{+}\left[t^{-\alpha}E_{1,1-\alpha}(t)\right] + \mathbb{N}^{+}\left[t^{-\beta}E_{1,1-\beta}(t)\right] + \mathbb{N}^{+}\left[e^{t}\right],$$

$$\left(\frac{s}{u}\right)^{\alpha}R(s,u) - \sum_{k=0}^{n}\frac{s^{\alpha-(k+1)}}{u^{\alpha-k}}f^{(k)}(0) + \left(\frac{s}{u}\right)^{\beta}R(s,u) - \sum_{k=0}^{n}\frac{s^{\beta-(k+1)}}{u^{\beta-k}}f^{(k)}(0) + R(s,u)$$

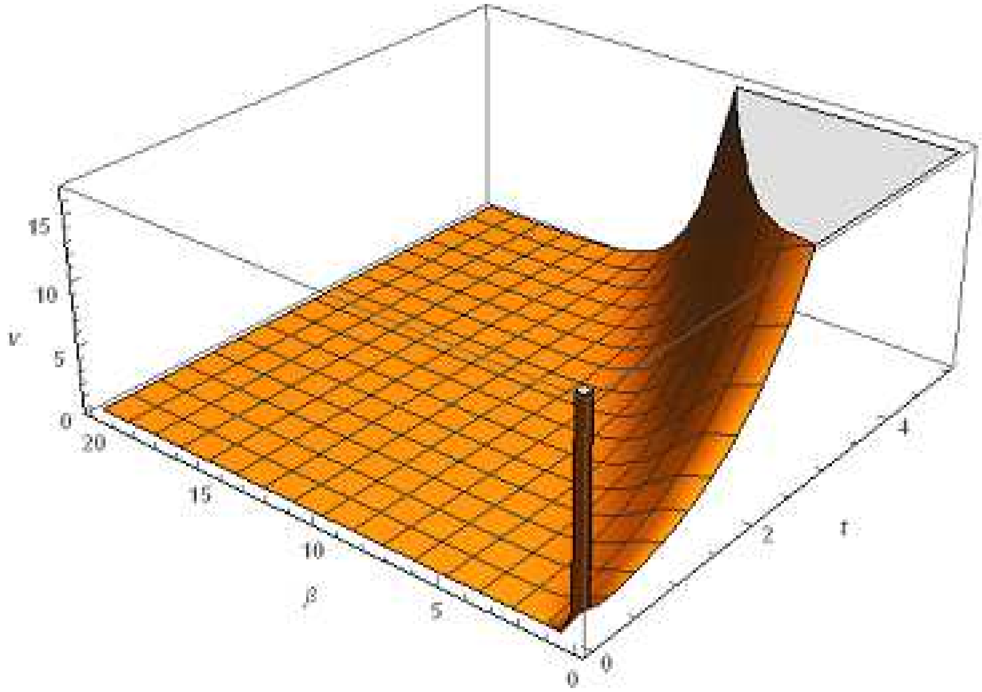

Figure 2b: A 3D plot of solution (23) as a function of t and β for $\alpha = 0.5$.

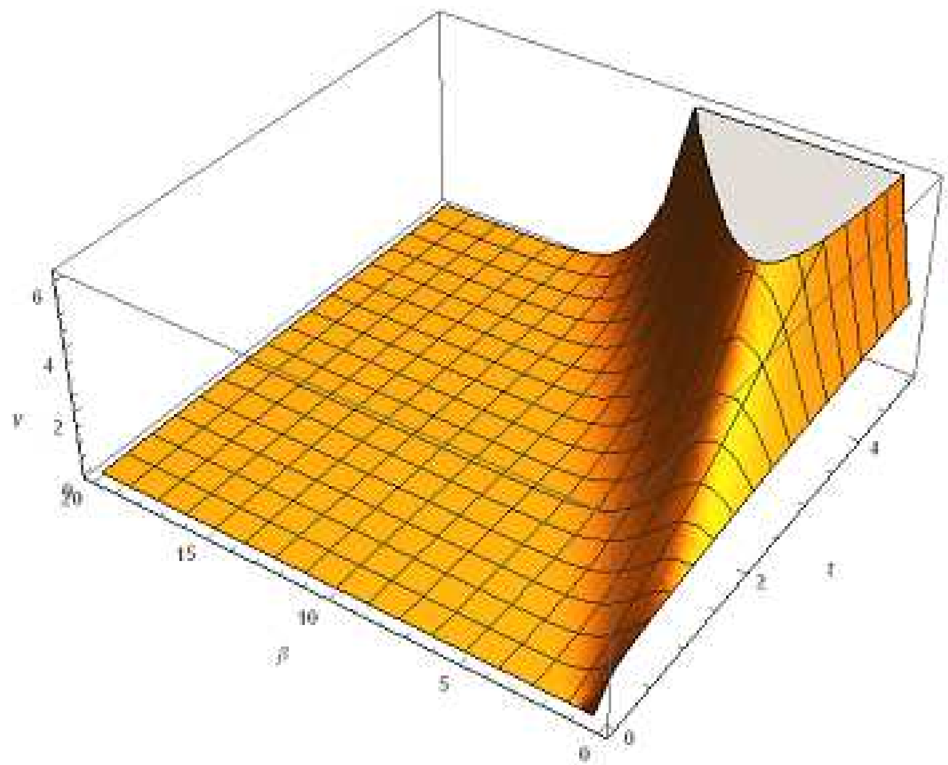

Figure 2c: A 3D plot of solution (23) as a function of t and β for $\alpha = 0.9$.

$$=\frac{1}{u}\left[\frac{\left(\frac{s}{u}\right)^{\alpha}}{\frac{s}{u}-1}\right]+\frac{1}{u}\left[\frac{\left(\frac{s}{u}\right)^{\beta}}{\frac{s}{u}-1}\right]+\frac{1}{u}\left[\frac{1}{\frac{s}{u}-1}\right],$$

$$\left(\frac{s}{u}\right)^{\alpha}R(s,u)-D^{\alpha-1}[V(t)]_{t=0}+\left(\frac{s}{u}\right)^{\beta}R(s,u)-D^{\beta-1}[V(t)]_{t=0}+R(s,u)$$

$$=\frac{1}{u}\left[\frac{1}{\frac{s}{u}-1}\right]\left[\left(\frac{s}{u}\right)^{\alpha}+\left(\frac{s}{u}\right)^{\beta}+1\right],$$

$$\left(\frac{s}{u}\right)^{\alpha}R(s,u)+\left(\frac{s}{u}\right)^{\beta}R(s,u)+R(s,u)=\frac{1}{u}\left[\frac{1}{\frac{s}{u}-1}\right]\left[\left(\frac{s}{u}\right)^{\alpha}+\left(\frac{s}{u}\right)^{\beta}+1\right],$$

$$R(s,u)=\frac{1}{u}\left[\frac{1}{\frac{s}{u}-1}\right],V(t)=\mathbb{N}^{-}\left\{\frac{1}{u}\left[\frac{1}{\frac{s}{u}-1}\right]\right\}=e^{t}. \tag{25}$$

Example 4. Solve the non-homogeneous fractional ordinary differential equation

$$D^{\alpha}[V(t)]-V(t)=\frac{t^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)}-\frac{t^{\beta-1}}{\Gamma(\beta)},\ \ 0<\alpha<1,\ \ \infty<\beta<\infty,\ t>0, \tag{26}$$

subject to initial condition $V(0)=0$.
Solution: By invoking the Natural transform on equation (26) and applying the initial conditions, using standard results from [1]

$$\mathbb{N}^{+}[D^{\alpha}[V(t)]]-\mathbb{N}^{+}[V(t)]=\mathbb{N}^{+}\left[\frac{t^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)}\right]-\mathbb{N}^{+}\left[\frac{t^{\beta-1}}{\Gamma(\beta)}\right],$$

$$\frac{s^{\alpha}}{u^{\alpha}}R(s,u)-\sum_{k=0}^{n}\frac{s^{\alpha-(k+1)}}{u^{\alpha-k}}f^{(k)}(0)-R(s,u)=\frac{1}{\Gamma(\beta-\alpha)}\mathbb{N}^{+}\left[t^{\beta-\alpha-1}\right]-\frac{1}{\Gamma(\beta)}\mathbb{N}^{+}\left[t^{\beta-1}\right],$$

$$\frac{s^{\alpha}}{u^{\alpha}}R(s,u)-D^{\alpha-1}[V(t)]_{t=0}-R(s,u)=\frac{1}{\Gamma(\beta-\alpha)}\Gamma(\beta-\alpha)\frac{u^{\beta-\alpha-1}}{s^{\beta-\alpha}}-\frac{1}{\Gamma(\beta)}\Gamma(\beta)\frac{u^{\beta-1}}{s^{\beta}},$$

but
$D^{\alpha-1}[V(t)]_{t=0}=0.$

$$\frac{s^{\alpha}}{u^{\alpha}}R(s,u)-R(s,u)=\frac{u^{\beta-\alpha-1}}{s^{\beta-\alpha}}-\frac{u^{\beta-1}}{s^{\beta}},$$

$$\left(\frac{s^{\alpha}}{u^{\alpha}}-1\right)R(s,u)=\left(\frac{u^{-\alpha}}{s^{-\alpha}}-1\right)\frac{u^{\beta-1}}{s^{\beta}},\ \ R(s,u)=\frac{u^{\beta-1}}{s^{\beta}}=\frac{1}{\Gamma(\beta)}\Gamma(\beta)\frac{u^{\beta-1}}{s^{\beta}}.$$

By taking the inverse Natural transform

$$V(t)=\frac{1}{\Gamma(\beta)}\mathbb{N}^{-}\left[\Gamma(\beta)\frac{u^{\beta-1}}{s^{\beta}}\right]=\frac{t^{\beta-1}}{\Gamma(\beta)},$$

$$\therefore V(t)=\frac{t^{\beta-1}}{\Gamma(\beta)}. \tag{27}$$

4 Conclusion

In this paper, the Natural transform was applied to solve single and multi-order nonhomogeneous fractional differential equations. Some basic concepts about the Natural transform and fractional calculus were also presented as background of the work.

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References

[1] F. B. M. Belgacem and R. Silambarasan, Theory of natural transform, *Math. Eng. Sci. Aerospace (MESA)* **3**(1), 99-124 (2012).
[2] M. R. Spiegel, *Theory and Problems of Laplace Transforms*, Schaums Outline Series, McGraw-Hill, New York, 1965.
[3] L. Debnath and D. Bhatta, *Integral Transforms and Their Applications*,2^{nd} edition, C. R. C. Press, London, 2007.
[4] J. L. Schiff, *Laplace Transforms Theory and Applications*, Auckland, New-Zealand.Springer, 2005.
[5] F. B. M. Belgacem, A. A. Karaballi and S. L. Kalla, Analytical investigations of the Sumudu transform and applications to integral production equations, *Math. Probl. Eng.* **3**, 103–118 (2003).
[6] F. B. M. Belgacem and A. A. Karaballi, Sumudu transform fundamental properties investigations an applications, *J. Appl. Math. Stoch. Anal.* **2006**, Article ID 91083 (2006).
[7] Z. H. Khan and W. A. Khan, N-transform properties and applications. *NUST J. Eng. Scie.* **1**(1), 127–133 (2008).
[8] F. B. M. Belgacem, Introducing and analysing deeper Sumudu properties, *Nonlin. Stud. J.***13**(1), 23–41 (2006).
[9] F. B. M. Belgacem, Sumudu transform applications to Bessel's functions and equations, *Appl. Math. Sci.* **4**(74), 3665–3686 (2010).
[10] A. Kilicman and H. Eltayeb, A note on the Sumudu transforms and differential equations, *Appl. Math. Sci.***4**(22), 1089–1098 (2010).
[11] A. Kilicman, H. Eltayeb and R. P. Agarwal, On Sumudu transform and system of differential equations, *Abstr. Appl. Anal.* **(2010)**, 1–11 (2010).
[12] D. Loonker and P. K. Banerji, Solution of fractional ordinary differential equations by natural transform, *Int. J. Math. Eng. Sci.*. **2**(12), 1–7 (2013).
[13] I. Podlubny, *Fractional Differential Equations, Vol. 198, Mathematics in Science and Engineering*, Academic Press, San Diego, USA, 1999.
[14] H. Bulut, H. M. Baskonus and F. B. M. Belgacem, The analytical solution of some fractional ordinary differential equations by the Sumudu transform method, *Abstr. Appl. Anal.* **2013**, 1–6 (2013).
[15] R. Silambarasan and F. B. M. Belgacem, Applications of the Natural transform to Maxwell's equations, *Progress of Electromagnetic Research Symposium Proc. Suzhou, China*, 899 – 902, (2011).
[16] A. Kilicman and H. Eltayeb, On a new integral transform and differential equations, *Math Probl. Eng.* **2010** 1–13, (2010).
[17] R. Magin, *Fractional Calculus in Bioengineering*, Begell House Publishers, 2006.
[18] A. Mathai and H. Haubold, *Special Functions for Applied Scientists*, Springer, 2008.