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Relative Annihilators in Lower BCK-Semilattices

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Abstract: As a generalization of annihilators, the notion of a relative annihilator is introduced, and their properties are investigated. Conditions for a relative annihilator to be an implicative (resp., positive implicative, commutative) ideal are discussed.

Keywords: Lower BCK-semilattice, relative annihilator, implicative ideal, positive implicative ideal, commutative ideal. *2010 Mathematics Subject Classification.* 06F35, 03G25.

1 Introduction

Aslam and Thaheem [2] discussed the annihilators of a subset of BCK-algebras, and Jun et al. [5] generalized it to BCI-algebras. Also the notion of an annihilator in BCK-algebras is studied in the papers [1], [3], [6] and [7].

In this manuscript we introduce the notion of the relative annihilator of a subset with respect to a subset in lower BCK-semilattices as an extension of annihilator, and we obtain some results. We show that the relative annihilator of an ideal with respect to an ideal in a lower BCK-semilattice is an ideal, and we discuss conditions for the relative annihilator of a subset with respect to a subset to be an implicative (resp., positive implicative, commutative) ideal.

2 Preliminaries

BCK/BCI-algebras form an important class of algebras for logic introduced by K. Iséki and was extensively investigated by several researchers.

An algebra $(X;*,0)$ of type $(2,0)$ is called a *BCI-algebra* if it satisfies the following conditions:

(I)$(\forall x,y,z\in X)\ (((x*y)*(x*z))*(z*y)=0)$,
(II)$(\forall x,y\in X)\ ((x*(x*y))*y=0)$,
(III)$(\forall x\in X)\ (x*x=0)$,
(IV)$(\forall x,y\in X)\ (x*y=0,\ y*x=0\ \Rightarrow\ x=y)$.

If a BCI-algebra X satisfies the following identity:

(V)$(\forall x\in X)\ (0*x=0)$,

then X is called a *BCK-algebra*. Any BCK/BCI-algebra X satisfies the following axioms:

(a1)$(\forall x\in X)\ (x*0=x)$,
(a2)$(\forall x,y,z\in X)\ (x\leq y\ \Rightarrow\ x*z\leq y*z,\ z*y\leq z*x)$,
(a3)$(\forall x,y,z\in X)\ ((x*y)*z=(x*z)*y)$,
(a4)$(\forall x,y,z\in X)\ ((x*z)*(y*z)\leq x*y)$

where $x\leq y$ if and only if $x*y=0$. A BCK-algebra X is called a *lower BCK-semilattice* (see [8]) if X is a lower semilattice with respect to the BCK-order.

A subset A of a BCK/BCI-algebra X is called an *ideal* of X (see [8]) if it satisfies:

$$0\in A, \tag{1}$$
$$(\forall x\in X)\,(\forall y\in A)\,(x*y\in A\ \Rightarrow\ x\in A)\,. \tag{2}$$

For any subset A of X, the ideal generated by A is defined to be the intersection of all ideals of X containing A, and it is denoted by $\langle A\rangle$. If A is finite, then we say that $\langle A\rangle$ is *finitely generated ideal* of X (see [8]).

A subset A of a BCK-algebra X is called a *commutative ideal* of X (see [8]) if it satisfies (1) and

$$(\forall x,y\in X)(\forall z\in A)\,((x*y)*z\in A\ \Rightarrow\ x*(y*(y*x))\in A) \tag{3}$$

A subset A of a BCK-algebra X is called a *positive implicative ideal* of X (see [8]) if it satisfies (1) and

$$(\forall x,y,z\in X)\,((x*y)*z\in A,\,y*z\in A\ \Rightarrow\ x*z\in A)\,. \tag{4}$$

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A subset A of a BCK-algebra X is called an *implicative ideal* of X (see [8]) if it satisfies (1) and

$$(\forall x,y\in X)(\forall z\in A)\,((x*(y*x))*z\in A\ \Rightarrow\ x\in A). \tag{5}$$

We refer the reader to the books [4,**?**] for further information regarding BCK/BCI-algebras.

3 Relative annihilators

In what follows, let X be a BCK-algebra unless otherwise specified. For $x,y\in X$, denote by $x\wedge y$ the greatest lower bound of x and y. For any nonempty subsets A and B of X, we denote

$$A\wedge B := \{a\wedge b \mid a\in A, b\in B\}.$$

If $A=\{a\}$, then $\{a\}\wedge B$ is denoted by $a\wedge B$.

Definition 1. *For any nonempty subsets A and B of X, we define a set*

$$(A:_\wedge B) := \{x\in X \mid x\wedge B\subseteq A\} \tag{6}$$

whenever $x\wedge B$ exists for all $x\in X$, and it is called the relative annihilator of B with respect to A.

If $0\in A$, then it is clear that $0\in(A:_\wedge B)$. Obviously, for any nonempty subsets A, B and C of X, we have

$$C\subseteq(A:_\wedge B)\ \Rightarrow C\wedge B\subseteq A. \tag{7}$$

Given a lower BCK-semilattice X, note that if $A=\{0\}$ in (6), then

$$\begin{aligned}(\{0\}:_\wedge B) &= \{x\in X \mid x\wedge B\subseteq\{0\}\}\\ &= \{x\in X \mid x\wedge b=0,\ \forall b\in B\}\\ &= B^*\end{aligned} \tag{8}$$

which is the annihilator of B (see [4]). Hence the relative annihilator of B with respect to A is a generalization of the annihilator of B.

Proposition 1. *For any nonempty subsets A, B and C of a lower BCK-semilattice X, we have*

(i) *If A is an ideal of X, then $A\subseteq(A:_\wedge B)$ and $B\subseteq(A:_\wedge(A:_\wedge B))$.*

(ii) *If $B_1\subseteq B_2$ in X, then $(A:_\wedge B_2)\subseteq(A:_\wedge B_1)$ and*

$$(A:_\wedge(B_1\cup B_2)) = (A:_\wedge B_1)\cap(A:_\wedge B_2).$$

(iii) $((A:_\wedge B):_\wedge C) = (A:_\wedge B\wedge C) = ((A:_\wedge C):_\wedge B)$.

(iv) $\left(\underset{\lambda\in\Lambda}{\cap} A_\lambda :_\wedge B\right) = \underset{\lambda\in\Lambda}{\cap}(A_\lambda :_\wedge B)$ *for any family $\{A_\lambda \mid \lambda\in\Lambda\}$ of subsets of X.*

(v) *If A is an ideal of X such that $A\subseteq B$, then $(A:_\wedge B)\cap B = A$.*

(vi) *If A is an ideal of X, then $(A:_\wedge(A:_\wedge B))\cap(A:_\wedge B) = A$.*

(vii) *If A is an ideal of X, then $(A:_\wedge X) = A$ and $(A:_\wedge A) = X$.*

(viii) *If A is an ideal of X, then $(A:_\wedge B) = (A:_\wedge(A:_\wedge(A:_\wedge B)))$.*

(ix) *If A is an ideal of X, then $(A:_\wedge B) = X\ \Leftrightarrow\ B\subseteq A$.*

Proof. (i) Let $x\in A$. Note that $x\wedge b\le x$ for all $b\in B$. Since A is an ideal, it follows that $x\wedge b\in A$ for all $b\in B$, that is, $x\wedge B\subseteq A$. Thus $x\in(A:_\wedge B)$, and so $A\subseteq(A:_\wedge B)$. Let $x\in B$ and $y\in(A:_\wedge B)$. Then $y\wedge b\in A$ for every element $b\in B$. Since $x\in B$, it follows that $x\wedge y\in A$. Thus $x\in(A:_\wedge(A:_\wedge B))$, and therefore $B\subseteq(A:_\wedge(A:_\wedge B))$.

(ii) Let $x\in(A:_\wedge B_2)$. Then $x\wedge B_1\subseteq x\wedge B_2\subseteq A$, and so $x\in(A:_\wedge B_1)$. Therefore $(A:_\wedge B_2)\subseteq(A:_\wedge B_1)$. Since $B_1\subseteq B_1\cup B_2$, we have

$$(A:_\wedge(B_1\cup B_2))\subseteq(A:_\wedge B_1) \text{ and }$$
$$(A:_\wedge(B_1\cup B_2))\subseteq(A:_\wedge B_2).$$

Thus

$$(A:_\wedge(B_1\cup B_2))\subseteq(A:_\wedge B_1)\cap(A:_\wedge B_2).$$

Now suppose that $x\in(A:_\wedge B_1)\cap(A:_\wedge B_2)$. Then $x\wedge B_1\subseteq A$ and $x\wedge B_2\subseteq A$. If $y\in B_1\cup B_2$, then $y\in B_1$ or $y\in B_2$. Hence $x\wedge y\in A$, and so $x\in(A:_\wedge(B_1\cup B_2))$, that is, $(A:_\wedge B_1)\cap(A:_\wedge B_2)\subseteq(A:_\wedge(B_1\cup B_2))$.

(iii) For any $x\in X$, we have

$$\begin{aligned}x\in((A:_\wedge B):_\wedge C) &\Leftrightarrow x\wedge C\subseteq(A:_\wedge B)\\ &\Leftrightarrow (\forall c\in C)\,(x\wedge c\in(A:_\wedge B))\\ &\Leftrightarrow (\forall c\in C)\,((x\wedge c)\wedge B\subseteq A)\\ &\Leftrightarrow (\forall c\in C)(\forall b\in B)\,((x\wedge c)\wedge b\in A)\\ &\Leftrightarrow (\forall c\in C)(\forall b\in B)\,(x\wedge(c\wedge b)\in A)\\ &\Leftrightarrow (\forall c\in C)(\forall b\in B)\,(x\wedge(b\wedge c)\in A)\\ &\Leftrightarrow x\wedge(B\wedge C)\subseteq A\\ &\Leftrightarrow x\in(A:_\wedge B\wedge C).\end{aligned}$$

Hence $((A:_\wedge B):_\wedge C) = (A:_\wedge B\wedge C)$. Similarly,

$$(A:_\wedge B\wedge C) = ((A:_\wedge C):_\wedge B).$$

(iv) For any $x\in X$, we have

$$\begin{aligned}x\in\left(\underset{\lambda\in\Lambda}{\cap}A_\lambda :_\wedge B\right) &\Leftrightarrow x\wedge B\subseteq\underset{\lambda\in\Lambda}{\cap}A_\lambda\\ &\Leftrightarrow (\forall b\in B)\left(x\wedge b\in\underset{\lambda\in\Lambda}{\cap}A_\lambda\right)\\ &\Leftrightarrow (\forall b\in B)(\forall\lambda\in\Lambda)\,(x\wedge b\in A_\lambda)\\ &\Leftrightarrow (\forall\lambda\in\Lambda)\,(x\wedge B\subseteq A_\lambda)\\ &\Leftrightarrow (\forall\lambda\in\Lambda)\,(x\in(A_\lambda :_\wedge B))\\ &\Leftrightarrow x\in\underset{\lambda\in\Lambda}{\cap}(A_\lambda :_\wedge B).\end{aligned}$$

Therefore $\left(\underset{\lambda\in\Lambda}{\cap}A_\lambda :_\wedge B\right) = \underset{\lambda\in\Lambda}{\cap}(A_\lambda :_\wedge B)$.

(v) Let A be an ideal and B a subset of X such that $A\subseteq B$. By using the part (i) we know that $A\subseteq(A:_\wedge B)$, and so $A\subseteq(A:_\wedge B)\cap B$. Now let $x\in(A:_\wedge B)\cap B$. Then

$x \in B$ and $x \in (A :_\wedge B)$, and thus $x \wedge b \in A$ for all $b \in B$. Since $x \in B$, it follows that $x = x \wedge x \in A$ which means that $(A :_\wedge B) \cap B \subseteq A$. Therefore, $(A :_\wedge B) \cap B = A$.

(vi) The result (i) implies that $A \subseteq (A :_\wedge B)$ and $A \subseteq (A :_\wedge (A :_\wedge B))$. Thus $A \subseteq (A :_\wedge (A :_\wedge B)) \cap (A :_\wedge B)$. Now let

$$x \in (A :_\wedge (A :_\wedge B)) \cap (A :_\wedge B).$$

Then $x \in (A :_\wedge (A :_\wedge B))$ and $x \in (A :_\wedge B)$. Since $x \in (A :_\wedge (A :_\wedge B))$, we have $x \wedge y \in A$ for all $y \in (A :_\wedge B)$. Also since $x \in (A :_\wedge B)$, we get $x = x \wedge x \in A$ which shows that

$$(A :_\wedge (A :_\wedge B)) \cap (A :_\wedge B) \subseteq A.$$

Therefore, $(A :_\wedge (A :_\wedge B)) \cap (A :_\wedge B) = A$.

(vii) By using part (i), we have $A \subseteq (A :_\wedge X)$. Now suppose that $y \in (A :_\wedge X)$. Then $y \wedge x \in A$ for all $x \in X$, and so $y = y \wedge y \in A$. Therefore $A = (A :_\wedge X)$. Obviously $(A :_\wedge A) = X$.

(viii) Suppose that $x \in (A :_\wedge B)$ and $y \in (A :_\wedge (A :_\wedge B))$. Then $y \wedge z \in A$ for every element $z \in (A :_\wedge B)$. Since $x \in (A :_\wedge B)$, it follows that $x \wedge y \in A$ and so that $x \in (A :_\wedge (A :_\wedge (A :_\wedge B)))$. Therefore,

$$(A :_\wedge B) \subseteq (A :_\wedge (A :_\wedge (A :_\wedge B))).$$

Conversely, let $x \in (A :_\wedge (A :_\wedge (A :_\wedge B)))$ and $b \in B$. Using (i) we have $B \subseteq (A :_\wedge (A :_\wedge B))$, and so $b \in (A :_\wedge (A :_\wedge B))$. Since $x \in (A :_\wedge (A :_\wedge (A :_\wedge B)))$, it follows that $x \wedge b \in A$, that is, $x \in (A :_\wedge B)$. Therefore $(A :_\wedge (A :_\wedge (A :_\wedge B))) \subseteq (A :_\wedge B)$.

(ix) Suppose that $(A :_\wedge B) = X$. Let b be an arbitrary element of B. Then clearly $b \in (A :_\wedge B)$, and so $b = b \wedge b \in A$. Therefore $B \subseteq A$.

Conversely, suppose that $B \subseteq A$. Let $x \in X$ and $b \in B$. Then $x \wedge b \leq b$, and thus $x \wedge b \in B \subseteq A$, that is, $x \in (A :_\wedge B)$. Thus $X \subseteq (A :_\wedge B)$, and so $X = (A :_\wedge B)$.

In [1, Propositions 3.7 and 3.8], Abujabal et al. discussed the following results.

If A and B are ideals of a commutative BCK-algebra X, then

$$(A :_\wedge C) \cap (B :_\wedge C) = (A \cap B :_\wedge C)$$

for every subset C of X.

If A is an ideal of a commutative BCK-algebra X, then

$$(A :_\wedge B \cup C) = (A :_\wedge B) \cap (A :_\wedge C)$$

for every subsets B and C of X.

We have more general form than two results above as a corollary of (ii) and (iv) in Proposition 1.

Corollary 1. *For any subsets A, B and C of a commutative BCK-algebra X, we have*

$$(A :_\wedge C) \cap (B :_\wedge C) = (A \cap B :_\wedge C)$$

and

$$(A :_\wedge B \cup C) = (A :_\wedge B) \cap (A :_\wedge C).$$

In [1, Proposition 3.5(iv)], Abujabal et al. discussed the following result.

Let A and B be ideals of a commutative BCK-algebra X. If $A \subseteq B$, then $(A :_\wedge B) \cap B = A$.

But, in the above Result, the condition "B is an ideal of X" is redundant. In fact, we have the following corollary of Proposition 1(v).

Corollary 2. *Let A be an ideal of a commutative BCK-algebra X. For any subset B of X, if $A \subseteq B$ then $(A :_\wedge B) \cap B = A$.*

In Proposition 1(i), if A is not an ideal of X then the inclusion $A \subseteq (A :_\wedge B)$ is not true in general as seen in the following example.

Example 1. Consider a lower BCK-semilattice $X = \{0,1,2,3,4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	0
2	2	1	0	2	1
3	3	3	3	0	3
4	4	4	4	4	0

For $A = \{0,2\}$ and $B = \{0,1,2\}$, we have $(A :_\wedge B) = \{0,3\}$ and $A \nsubseteq (A :_\wedge B)$. Note that A is not an ideal of X.

In Proposition 1(i), the equality $A = (A :_\wedge B)$ does not hold in general as seen in the following example.

Example 2. let $X = \{0,1,2,3,4\}$ be a set with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	0
2	2	1	0	2	0
3	3	3	3	0	3
4	4	4	4	4	0

Then X is a lower BCK-semilattice. For an ideal $A = \{0,1,2\}$ of X, if we take $B = \{3\}$, then

$$(A :_\wedge B) = \{x \in X \mid x \wedge B \subseteq A\} = \{0,1,2,4\} \neq A.$$

We now provide a condition for the equality $A = (A :_\wedge B)$ to be hold.

Proposition 2. *Let X be a lower BCK-semilattice. If A is an ideal of X, then $A = (A :_\wedge B)$ for some singleton subset B of $(A :_\wedge B)$.*

Proof. Let $x \in (A :_\wedge B)$ and take $B = \{x\}$. Then $x = x \wedge x \in A$, and thus $(A :_\wedge B) \subseteq A$. Since $A \subseteq (A :_\wedge B)$ by Proposition 1(i), we have $A = (A :_\wedge B)$.

Question 1. For any nonempty subset A of a lower BCK-semilattice X, does the following condition hods?

$$(\forall x, y \in X)\,(x \leq y \;\Rightarrow\; x \wedge A \subseteq y \wedge A). \qquad (9)$$

The answer to the question above is not valid in general as seen in the following example.

*Example 3.*Consider a lower BCK-semilattice $X = \{0,1,2,3,4\}$ with the following Cayley table

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	1	0	1	0
3	3	3	3	0	0
4	4	4	4	3	0

For $A = \{2,4\}$, we have

$$1 \wedge A = 1 \wedge \{2,4\} = \{1\} \text{ and } 2 \wedge A = 2 \wedge \{2,4\} = \{2\}.$$

Note that $1 \leq 2$, but $1 \wedge A \nsubseteq 2 \wedge A$.

If we strength conditions, then we have

Proposition 3.*If A is an ideal of a lower BCK-semilattice X, then the condition (9) holds.*

*Proof.*Let $x,y \in X$ be such that $x \leq y$. Suppose that $z \in x \wedge A$. Then there exists an element $a \in A$ such that $z = x \wedge a$. Since $x \wedge a \leq a$ and A is an ideal of X, it follows that $z = x \wedge a \in A$. The condition $x \leq y$ induces $x = x \wedge y$, and so

$$z = x \wedge a = (x \wedge y) \wedge a = y \wedge (x \wedge a) \in y \wedge A.$$

This shows that $x \wedge A \subseteq y \wedge A$ for all $x,y \in X$ with $x \leq y$.

Corollary 3.*If A is an ideal of a commutative BCK-algebra X, then the condition (9) holds.*

Theorem 1.*For any nonempty subset A and an ideal B of a lower BCK-semilattice X, the relative annihilator of B with respect to A is a subalgebra of X.*

*Proof.*Let $x,y \in (A :_\wedge B)$. Then $x \wedge B \subseteq A$ and $y \wedge B \subseteq A$. Since $x * y \leq x$, we get $(x * y) \wedge B \subseteq x \wedge B \subseteq A$ by Proposition 3. Therefore $x * y \in (A :_\wedge B)$, which shows that the relative annihilator of B with respect to A is a subalgebra of X.

Corollary 4.*For any nonempty subset A and an ideal B of a commutative BCK-algebra X, the relative annihilator of B with respect to A is a subalgebra of X.*

The following example shows that there exist nonempty subsets A and B of X such that the relative annihilator of B with respect to A is not an ideal of X.

*Example 4.*Consider a lower BCK-semilattice $X = \{0,1,2,3,4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	1
2	2	1	0	0	2
3	3	1	1	0	3
4	4	4	4	4	0

For subsets $A = \{0,2,4\}$ and $B = \{0,3\}$ of X, we have $(A :_\wedge B) = \{0,2,4\}$, which is not an ideal of X.

For a nonempty subset B of a lower BCK-semilattice X, consider the following condition:

$$(\forall x,y \in X)(\forall b \in B)\left((x \wedge b) * (y \wedge b) \leq (x * y) \wedge b\right). \quad (10)$$

We provide conditions for the relative annihilator of a set with respect to a set to be an ideal.

Theorem 2.*Let B be a nonempty subset of a lower BCK-semilattice X in which the condition (10) is valid. If A is an ideal of X, then the relative annihilator $(A :_\wedge B)$ of B with respect to A is an ideal of X.*

*Proof.*Assume that A is an ideal of X. Since $0 \wedge B = \{0\} \subseteq A$, we have $0 \in (A :_\wedge B)$. Let $x,y \in X$ be such that $x * y \in (A :_\wedge B)$ and $y \in (A :_\wedge B)$. Then $(x * y) \wedge B \subseteq A$ and $y \wedge B \subseteq A$, that is, $(x * y) \wedge b \in A$ and $y \wedge b \in A$ for all $b \in B$. Since A is an ideal of X, it follows from (10) that

$$(x \wedge b) * (y \wedge b) \in A$$

and that $x \wedge b \in A$ for all $b \in B$, that is, $x \wedge B \subseteq A$. Hence $x \in (A :_\wedge B)$ and $(A :_\wedge B)$ is an ideal of X.

Since every commutative BCK-algebra X is a lower BCK-semilattice and satisfies the condition (10), we have the following corollary.

Corollary 5([1]). *Let B be a nonempty subset of a commutative BCK-algebra X. If A is an ideal of X, then the relative annihilator $(A :_\wedge B)$ of B with respect to A is an ideal of X.*

The converse of Theorem 2 is not true in general, that is, for any subset B of a lower BCK-semilattice X satisfying the condition (10), there exists a subset A of X such that the relative annihilator $(A :_\wedge B)$ of B with respect to A is an ideal of X, but A is not an ideal of X.

*Example 5.*Consider a lower BCK-semilattice $X = \{0,1,2,3\}$ with the following Cayley table.

$*$	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	2	1	0

Then a subset $B = \{2\}$ of X satisfies the condition (10). Let $A = \{0,1,3\}$ be a subset of X. Then $(A :_\wedge B) = \{0,1\}$ which is an ideal of X. But A is not an ideal of X.

Lemma 1([8]). *Let A and B be ideals of X such that $A \subseteq B$. If A is a positive implicative (resp., commutative and implicative) ideal of X, then so is B.*

Using Proposition 1, Theorem 2 and Lemma 1, we have the following theorem.

Theorem 3. *For a nonempty subset B of a lower BCK-semilattice X satisfying the condition (10), if A is a positive implicative (resp., commutative and implicative) ideal of X, then so is the relative annihilator $(A :_\wedge B)$ of B with respect to A.*

The converse of Theorem 3 is not true in general, that is, for any subset B of a lower BCK-semilattice X satisfying the condition (10), there exists a subset A of X such that the relative annihilator $(A :_\wedge B)$ of B with respect to A is a positive implicative (resp., commutative and implicative) ideal of X, but A is not a positive implicative (resp., commutative and implicative) ideal of X.

Example 6.(1) Let $X = \{0,1,2,3,4\}$ be a set with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	2	0	0	2
3	3	3	2	0	3
4	4	4	4	4	0

Then X is a lower BCK-semilattice. Note that $A = \{0,1\}$ is an ideal which is not positive implicative, and the set and the set $B = \{4\}$ satisfies the condition (10). Then

$$(A :_\wedge B) = \{x \in X \mid x \wedge B \subseteq A\} = \{0,1,2,3\}$$

and it is a positive implicative ideal of X.

(2) Consider a lower BCK-algebra $X = \{0,1,2,3,4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	1
2	2	2	0	2	2
3	3	3	3	0	3
4	4	4	4	4	0

Then the set $A = \{0,3\}$ is an ideal which is neither commutative nor implicative, and the set $B = \{4\}$ satisfies the condition (10). Then

$$(A :_\wedge B) = \{x \in X \mid x \wedge B \subseteq A\} = \{0,1,2,3\}$$

and it is both a commutative ideal and an implicative ideal of X.

Theorem 4. *If A and B are ideals of a lower BCK-semilattice X, then the relative annihilator $(A :_\wedge B)$ of B with respect to A is an ideal of X.*

Proof. Obviously, $0 \in (A :_\wedge B)$. Let $x, y \in X$ be such that $x * y \in (A :_\wedge B)$ and $y \in (A :_\wedge B)$. Then $(x * y) \wedge B \subseteq A$ and $y \wedge B \subseteq A$, that is,

$$(x * y) \wedge b \in A \tag{11}$$

and

$$y \wedge b \in A \tag{12}$$

for all $b \in B$. Since $x \wedge b \leq b$ and B is an ideal of X, we have $x \wedge b \in B$. It follows from (12) that

$$y \wedge (x \wedge b) \in A. \tag{13}$$

Note that $(x \wedge b) * ((x \wedge b) * y)$ is a lower bound of y and $x \wedge b$. Thus

$$(x \wedge b) * ((x \wedge b) * y) \leq y \wedge (x \wedge b),$$

and so

$$(x \wedge b) * ((x \wedge b) * y) \in A. \tag{14}$$

Since $x \wedge b \leq b$, we have

$$(x \wedge b) * y \leq b * y \leq b$$

and since $x \wedge b \leq x$, we get

$$(x \wedge b) * y \leq x * y.$$

Hence $(x \wedge b) * y \leq (x * y) \wedge b \in A$ by (11), and so $(x \wedge b) * y \in A$. Since A is an ideal of X, it follows from (14) that $x \wedge b \in A$ and so that $x \wedge B \subseteq A$, that is, $x \in (A :_\wedge B)$. Therefore the relative annihilator $(A :_\wedge B)$ of B with respect to A is an ideal of X.

Corollary 6. *If A and B are ideals of a commutative BCK-algebra X, then the relative annihilator $(A :_\wedge B)$ of B with respect to A is an ideal of X.*

Using Proposition 1, Theorem 4 and Lemma 1, we have the following theorem.

Theorem 5. *For ideals A and B of a lower BCK-semilattice X, if A is positive implicative (resp., commutative and implicative), then the relative annihilator $(A :_\wedge B)$ of B with respect to A is a positive implicative (resp., commutative and implicative) ideal of X.*

The converse of Theorem 5 is not true in general, that is, for ideals A and B of a lower BCK-semilattice X such that the relative annihilator $(A :_\wedge B)$ of B with respect to A is a positive implicative (resp., commutative and implicative) ideal of X, A may not be positive implicative (resp., commutative and implicative).

Example 7.(1) Let $X = \{0,1,2,3,4\}$ be a set with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	1
2	2	1	0	2	2
3	3	3	3	0	3
4	4	4	4	4	0

Then X is a lower BCK-semilattice. Note that $A = \{0,3\}$ and $B = \{0,4\}$ are ideals of X in which A is not positive implicative. Then

$$(A :_\wedge B) = \{x \in X \mid x \wedge B \subseteq A\} = \{0,1,2,3\}$$

and it is a positive implicative ideal of X.

(2) Consider a lower BCK-semilattice $X = \{0,1,2,3,4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	2	0
3	3	3	3	0	3
4	4	4	4	4	0

Note that $A = \{0,1\}$ and $B = \{0,1,3\}$ are ideals of X in which A is neither commutative nor implicative. Then

$$(A :_\wedge B) = \{x \in X \mid x \wedge B \subseteq A\} = \{0,1,2,4\}$$

and it is both a commutative ideal and an implicative ideal of X.

The following example shows that there exist subsets A and B of X such that $B \nsubseteq (A :_\wedge B)$.

*Example 8.*Consider a BCK-algebra $X = \{0,1,2,3,4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	0	2
3	3	2	1	0	3
4	4	4	4	4	0

For subsets $A = \{0,2\}$ and $B = \{0,1,2,3\}$ of X, we have

$$(A :_\wedge B) = \{0,2,4\},$$

and thus $B \nsubseteq (A :_\wedge B)$.

Theorem 6.*If B_1 and B_2 are ideals of a lower BCK-semilattice X such that $B_1 \cap B_2 = \{0\}$, then $B_1 \subseteq (A :_\wedge B_2)$ for any subset A of X with $0 \in A$.*

*Proof.*Let B_1 and B_2 be ideals of a lower BCK-semilattice X such that $B_1 \cap B_2 = \{0\}$. For any $b_1 \in B_1$ and $b_2 \in B_2$, we have $b_1 \wedge b_2 \leq b_2$ and so $b_1 \wedge b_2 \in B_2$ since B_2 is an ideal of X. Similarly we get $b_1 \wedge b_2 \in B_1$. Thus $b_1 \wedge b_2 \in B_1 \cap B_2 = \{0\}$, and so $b_1 \wedge b_2 = 0 \in A$. It follows that $b_1 \in (A :_\wedge B_2)$. Therefore $B_1 \subseteq (A :_\wedge B_2)$.

The following example shows that the converse of Theorem 6 is not true in general.

*Example 9.*Let $X = \{0,1,2,3\}$ be a set with the following Cayley table.

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	2	0	2
3	3	3	3	0

Then X is a lower BCK-semilattice. Let $A = \{0,1,3\}$, $B_1 = \{0,1\}$ and $B_2 = \{0,1,2\}$. Then

$$(A :_\wedge B_2) = \{x \in X \mid x \wedge B_2 \subseteq A\} = \{0,1,3\},$$

and so $B_1 \subseteq (A :_\wedge B_2)$, but $B_1 \cap B_2 = \{0,1\} \neq \{0\}$.

4 Conclusions and future works

As we mentioned in the abstract, in this article the notions of a relative annihilator is introduced as a generalization of annihilators and then their properties are investigated. We obtain some related results and conditions for a relative annihilator to be an implicative (resp., positive implicative, commutative) ideal are discussed.

Now there are some ideas and questions:

(i) How we can define some other types of relative annihilators, e.g. S-relative annihilator, I-relative annihilator, PI-relative annihilator and so on.

(ii) Can we obtain some relationship between different types of relative annihilator.

(iii) Can we generalized these ideas to hyper BCK (K)-algebra.

We will try to work on these ideas and give the results in the forthcoming papers.

References

[1] H. A. S. Abyjabal, M. A. Obaid, M. Aslam and A. B. Thaheem, On annihilators of *BCK*-algebras, Czechoslovak Math. J. 45(120) (1995), no. 4, 727–735.

[2] M. Aslam and A. B. Thaheem, On certain ideals in *BCK*-algebras, Math. Japon. 36 (1991), no. 5, 895–906.

[3] R. Halas, Annihilators in *BCK*-algebras, Czechoslovak Math. J. 53(128) (2003), no. 4, 1001–1007.

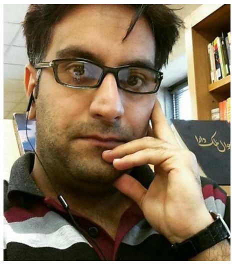
[4] Y. Huang, *BCI*-algebra, Science Press, Beijing 2006.

[5] Y. B. Jun, E. H. Roh and J. Meng, Annihilators in *BCI*-algebras, Math. Japon. 43 (1996), no. 3, 559–562.

[6] M. Kondo, Annihilators in *BCK*-algebras, Math. Japon. 49 (1999), no. 3, 407–410.

[7] M. Kondo, Annihilators in *BCK*-algebras II, Mem. Fac. Sci. Eng. Shimane Univ. Ser. B Math. Sci. 31 (1998), 21–25.

[8] J. Meng and Y. B. Jun, *BCK*-algebras, Kyung Moon Sa Co., Seoul 1994.



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