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The Distribution Having Power Hazard Function (DPHF) Based on Orderd Random Variables

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Abstract: In this paper, recurrence relations for single and product moments of generalized order statistics *(gos)* from the distribution having power hazard function (DPHF) have been established. These relations are deduced for moments of order statistics and upper record values. Further,this distribution has been characterized through the recurrence relation for a single moments and minimal order statistics from (DPHF).

Keywords: Single and product moments, order statistics, upper record values and characterization.

1 Introduction

The concept of generalized order statistics (gos) has been introduced and extensively studied by Kamps [5].

Let $n \in N$, be a given integer and $\widetilde{m} = (m_1, m_2, ..., m_{n-1}) \in \Re^{n-1}, k \geq 1$ be the parameters such that $\gamma_i = k + n - i + \sum_{j=1}^{n-1} m_j > 0$ for $1 \leq i \leq n-1$,

Then $X(1,n,\widetilde{m},k), X(2,n,\widetilde{m},k), ..., X(r,n,\widetilde{m},k)$ are called gos from continuous population with the cummulative distribution function (cdf) $F(x)$ and the probability density function *(pdf) f(x)*if their joint pdf has the form

$$k\left(\prod_{j=1}^{n=1}\gamma_j\right)\left(\prod_{i=1}^{n=1}[1-F(x_i)]^{m_i} f(x_i)\right)[1-F(x_n)]^{k-1} f(x_n) \tag{1}$$

on the cone $F^{-1}(0+) < x_1 \leq x_2 \leq ... \leq x_n < F^{-1}(1)$ of $\Re^n$,

Choosing the parameters appropriatly, models such as ordinary order statistics $(\gamma_i = n-i-1; i = 1,2,...,n,$ i.e $m_1 = m_2 = m_{n-1} = 0)$, k-*th* record values $(\gamma_i = k$ i.e. $m_1 = m_2 = m_{n-1} = -1, k \in N)$, sequential order statistics $(\gamma_i = (n-i+1), \alpha_i; \alpha_1, \alpha_2, ..., \alpha_n > 0)$, order statistics with non-integral sample size $(\gamma_i = \alpha - i + 1; \alpha > 0)$, Pfefier's record values $(\gamma_i = \beta_i; \beta_1, \beta_2, ...\beta_n > 0)$ and progressive type II censored order statistics $(m_i \in N, k \in N)$ are obtained by Kamps [5].
For simplicity we shall assume $m_1 = m_2 = m_{n-1} = -m$.

The *pdf* of $X(r,n,m,k)$ is

$$f_{X(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!}[\overline{F}(x)]^{\gamma_r - 1} g_m^{r-1}[F(x)] f(x), \tag{2}$$

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and the joint *pdf* of X(r,n,m,k) and X(s,n,m,k) ,$1 \leq r < s \leq n$, is,

$$f_{x(r,n,m,k),x(s,n,m,k)}(x,y) = \frac{C_{s-1}}{(r-1)!(s-r-1)!}[\overline{F}(x)]^m g_m^{r-1}[F(x)] \times [h_m(F(y)) - h_m(F(x))]^{s-r-1}[\overline{F}(y)]^{\gamma_s-1} f(x)f(y), \quad x < y, \quad (3)$$

where

$$\gamma_i = k + (n-i) + \sum_{j=r}^{n-1} m_j = k + (n-i) + (m+1), \; C_{r-1} = \prod_{i=1}^{r} \gamma_i,$$

$$h_m(x) = \begin{cases} -\frac{1}{m+1}(1-x)^{m+1} & m \neq -1 \\ -\log(1-x) & m = -1 \end{cases}$$

and

$$g_m(x) = h_m(x) - h_m(0) \qquad x \in [0,1).$$

The result is given in the paper can be used to compute the moments of ordered random variables, if the parent distribution follows the (DPHF). Since recurrence relations reduce the amount of direct computation and hence reduce the time and labour.

The recurrence relations based on generalized order statistics have received considerable attention in recent years. Many authors derived the recurrence relations for generalized order statistics for different distributions. See, Ahmad and Fawzy [1], AL-Hussaini *et al.* [2], Kumar and Khan [11], Khan and Khan [7,8] and Khan *et al.* [9,10] among others.

The power hazard function has been defined by Mugdadi [13],

$$h(x) = \alpha x^{\beta}, \qquad (x > 0, \alpha > 0, \beta > -1). \quad (4)$$

Corresponding to this hazard function, its cumulative distribution function $(cdf) F(x)$ and probability density function $(pdf)\ f(x)$ are given respectively by,

$$F(x) = 1 - exp\left(-\frac{\alpha}{\beta+1} x^{\beta+1}\right), \quad (5)$$

$$f(x) = \alpha x^{\beta} exp\left(-\frac{\alpha}{\beta+1} x^{\beta+1}\right), \qquad (x > 0, \alpha > 0, \beta > -1). \quad (6)$$

The distribution with density defined in (5) will be referred to the distribution has power hazard function (DPHF). For $-1 < \beta < 0$, the DPHF has a decreasing hazard function and $\beta > 0$, the DPHF has an increasing hazard function. For more details properties of the DPHF, see Ismail [4].

It is clear that some well- known life time distributions as the Weibull, Rayleigh and exponential are special cases of the DPHF distribution as follows,

If $\beta = \alpha - 1$ then DPHF reduces to Weibull$(\alpha, 1)$.

If $\alpha = \frac{1}{\theta^2}$ then DPHF reduces to Rayleigh (θ).

If $\beta = 0$ then DPHF reduces to exponential distribution with mean $\left(\frac{1}{\alpha}\right)$.

Note that for distribution having power hazard function defined in (5)

$$f(x) = \alpha x^{\beta} \overline{F}(x). \quad (7)$$

The relation in (7) will be exploited in this paper to derive some recurrence relations for the moments of from the distribution having power hazard function.

It appears from literature that no attention has been paid on the characterization of distribution having power hazard function (DPHF) based on (gos) .

The contents of this paper are organised as follows. First, the recurrence relations for single moments is presented in section 2 and its special cases for order statistics and upper record values are discussed. The recurrence relations for product moments and its reduces cases are presented in section 3. The characterizations results are discussed in section 4. Finally, conclusion is discussed in section 5.

2 Relations for Single Moments

Theorem 2.1: Let X be a non- negative continuous random variable follows the distribution having power hazard function given in (6). Suppose that $j > 0$ and $1 \geq r \geq n$, then

$$E[X^j(r,n,m,k)] = \frac{\alpha}{j+\beta+1}\left[\gamma_r E[X^{j+\beta+1}(r,n,m,k)] - E[X^{j+\beta+1}(r-1,n,m,k)\right]. \tag{8}$$

Proof: From (2), we have,

$$E[X^j(r,n,m,k)] = \frac{C_{r-1}}{(r-1)!}\int_0^\infty x^j[\overline{F}(x)]^{\gamma_r-1}g_m^{r-1}[F(x)]f(x)dx \tag{9}$$

$$E[X^j(r,n,m,k)] = \frac{C_{r-1}}{(r-1)!}\int_0^\infty x^j[\overline{F}(x)]^{\gamma_r}\alpha x^\beta g_m^{r-1}[F(x)]f(x)dx$$

$$E[X^j(r,n,m,k)] = \frac{\alpha C_{r-1}}{(r-1)!}\int_0^\infty x^{j+\beta}[\overline{F}(x)]^{\gamma_r}g_m^{r-1}[F(x)]f(x)dx.$$

Integrating by part is taking $x^{j+\beta}$ as the part to be integrated and simplifying the resulting expression , as follows

$$E[X^j(r,n,m,k)] = \frac{\alpha\gamma_r}{j+\beta+1}\left[E[X^{j+\beta+1}(r,n,m,k)]\right] - \frac{\alpha}{j+\beta+1}\left[E[X^{j+\beta+1}(r-1,n,m,k)]\right]$$

$$E[X^j(r,n,m,k)] = \frac{\alpha}{j+\beta+1}\left[\gamma_r E[X^{j+\beta+1}(r,n,m,k)] - E[X^{j+\beta+1}(r-1,n,m,k)]\right], \tag{10}$$

and hence the Theorem.

Remark 2.1: Setting $m = 0,\ \ k = 1$ in (10), the result reduces for order statistics as follows,

$$E[X^j_{r:n}] = \frac{\alpha}{j+\beta+1}\left[(n-r+1)E[X^{j+\beta+1}_{r:n}] - E[X^{j+\beta+1}_{r-1:n}]\right].$$

Remark 2.2: Setting $m = -1,\ \ k \geq 1$ in (10), the result reduces for order statistics as follows,

$$E[X^k_n]^j = \frac{\alpha k}{j+\beta+1}\left[E[X^k_n]^{j+\beta+1} - E[X^{(k)}_{n-1}]^{j+\beta+1}\right],$$

as obtain by Khan [6].

3 Relations for Product Moments

Theorem 3.1: Let X be a non- negative continuous random variable follows the distribution having power hazard function given in (6). Suppose that $i, j > 0$ and $1 \leq r \leq s \leq n$ then,

$$E[X^i(r,n,m,k)X^j(s,n,m,k)] = \frac{\alpha}{j+\beta+1}\left[\gamma_r E[X^i(r,n,m,k)X^{j+\beta+1}(s,n,m,k)]. - E[X^i(r,n,m,k)X^{j+\beta+1}(s-1,n,m,k)]\right] \tag{11}$$

Proof: From (3), we have

$$E[X^i(r,n,m,k)X^j(s,n,m,k)] = \frac{C_{s-1}}{(r-1)!(s-r-1)!}\int_0^\infty x^i[\overline{F}(x)]^m g_m^{r-1}[F(x)]f(x)I(x)dx, \tag{12}$$

where

$$\begin{aligned} I(x) &= \int_x^\infty y^j[\overline{F}(y)]^{\gamma_s-1}[h_m(F(y)) - h_m(F(x))]^{s-r-1}f(y)dy \\ &= \int_x^\infty y^j[\overline{F}(y)]^{\gamma_s}[h_m(F(y)) - h_m(F(x))]^{s-r-1}\frac{f(y)}{\overline{F}(x)}dy \\ &= \alpha\int_x^\infty y^{j+\beta}[\overline{F}(y)]^{\gamma_s-1}[h_m(F(y)) - h_m(F(x))]^{s-r-1}dy \end{aligned}$$

solving the integral in $I(x)$ by parts,

$$I(x) = \frac{\alpha\gamma_r}{j+\beta+1}\int_x^\infty y^{j+\beta}[\overline{F}(y)]^{\gamma_s-1}[h_m(F(y))-h_m(F(x))]^{s-r-1}f(x)dy$$
$$-\frac{\alpha(s-r-1)}{j+\beta+1}\int_x^\infty y^{j+\beta}[\overline{F}(y)]^{\gamma_s-1}[h_m(F(y))-h_m(F(x))]^{s-r-2}f(x)dy,$$

substituting the resulting expression in (12) and simplifying, it leads to (11)

$$E[X^i(r,n,m,k)X^j(s,n,m,k)] = \frac{\alpha\gamma_s}{j+\beta+1}E[X^i(r,n,m,k)X^{j+\beta+1}(s,n,m,k)]$$
$$-\frac{\alpha}{j+\beta+1}E[X^i(r,n,m,k)X^{j+\beta+1}(s-1,n,m,k)], \tag{13}$$

and hence the result.

Remark 3.1: Setting $m=0,\ \ k=1$ in (13), the recurrence relation for the product moments order statistics from DPHF as follows,

$$E[X^{i,j}_{r,s:n}] = \frac{\alpha(n-s-1)}{j+\beta+1}E[X^{i,j+\beta+1}_{r,s:n}] - \frac{\alpha}{j+\beta+1}E[X^{i,j+\beta+1}_{r,s-1:n}].$$

Remark 3.2: Setting $m=-1,\ \ k\geq 1$ in (13),we get the recurrence relation for the product moments of upper k-record from DPHF as follows,

$$E\left[(y_m^{(k)})^i(y_n^{(k)})^j\right] = \frac{\alpha k}{j+\beta+1}\left\{E\left[(y_m^{(k)})^i(y_{n-1}^{(k)})^{j+\beta+1}\right] - E\left[(y_m^{(k)})^i(y_{n-1}^{k})^{j+\beta+1}\right]\right\},$$

as obtain by Khan [6].

4 Characterizations

This section discusses the characterization results of DPHF. Characterization of a probability distribution plays an important role in probability and statistics. A probability distribution can be characterized through various method, In recent years, there has been a great interest in the characterizations of probability distributions through recurrence relations based on (gos).

Theorem 4.1 is characterized based on the following result of Lin [12], which is given in proposition 1.

Proposition 1

Let n_0 be any fixed non-negative integer and let a,b be real numbers such that $-\infty < a < b < \infty$. Let $g(x) \geq 0$ be an absolutely continuous function with $g'(x) \neq 0$ almost everywhere on (a,b). Then the sequence of functions $\left\{[g(x)]^n \exp^{-g(x)}, n \geq n_0\right\}$ is complete in $L(a,b)$ if and only if $g(x)$ is strictly monotone on (a,b).

Theorem 4.1: The necessary and sufficient condition for a random variable to be distributed with pdf given by (6) is that,

$$E[X^j(r,n,m,k)] = \frac{\alpha}{j+\beta+1}\left[\gamma_r E[X^{j+\beta+1}(r,n,m,k)] - E[X^{j+\beta+1}(r-1,n,m,k)\right], \tag{14}$$

if and only if

$$F(x) = 1 - exp\left(-\frac{\alpha}{\beta+1}x^{\beta+1}\right), (x>0, \alpha>0, \beta>-1).$$

Proof: The necessary part follows immediately from (14). On other hand if the recurrence relation (14) is satisfied, then on rearranging the terms in (14)

$$\frac{C_{r-1}}{(r-1)!}\int_0^\infty x^j[\overline{F}(x)]^{\gamma_r-1}g_m^{r-1}[F(x)]f(x)dx = \frac{\alpha}{j+\beta+1}\frac{C_{r-1}\gamma_r}{(r-1)!}\int_0^\infty x^{j+\beta+1}[\overline{F}(x)]^{\gamma_r-1}g_m^{r-1}[F(x)]f(x)dx$$
$$-\frac{\alpha}{j+\beta+1}\frac{C_{r-1}r-1}{(r-1)!}\int_0^\infty x^{j+\beta+1}[\overline{F}(x)]^{\gamma_r+m}g_m^{r-2}[F(x)]f(x)dx$$
$$\frac{C_{r-1}}{(r-1)!}\int_0^\infty x^j[\overline{F}(x)]^{\gamma_r-1}g_m^{r-1}[F(x)]f(x)d(x) = \frac{\alpha}{j+\beta+1}\frac{C_{r-1}}{(r-1)!}\int_0^\infty x^{j+\beta+1}[\overline{F}(x)]^{\gamma_r}g_m^{r-2}[F(x)]f(x)dx$$
$$\left[\frac{\gamma_r[F(x)]}{\overline{F}(x)} - (r-1)[\overline{F}(x)]^m\right]dx. \tag{15}$$

Let

$$h(x) = -[F\overline{(x)}]^{\gamma_r} g_m^{r-1}[F(x)]. \quad (16)$$

Differentiating (16) both side with respect to x ,

$$h'(x) = -[\overline{F}(x)]^{\gamma_r} g_m^{r-2}[F(x)] f(x \left\{ \frac{\gamma_r[F(x)]}{\overline{F}(x)} - (r-1)[\overline{F}(x)]^m \right\}.$$

Thus

$$\frac{C_{r-1}}{(r-1)!}\int_0^\infty x^j [\overline{F}(x)]^{\gamma_r-1} g_m^{r-1}[F(x)] f(x)dx = \frac{\alpha}{j+\beta+1}\frac{C_{r-1}}{(r-1)!}\int_0^\infty x^{j+\beta+1}h'(x)dx. \quad (17)$$

Integrating RHS in (17) by parts and using the value of $h(x)$ from (16),

$$\frac{C_{r-1}}{(r-1)!}\int_0^\infty x^j [\overline{F}(x)]^{\gamma_r-1} g_m^{r-1}[F(x)] f(x)dx = \frac{\alpha C_{r-1}}{(r-1)!}\int_0^\infty x^{j+\beta} [\overline{F}(x)]^{\gamma_r} g_m^{r-1}[F(x)]dx$$

$$\frac{C_{r-1}}{(r-1)!}\int_0^\infty x^j [\overline{F}(x)]^{\gamma_r} g_m^{r-1}[F(x)] \left[f(x) - \alpha x^\beta \overline{F}(x) \right] dx = 0. \quad (18)$$

It follows from the above proposition,

$$f(x) = \alpha x^\beta \overline{F}(x),$$

which proves that $f(x)$ has the from as in (6).*i.e*,

$$F(x) = 1 - exp\left(-\frac{\alpha}{\beta+1}x^{\beta+1}\right), \qquad (x > 0, \alpha > 0, \beta > -1).$$

Theorem 4.2 is characterized based on the moments of minimal order statistics. Putting $n = 1$ in (14).

Theorem 4.2: Let j be a non- negative integer. A necessary and sufficient condition for a random variable X to be distributed with pdf given by (6) is that,

$$E[X_{1:n}^j] = \frac{\alpha n}{(j+\beta+1)} E[X_{1:n}^{j+\beta+1}]. \quad (19)$$

Proof: The necessary part follows immediately from (8). On the other hand if the recurrence relation (19) is satisfied, then

$$\int_0^\infty x^j [\overline{F}(x)]^{n-1} f(x)dx = \frac{\alpha n}{(j+\beta+1)}\int_0^\infty x^{j+\beta+1} [\overline{F}(x)]^{n-1} f(x)dx.$$

Integrating the integrals on the right-hand side of the above expression by parts, we get

$$\int_0^\infty x^j [\overline{F}(x)]^{n-1} f(x)dx = \alpha \int_0^\infty x^{j+\beta} [\overline{F}(x)]^n dx,$$

which further reduces to

$$\int_0^\infty x^j [\overline{F}(x)]^{n-1}[f(x) - \alpha x^\beta \overline{F}(x)]dx = 0 \qquad ,n = 1,2\ldots \quad (20)$$

Now applying a generalization of the Muntz-Szasz Theorem (Hwang and Lin [3]) to equation (20), it gives

$$f(x) = \alpha x^\beta \overline{F}(x)$$

which proves that $f(x)$ has the from as in (6). *i.e*,

$$F(x) = 1 - exp\left(-\frac{\alpha}{\beta+1}x^{\beta+1}\right), \qquad (x > 0,\ \alpha > 0,\ \beta > -1).$$

5 Conclusion

Characterization of probability distribution plays an important role in probability and statistics. Before a particular probability distribution model is applied to fit the real data, it is necessary to confirm whether the given probability distribution satisfies the underlying requirements by its characterization. A probability distribution can be characterized through various method.In this paper, characterizations result based on recurrence relations of single moments of generalized order statistics and minimal order statistics from (DPHF) are established.

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