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Bayesian Analysis of Topp-Leone Distribution under Different Loss Functions and Different Priors

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Abstract: Topp-Leone distribution is a continuous unimodal distribution with wide range of applications in reliability fields and is used for modeling lifetime phenomena. Topp-Leone distribution has a J-shaped density function with a hazard function of bathtub-shaped. This distribution has attracted recent attention for the statistician but has not been discussed in detail in Bayesian approach. The present study focus on the Bayesian estimation of shape parameter of Topp-Leone Distribution under various simple and mixture priors along with different loss functions. The prior predictive and posterior predictive distribution has also been derived. The simulation study has been conducted to compare the different Baye's estimators under different loss functions. A real life example has also been discussed to compare the performance of these estimates.

Keywords: Bayesian Estimation, K-LF, generalized entropy LF, El-Sayyad LF, Relative quadratic LF.

1 Introduction

Topp-Leone distribution was introduced and discussed by Topp and Leone [1] and used it as a model for failure data and defined its probability density function as

$$f(x)=\frac{2\lambda}{\theta}\left(1-\frac{x}{\theta}\right)\left(\frac{2x}{\theta}-\frac{x^2}{\theta^2}\right)^{\lambda-1} \quad ; 0<x<\theta<\infty ; \lambda>0 \tag{1}$$

The cumulative distribution function of (1.1) is

$$F(x)=\begin{cases}\left(\frac{2x}{\theta}-\frac{x^2}{\theta^2}\right) & ;if\ 0\le x\le\theta<\infty \\ 0 & ;if\ x<0 \\ 1 & ;if\ x>\theta\end{cases}$$

On implementing the restriction $0<\lambda<1$, the pdf of Topp-Leone distribution becomes a convex and decreasing J-shaped function. For $\theta=1$, the distribution reduces to standard Topp-Leone distribution. The cumulative distribution function when elevated to a power of $\lambda>0$ in left triangular distribution gives rise to new distribution known as Topp-Leone distribution. Nadarajah and Kotz [2, 3] discussed its different structural properties. The reliability measures and asymptotic distribution of order statistics of the distribution were discussed by Ghitany et al. [4, 5]. The two-sided generalized version of the distribution was studied by Vicari et al. [6] along with its structural properties, and estimation procedures.

Genc [7] studied the moments of order statistics from Topp–Leone distribution and obtained its single moments and a moment relation. Al-Zahrani [8] discussed a class of goodness-of-fit tests for the Topp-Leone distribution with estimated parameters.The researchers are greatly interested in studying the different characteristics of the distribution. Feroze et al. [9] studied Bayesian analysis of failure rate (shape parameter) for Topp-Leone distribution under different loss functions and a couple of non informative priors using singly type II censored samples doubly type II censored samples. El-Sayed et al. [10] discussed bayesian and non-bayesian estimation of Topp-Leone distribution based on lower record values. Mir Mostafaee [11] presented recurrence relations for the moments of order statistics from Topp-Leone distribution without any restrictionfor the shape parameter. Sultan et al. [12] obtained the Baye's estimates under different informative and

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non-informative priors of shape parameter of Topp-Leone Distribution using Bayesian approximation techniques.

2 Loss function

In decision theory the problem to estimate parameters can be formulated by using different possible actions. The appropriate decisions need to be judged carefully. The loss which occurs with the action triggered by erroneous estimation, i.e., by replacing parameter ϑ with decision element ν (say) is determined by loss function. The number of loss functions has been suggested by different authors. Sultan et al. [13] studied the classical estimates and Bayesian estimates of the parameters of lognormal distribution by using different prior distributions and loss functions.

In our presented study, the loss functions used are defined below:

K-LF:Wasan [14] proposed K-loss function and defined it as $l(\hat{\lambda},\lambda)=\frac{(\hat{\lambda}-\lambda)^2}{\hat{\lambda}\lambda}$. The Baye's estimate and posterior risk under K-LF is given by

$$\hat{\lambda}=\sqrt{E(\lambda\,|\,x)/E(\lambda^{-1}\,|\,x)}\ \ \&\ \ \rho(\hat{\lambda})=2\{E(\lambda\,|\,x)E(\lambda^{-1}\,|\,x)-1\}$$

The K-LF is well defined for measuring inaccuracy for an estimator of a scale parameter of a distribution defined on $R^{+}(0,\infty)$.

RQLF: Zellner [15] defined RQLF of the form $l(\hat{\lambda},\lambda)=\left(\frac{\hat{\lambda}-\lambda}{\lambda}\right)^2$ and obtained Baye's estimate and posterior risk as

$$\hat{\lambda}=\frac{E(\lambda^{-1}\,|\,x)}{E(\lambda^{-2}\,|\,x)}\ \ \&\ \ \rho(\hat{\lambda})=1-\frac{\left[E(\lambda^{-1}\,|\,x)\right]^2}{E(\lambda^{-2}\,|\,x)}$$

GELF: The generalized entropy loss function proposed by Calabria and Pulcini [16] is the generalization of the entropy loss function which is given as $L(\hat{\lambda}-\lambda)\propto\left(\frac{\hat{\lambda}}{\lambda}\right)^c-c\ln\left(\frac{\hat{\lambda}}{\lambda}\right)-1$ and the Baye's estimate and posterior risk under the generalized entropy loss function is given by

$$\hat{\lambda}=\left\{E(\lambda^{-c_1}\,|\,x)\right\}^{-1/c_1}\ \ \&\ \ \rho(\hat{\lambda})=\ln\{E(\lambda^{-c_1}\,|\,x)\}+c_1E(\ln(\lambda\,|\,x))\text{ provided }E(\lambda^{-c_1})\text{ exists and is finite.}$$

El-sayyad LF: El-sayyad [17] proposed a loss function of the form $l(\hat{\lambda},\lambda)=\lambda^{\alpha}(\hat{\lambda}^{\beta}-\lambda^{\beta})^2$ where α,β are constants and the Baye's estimate under this loss function is obtained as

$$\hat{\lambda}=\left\{\frac{E(\lambda^{\alpha+\beta}\,|\,x)}{E(\lambda^{\alpha}\,|\,x)}\right\}^{1/\beta}$$

3 Posterior Distribution of TL-Distribution under Extension of Jeffrey's prior

The likelihood function of (1) is given as

$$L(x\,|\,\lambda)\propto\lambda^n e^{-\lambda S}\ where\ S=-\sum_{i=1}^{n}\ln\left(\frac{2x}{\theta}-\frac{x^2}{\theta^2}\right)$$

The extension of Jeffrey's prior is defined as $g(\lambda)\propto\left|I(\lambda)\right|^{m_1}$, $m_1\in R^{+}$; $g(\lambda)=\left(\frac{1}{\lambda}\right)^{m_1}$, $m_1\in R^{+}$

Thus the posterior distribution is $P(\lambda\,|\,x)\propto\lambda^n e^{-\lambda S}\left(\frac{1}{\lambda}\right)^{m_1}$

$$\Rightarrow P(\lambda \mid x) = \frac{S^{n-m_1+1}}{\Gamma(n-m+1)} \lambda^{n-m_1} e^{-\lambda S} \tag{2}$$

3.1 Baye's estimators and risk functions under extension of Jeffrey's prior using KL-LF, RQLF, GELF, EL-Sayyad LF

In this section the Baye's estimates and posterior risks under different loss functions are discussed.

Under KL-LF: $\hat{\lambda} = \sqrt{E(\lambda \mid x) / E(\lambda^{-1} \mid x)}$ & $\rho(\hat{\lambda}) = 2\{E(\lambda \mid x)\, E(\lambda^{-1} \mid x) - 1\}$

$$E(\lambda \mid x) = \frac{n-m_1+1}{S} \;\&\; E(\lambda^{-1} \mid x) = \frac{S}{n-m_1}$$

$$\therefore\ \hat{\lambda}_{KLF} = \frac{\sqrt{(n-m_1+1)(n-m_1)}}{S} \;\&\; \rho(\hat{\lambda}_{KLF}) = \frac{2}{n-m_1}$$

Under RQLF: $\hat{\lambda} = \dfrac{E(\lambda^{-1} \mid x)}{E(\lambda^{-2} \mid x)}$ & $\rho(\hat{\lambda}) = 1 - \dfrac{\left[E(\lambda^{-1} \mid x)\right]^2}{E(\lambda^{-2} \mid x)}$

$$E(\lambda^{-2} \mid x) = \frac{S^2}{(n-m_1)(n-m_1-1)}$$

$$\therefore\ \hat{\lambda}_{RQLF} = \frac{(n-m_1-1)}{S} \;\&\; \rho(\hat{\lambda}_{RQLF}) = 1 - \frac{n-m_1-1}{(n-m_1)^2}$$

Under GELF: $\hat{\lambda} = \left\{ E(\lambda^{-c_1} \mid x) \right\}^{-1/c_1}$ & $\rho(\hat{\lambda}) = \ln\{E(\lambda^{-c_1} \mid x)\} + c_1 E(\ln(\lambda \mid x))$

$$E(\lambda^{-c_1} \mid x) = \frac{(n-m_1-c_1)\Gamma(n-m_1-c_1)}{\Gamma(n-m_1+1) S^{-c_1}};\quad E(\ln\lambda \mid x) = \phi(n-m_1+1) - \ln S$$

Where $\phi(n-m_1+1)$ is a digamma function.

$$\therefore\ \hat{\lambda}_{GELF} = \left\{ \frac{(n-m_1-c_1)\Gamma(n-m_1-c_1)}{\Gamma(n-m_1+1) S^{-c_1}} \right\}^{-1/c_1}$$

$$\rho(\hat{\lambda}_{GELF}) = \ln\left\{ \frac{(n-m_1-c_1)\Gamma(n-m_1-c_1)}{\Gamma(n-m_1+1) S^{-c_1}} \right\} + c_1(\phi(n-m_1+1) - \ln S)$$

El-sayyad LF: $\hat{\lambda} = \left\{ \dfrac{E(\lambda^{\alpha+\beta} \mid x)}{E(\lambda^{\alpha} \mid x)} \right\}^{1/\beta}$

$$E(\lambda^{\alpha+\beta} \mid x) = \frac{(n-m_1+\alpha+\beta)\Gamma(n-m_1+\alpha+\beta)}{\Gamma(n-m_1+1) S^{\alpha+\beta}} \;\&\; E(\lambda^{\alpha} \mid x) = \frac{(n-m_1+\alpha)\Gamma(n-m_1+\alpha)}{\Gamma(n-m_1+1) S^{\alpha}}$$

$$\therefore\ \hat{\lambda}_{EL-LF} = \left\{ \frac{(n-m_1+\alpha+\beta)\Gamma(n-m_1+\alpha+\beta)}{(n-m_1+\alpha)\Gamma(n-m_1+\alpha) S^{\beta}} \right\}^{1/\beta} \;\&$$

The risk function in case of El-Sayyad loss function is obtained as

$$\rho(\hat{\lambda}) = \int_0^\infty \lambda^a (\hat{\lambda}^\beta - \lambda^\beta)^2 P(\lambda \mid x)\, d\lambda$$

$$\Rightarrow \rho(\hat{\lambda}_{EL\_SLF}) = \frac{1}{\Gamma(n-m_1+1)S^{\alpha+2\beta}}\left\{(n-m_1+\alpha+2\beta)\Gamma(n-m_1+\alpha+2\beta) - \frac{\left[(n-m_1+\alpha+\beta)\Gamma(n-m_1+\alpha+\beta)\right]^2}{(n-m_1+\alpha)\Gamma(n-m_1+\alpha)}\right\}$$

4 Posterior Distribution of TL-Distribution under Erlang Prior

The Erlang prior is defined as $g(\lambda) \propto \lambda^{b_1} e^{-\lambda/a_1}$; where $a_1, b_1 > 0$ are hyper parameters

Thus the posterior distribution is as $P(\lambda \mid x) \propto \lambda^{n+b_1} e^{-\lambda(S+1/a_1)}$

$$\Rightarrow P(\lambda \mid x) = \frac{(S+1/a_1)^{n+b_1-1}}{\Gamma(n+b_1)} \lambda^{n+b_1-1} e^{-\lambda(S+1/a_1)} \tag{3}$$

4.1 Baye's estimators and risk functions under Erlang prior using KL-LF, RQLF, GELF, EL-Sayyad LF

Under KL-LF: $\hat{\lambda}_{KLF} = \frac{\sqrt{(n+b_1+1)(n+b_1)}}{(S+1/a_1)}$ & $\rho(\hat{\lambda}_{KLF}) = \frac{2}{n+b_1-1}$

Under RQLF: $\hat{\lambda}_{RQLF} = \frac{(n+b_1-2)}{S+1/a_1}$ & $\rho(\hat{\lambda}_{RQLF}) = \frac{1}{n+b_1-1}$

Under GELF: $\hat{\lambda}_{GELF} = \left\{\frac{\Gamma(n+b_1-c_1)}{\Gamma(n+b_1)(S+1/a_1)^{-c_1}}\right\}^{-1/c_1}$

$$\rho(\hat{\lambda}_{GELF}) = \ln\left\{\frac{\Gamma(n+b_1-c_1)\ (S+1/a_1)^{c_1}}{\Gamma(n+b_1)}\right\} + c_1(\phi(n+b_1) - \ln(S+1/a_1))$$

El-sayyad LF: $\hat{\lambda}_{EL-SLF} = \left\{\frac{\Gamma(n+b_1+\alpha+\beta)}{\Gamma(n+b_1+\alpha)(S+1/a_1)^{\beta}}\right\}^{1/\beta}$

The risk function in case of El-Sayyad loss function is obtained as

$$\rho(\hat{\lambda}_{EL-SLF}) = \frac{1}{\Gamma(n+b_1)(S+1/a_1)^{\alpha+2\beta}}\left\{\Gamma(n+b_1+\alpha+2\beta) - \frac{\left[\Gamma(n+b_1+\alpha+\beta)\right]^2}{\Gamma(n+b_1+\alpha)}\right\}.$$

5 Posterior Distribution of TL-Distribution under Mixture of Gamma and Jeffrey's prior

The mixture of gamma and Jeffrey's prior is defined as

$$g(\lambda) \propto \upsilon \frac{l^r}{\Gamma r} \lambda^{r-1} e^{-l\lambda} + (1-\upsilon)\frac{1}{\lambda}; \quad 0<\upsilon<1 \text{ and } l, r > 0 \text{ are hyper parameters.}$$

Thus the posterior distribution is as $P(\lambda \mid x) \propto \upsilon \frac{l^r}{\Gamma r} \lambda^{n+r-1} e^{-\lambda(S+l)} + (1-\upsilon)\lambda^{n-1} e^{-\lambda S}$

$$\Rightarrow P(\lambda \mid x) == \frac{1}{\varpi}\left\{\frac{\upsilon l^r}{\Gamma r} \lambda^{n+r-1} e^{-\lambda(S+l)} + (1-\upsilon)\lambda^{n-1} e^{-\lambda S}\right\} \tag{4}$$

where $\varpi = \upsilon \frac{l^r \Gamma(n+r)}{(S+l)^{n+r}\Gamma r} + (1-\upsilon)\frac{\Gamma n}{S^n}$.

5.1 Baye's estimators and risk functions under mixture of gamma and Jeffrey's prior using KL-LF, RQLF, GELF, El-Sayyad LF

Under KL-LF:
$$\hat{\lambda}_{KLF}=\sqrt{\frac{\upsilon\frac{l^r\Gamma(n+r+1)}{(S+l)^{n+r+1}\Gamma r}+(1-\upsilon)\frac{\Gamma(n+1)}{S^{n+1}}}{\upsilon\frac{l^r\Gamma(n+r-1)}{(S+l)^{n+r-1}\Gamma r}+(1-\upsilon)\frac{\Gamma(n-1)}{S^{n-1}}}}\ \&$$

$$\rho(\hat{\lambda}_{KLF})=2\left\{\frac{\left[\frac{\upsilon l^r\Gamma(n+r-1)}{(S+l)^{n+r-1}\Gamma r}+(1-\upsilon)\frac{\Gamma(n-1)}{S^{n-1}}\right]\left[\frac{\upsilon l^r\Gamma(n+r+1)}{(S+l)^{n+r+1}\Gamma r}+(1-\upsilon)\frac{\Gamma(n+1)}{S^{n+1}}\right]}{\varpi^2}-1\right\}$$

Under RQLF:
$$\hat{\lambda}_{RQLF}=\frac{\frac{\upsilon l^r\Gamma(n+r-1)}{(S+l)^{n+r-1}\Gamma r}+(1-\upsilon)\frac{\Gamma(n-1)}{S^{n-1}}}{\upsilon\frac{l^r\Gamma(n+r-2)}{(S+l)^{n+r-2}\Gamma r}+(1-\upsilon)\frac{\Gamma(n-2)}{S^{n-2}}}\ \&$$

$$\rho(\hat{\lambda}_{RQLF})=1-\frac{\left\{\upsilon\frac{l^r\Gamma(n+r-1)}{(S+l)^{n+r-1}\Gamma r}+(1-\upsilon)\frac{\Gamma(n-1)}{S^{n-1}}\right\}^2}{\varpi\left\{\upsilon\frac{l^r\Gamma(n+r-2)}{(S+l)^{n+r-2}\Gamma r}+(1-\upsilon)\frac{\Gamma(n-2)}{S^{n-2}}\right\}}$$

Under GELF:
$$\hat{\lambda}_{GELF}=\left\{\frac{1}{\varpi}\left\{\upsilon\frac{l^r\Gamma(n+r-c_1)}{(S+l)^{n+r-c_1}\Gamma r}+(1-\upsilon)\frac{\Gamma(n-c_1)}{S^{n-c_1}}\right\}\right\}^{-1/c_1}$$

$$\rho(\hat{\lambda}_{GELF})=\ln\left\{\frac{1}{\varpi}\left\{\frac{\upsilon l^r\Gamma(n+r-c_1)}{(S+l)^{n+r-c_1}\Gamma r}+\frac{(1-\upsilon)\Gamma(n-c_1)}{S^{n-c_1}}\right\}\right\}+c_1\left(\frac{\upsilon l^r\Gamma(n+r)[\phi(n+r)-\ln(S+l)]}{\varpi(S+l)^{n+r}\Gamma r}+\frac{(1-\upsilon)\Gamma(n)[\phi(n)-\ln S]}{\varpi S^n}\right)$$

El-sayyad LF:
$$\hat{\lambda}_{EL-SLF}=\left\{\frac{\frac{\upsilon l^r\Gamma(n+r+\alpha+\beta)}{(S+l)^{n+r+\alpha+\beta}\Gamma r}+\frac{(1-\upsilon)\Gamma(n+\alpha+\beta)}{S^{n+\alpha+\beta}}}{\frac{\upsilon l^r\Gamma(n+r+\alpha)}{(S+l)^{n+r+\alpha}\Gamma r}+\frac{(1-\upsilon)\Gamma(n+\alpha)}{S^{n+\alpha}}}\right\}^{1/\beta}$$

The risk function in case of El-Sayyad loss function is obtained as

$$\Rightarrow\rho(\hat{\lambda}_{EL-LSF})=\frac{1}{\varpi}\left\{\left[\frac{\upsilon l^r\Gamma(n+r+\alpha+2\beta)}{(S+l)^{n+r+\alpha+2\beta}\Gamma r}+\frac{(1-\upsilon)\Gamma(n+\alpha+2\beta)}{S^{n+\alpha+2\beta}}\right]-\frac{\left[\frac{\upsilon l^r\Gamma(n+r+\alpha+\beta)}{(S+l)^{n+r+\alpha+\beta}\Gamma r}+\frac{(1-\upsilon)\Gamma(n+\alpha+\beta)}{S^{n+\alpha+\beta}}\right]^2}{\frac{\upsilon l^r\Gamma(n+r+\alpha)}{(S+l)^{n+r+\alpha}\Gamma r}+\frac{(1-\upsilon)\Gamma(n+\alpha)}{S^{n+\alpha}}}\right\}$$

6 Prior Predictive Distribution under Extension of Jeffrey's prior

The prior predictive distribution is defined as

$$g(y)=\int_0^\infty f(y\,|\,\lambda)\;g(\lambda)\,d\lambda$$

Under extension of Jeffrey's prior predictive distribution is

$$g(y)=\int_0^\infty\frac{2\lambda}{\theta}\left(1-\frac{y}{\theta}\right)\left(\frac{2y}{\theta}-\frac{y^2}{\theta^2}\right)^{\lambda-1}\frac{1}{\lambda^{m_1}}d\lambda$$

$$g(y)=\frac{2\left(1-\frac{y}{\theta}\right)\Gamma(2-m_1)}{\theta\left(\frac{2y}{\theta}-\frac{y^2}{\theta^2}\right)\left\{-\ln\left(\frac{2y}{\theta}-\frac{y^2}{\theta^2}\right)\right\}}$$

7 Prior Predictive Distribution under Erlang Prior

Under Erlang prior, prior predictive distribution is as

$$g(y)=\int_0^\infty \frac{2\lambda}{\theta}\left(1-\frac{y}{\theta}\right)\left(\frac{2y}{\theta}-\frac{y^2}{\theta^2}\right)^{\lambda-1}\frac{1}{a_1\Gamma b_1}\left(\frac{\lambda}{a_1}\right)^{b_1-1}e^{-\lambda/a_1}\,d\lambda$$

$$g(y)=\frac{2\left(1-\frac{y}{\theta}\right)b_1}{\theta\, a_1^{\,b_1}\left(\frac{2y}{\theta}-\frac{y^2}{\theta^2}\right)\left\{\frac{1}{a_1}-\ln\left(\frac{2y}{\theta}-\frac{y^2}{\theta^2}\right)\right\}^{b_1+1}}$$

8 Prior Predictive Distribution under Mixture of Gamma and Jeffrey's prior

Under mixture of gamma and Jeffrey's prior, prior predictive distribution is obtained as

$$g(y)=\int_0^\infty \frac{2\lambda}{\theta}\left(1-\frac{y}{\theta}\right)\left(\frac{2y}{\theta}-\frac{y^2}{\theta^2}\right)^{\lambda-1}\left[\frac{\upsilon\, l^r}{\Gamma r}\lambda^{r-1}e^{-l\lambda}+\frac{(1-\upsilon)}{\lambda}\right]d\lambda$$

$$g(y)=\frac{\upsilon\,2\left(1-\frac{y}{\theta}\right)r\,l^r}{\theta\left(\frac{2x}{\theta}-\frac{x^2}{\theta^2}\right)\left\{l-\ln\left(\frac{2y}{\theta}-\frac{y^2}{\theta^2}\right)\right\}^{r+1}}-\frac{(1-\upsilon)\,2\left(1-\frac{y}{\theta}\right)}{\theta\left(\frac{2x}{\theta}-\frac{x^2}{\theta^2}\right)\ln\left(\frac{2y}{\theta}-\frac{y^2}{\theta^2}\right)}$$

9 Posterior Predictive Distribution under Extension Of Jeffrey's Prior

The posterior predictive distribution for future observations $y=x_{n+1}$ is defined as

$$P_1(y\,|\,\mathrm{x})=\int_0^\infty f(y\,|\,\lambda)\;P(\lambda\,|\,x)\,d\lambda$$

Under extension of Jeffrey's prior, posterior predictive distribution is

$$P_1(y\,|\,\mathrm{x})=\int_0^\infty \frac{2\lambda}{\theta}\left(1-\frac{y}{\theta}\right)\left(\frac{2y}{\theta}-\frac{y^2}{\theta^2}\right)^{\lambda-1}\frac{S^{n-m_1+1}}{\Gamma(n-m+1)}\lambda^{n-m_1}e^{-\lambda S}d\lambda$$

$$P_1(y\,|\,\mathrm{x})=\frac{2\left(1-\frac{y}{\theta}\right)(n-m_1+1)S^{n-m_1+1}}{\theta\left(\frac{2y}{\theta}-\frac{y^2}{\theta^2}\right)\left\{S-\ln\left(\frac{2y}{\theta}-\frac{y^2}{\theta^2}\right)\right\}^{n-m_1+2}}$$

10 Posterior Predictive Distribution under ErlangPrior

Under Erlang prior, posterior predictive distribution is as

$$P_1(y\,|\,\mathrm{x})=\int_0^\infty \frac{2\lambda}{\theta}\left(1-\frac{y}{\theta}\right)\left(\frac{2y}{\theta}-\frac{y^2}{\theta^2}\right)^{\lambda-1}\frac{(S+1/a_1)^{n+b_1-1}}{\Gamma(n+b_1)}\lambda^{n+b_1-1}e^{-\lambda(S+1/a_1)}\,d\lambda$$

$$g(y)=\frac{2\left(1-\frac{y}{\theta}\right)(S+1/a_1)^{n+b_1-1}(n+b_1)}{\theta\left(\frac{2y}{\theta}-\frac{y^2}{\theta^2}\right)\left\{S+\frac{1}{a_1}-\ln\left(\frac{2y}{\theta}-\frac{y^2}{\theta^2}\right)\right\}^{n+b_1+1}}$$

11 Posterior Predictive Distribution under Mixture of Gamma and Jeffrey's prior

Under mixture of gamma and Jeffrey's prior, posterior predictive distribution is obtained as

$$g(y)=\int_0^\infty \frac{2\lambda}{\theta}\left(1-\frac{y}{\theta}\right)\left(\frac{2y}{\theta}-\frac{y^2}{\theta^2}\right)^{\lambda-1}\frac{1}{\varpi}\left\{\frac{\upsilon l^r}{\Gamma r}\lambda^{n+r-1}e^{-\lambda(S+l)}+(1-\upsilon)\lambda^{n-1}e^{-\lambda S}\right\}d\lambda$$

$$g(y)=\frac{2\left(1-\frac{y}{\theta}\right)}{\theta\left(\frac{2x}{\theta}-\frac{x^2}{\theta^2}\right)\varpi}\left\{\frac{\upsilon l^r\Gamma(n+r+1)}{\left[S+l-\ln\left(\frac{2y}{\theta}-\frac{y^2}{\theta^2}\right)\right]^{n+r+1}\Gamma r}+\frac{(1-\upsilon)\Gamma(n+1)}{\left[S-\ln\left(\frac{2y}{\theta}-\frac{y^2}{\theta^2}\right)\right]^{n+1}}\right\}$$

12 Simulation Study

In our simulation study we have generated a sample of sizes n=25, 50, 100 to observe the effect of small, medium, and large samples on the estimators. The results are replicated 1000 times and the average of the results has been presented in the tables. To examine the performance of Bayesian estimates for shape parameter of Topp-Leone distribution under different loss functions the risks for each estimates are presented in parenthesis in the below tables.

Table1: Baye's estimators and posterior risk under Jeffrey's prior

n			m_1=0.5	m_1=1.0	m_1=1.5
25	$\hat{\lambda}_{KLF}$		0.9581 (0.08511)	0.9388 (0.08333)	0.9388 (0.08163)
	$\hat{\lambda}_{RQLF}$		0.9007 (0.96084)	0.8815 (0.96006)	0.8815 (0.95925)
	$\hat{\lambda}_{GELF}$	C_1=1	0.9391 (0.02112)	0.9199 (0.02068)	0.9582 (0.02026)
		C_1=-1	0.9774 (0.02054)	0.9582 (0.02013)	0.9007 (0.01973)
	$\hat{\lambda}_{EL-SLF}$		1.0252 (0.08984)	1.0061 (0.08484)	0.9869 (0.08003)
50	$\hat{\lambda}_{KLF}$		0.7821 (0.04123)	0.7743 (0.04081)	0.7664 (0.04040)
	$\hat{\lambda}_{RQLF}$		0.7587 (0.98020)	0.7508 (0.98001)	0.7431 (0.97981)
	$\hat{\lambda}_{GELF}$	C_1=1	0.7743 (0.01027)	0.7665 (0.01016)	0.7587 (0.01006)
		C_1=-1	0.7899 (0.01013)	0.7821 (0.01003)	0.7743 (0.00993)
	$\hat{\lambda}_{EL-SLF}$		0.8095 (0.01835)	0.8017 (0.01782)	0.7938 (0.01730)
100	$\hat{\lambda}_{KLF}$		0.8909 (0.02030)	0.8865 (0.02020)	0.8821 (0.02010)
	$\hat{\lambda}_{RQLF}$		0.8776 (0.99005)	0.8731 (0.99000)	0.8687 (0.98995)
	$\hat{\lambda}_{GELF}$	C_1=1	0.8865 (0.00502)	0.8821 (0.00501)	0.8776 (0.00500)
		C_1=-1	0.8954 (0.00503)	0.8910 (0.00503)	0.8865 (0.00498)
	$\hat{\lambda}_{EL-SLF}$		0.9065 (0.041316)	0.9021 (0.04027)	0.8976 (0.03925)

Table2: Baye's estimate and posterior risk under erlang prior

n			$a_1=1;b_1=1.5$	$a_1=2;b_1=2.5$	$a_1=3;b_1=3.5$
25	$\hat{\lambda}_{KLF}$		0.9965 (0.07843)	1.0529 (0.07547)	1.0973 (0.07272)
	$\hat{\lambda}_{RQLF}$		0.9044 (0.03921)	0.9590 (0.03773)	1.0029 (0.03636)
	$\hat{\lambda}_{GELF}$	$C_1=1$	0.9413 (0.01947)	0.9966 (0.01874)	1.0407 (0.01764)
		$C_1=-1$	0.9782 (0.01898)	1.0342 (0.01829)	1.0786 (0.01807)
	$\hat{\lambda}_{EL-SLF}$		1.0243 (0.03245)	1.0812 (0.03227)	1.1258 (0.03220)
50	$\hat{\lambda}_{KLF}$		0.8008 (0.03960)	0.8226 (0.03883)	0.84032 (0.03809)
	$\hat{\lambda}_{RQLF}$		0.7624 (0.01980)	0.7838 (0.01941)	0.80145 (0.01904)
	$\hat{\lambda}_{GELF}$	$C_1=1$	0.7778 (0.00986)	0.7993 (0.00967)	0.81701 (0.00949)
		$C_1=-1$	0.7932 (0.00974)	0.8149 (0.00955)	0.8325 (0.00937)
	$\hat{\lambda}_{EL-SLF}$		0.8124 (0.04092)	0.8342 (0.04084)	0.8520 (0.04081)
100	$\hat{\lambda}_{KLF}$		0.9007 (0.01990)	0.9136 (0.01970)	0.9238 (0.01951)
	$\hat{\lambda}_{RQLF}$		0.8787 (0.00995)	0.8914 (0.00985)	0.9016 (0.00975)
	$\hat{\lambda}_{GELF}$	$C_1=1$	0.8875 (0.00496)	0.9003 (0.00491)	0.9105 (0.00487)
		$C_1=-1$	0.8963 (0.00493)	0.9092 (0.00488)	0.9194 (0.00483)
	$\hat{\lambda}_{EL-SLF}$		0.9074 (0.06825)	0.9203 (0.06765)	0.9305 (0.06523)

Table3: Baye's estimators and posterior risk under mixture of gamma and Jeffrey's prior

n			$v=0.3$			$v=0.9$		
			$l=1;r=1.5$	$l=2;r=2.5$	$l=3;r=3.5$	$l=1;r=1.5$	$l=2;r=2.5$	$l=3;r=3.5$
25	$\hat{\lambda}_{KLF}$		0.9566 (0.95109)	0.8827 (0.18490)	0.5336 (0.15390)	1.0462 (0.90103)	0.7489 (0.07805)	0.3105 (0.05129)
	$\hat{\lambda}_{RQLF}$		0.8841 (1)	0.8889 (1)	0.9034 (1)	0 .8982 (1)	0.9049 (1)	0.9105 (1)
	$\hat{\lambda}_{GELF}$	$C_1=1$	0.9224 (0.04925)	0.9266 (0.02516)	0.9389 (0.01988)	0.9356 (0.05159)	0.9409 (0.01938)	0.9449 (0.01582)
		$C_1=-1$	0.9606 (0.02133)	0.9642 (0.02117)	0.9744 (0.02079)	0.9729 (0.02057)	0.9768 (0.02029)	0.9794 (0.02006)
	$\hat{\lambda}_{EL-SLF}$		1.0082 (0.08510)	1.0112 (0.08450)	1.0186 (0.08149)	1.0196 (0.08613)	1.0216 (0.07994)	1.0224 (0.07659)
50	$\hat{\lambda}_{KLF}$		0.7868 (0.19077)	0.7314 (0.10669)	0.4564 (0.03105)	0.2580 (0.04020)	0.6176 (0.03952)	0.2580 (0.03238)
	$\hat{\lambda}_{RQLF}$		0.7520 (1)	0.7547 (1)	0.7642 (1)	0.7691 (1)	0.7641 (1)	0.7691 (1)
	$\hat{\lambda}_{GELF}$	$C_1=1$	0.7677 (0.00313)	0.7703 (0.00307)	0.7794 (0.00263)	0.7841 (0.01085)	0.7794 (0.01061)	0.7841 (0.00964)
		$C_1=-1$	0.7833 (0.01792)	0.7859 (0.01427)	0.7946 (0.01702)	0.7991 (0.01645)	0.7947 (0.00988)	0.7991 (0.00972)
	$\hat{\lambda}_{EL-SLF}$		0.7911 (0.01810)	0.7937 (0.01797)	0.8022 (0.01778)	0.8065 (0.01963)	0.8023 (0.01854)	0.8065 (0.01840)
100	$\hat{\lambda}_{KLF}$		0.8866 (0.02017)	0.8869 (0.02012)	0.6902 (0.02006)	0.8873 (0.02994)	0.8881 (0.00287)	0.4711 (0.02003)
	$\hat{\lambda}_{RQLF}$		0.8733 (1)	0.8735 (1)	0.8749 (1)	0.8740 (1)	0.8750 (1)	0.8763 (1)
	$\hat{\lambda}_{GELF}$	$C_1=1$	0.8822 (0.00203)	0.8824 (0.00201)	0.8837 (0.00197)	0.8828 (0.00500)	0.8838 (0.00496)	0.8850 (0.00491)
		$C_1=-1$	0.8911 (0.00201)	0.8913 (0.00200)	0.8925 (0.00413)	0.8917 (0.00497)	0.8925 (0.00492)	0.8937 (0.00489)
	$\hat{\lambda}_{EL-SLF}$		0.8978 (0.01527)	0.8980 (0.01525)	0.8990 (0.01513)	0.8983 (0.01521)	0.8991 (0.01512)	0.9002 (0.01501)

Real life example: To examine the applicability of the results real life data sets are analyzed. The data reported by Butler [18] has been used to show the use of above findings.

Survival time of 30 light bulbs in months														
0.020	0.025	0.059	0.062	0.145	0.186	0.196	0.197	0.205	0.210	0.262	0.314	0.511	0.604	0.678
0.695	0.740	0.760	0.846	0.860	0.914	0.992	1.181	1.194	0.309	1.995	20255	2.509	2.910	5.543

The Bayes estimates and posterior risk (given in parenthesis) under KLF, RQLF, GELF, EL-SLF based on non-informative and informative priors have been presented in the below tables.

Table4: Baye's estimators and posterior risk under Jeffrey's prior

		m_1=0.5	**m_1=1.0**	**m_1=1.5**
$\hat{\lambda}_{KLF}$		0.7140 (0.0707)	0.7021 (0.0689)	0.6902 (0.0502)
$\hat{\lambda}_{RQLF}$		0.7260 (0.9672)	0.7141 (0.9567)	0.7022 (0.9461)
$\hat{\lambda}_{GELF}$	**C_1=1**	0.00078 (0.0178)	0.00079 (0.0171)	0.00081 (0.0164)
	C_1=-1	0.0008 (0.0174)	0.00082 (0.0167)	0.00083 (0.0160)
$\hat{\lambda}_{EL-SLF}$		0.7557 (0.0224)	0.7438 (0.0214)	0.7319 (0.0204)

Table5: Baye's estimators and posterior risk under Erlang prior

		a_1=1;b_1=1.5	***a_1=2;b_1=2.5***	***a_1=3;b_1=3.5***
$\hat{\lambda}_{KLF}$		0.7439 (0. 0655)	0.7762 (0.0634)	0.8029 (0.0615)
$\hat{\lambda}_{RQLF}$		0.6859 (0.0327)	0.7175 (0.03174)	0.7439 (0.0307)
$\hat{\lambda}_{GELF}$	**C_1=1**	0.7091 (0.0163)	0.7410 (0.0157)	0.7675 (0.0153)
	C_1=-1	0.7324 (0.0159)	0.7645 (0.0154)	0.7911 (0.0149)
$\hat{\lambda}_{EL-SLF}$		0.7614 (0.0369)	0.7939 (0.0358)	0.8206 (0.0348)

Table6: Baye's estimators and posterior risk under mixture of gamma and Jeffrey's prior

	v=0.3				***v=0.9***		
		***l*=1;*r*=1.5**	***l*=2;*r*=2.5**	***l*=3;*r*=3.5**	***l*=1;*r*=1.5**	***l*=2;*r*=2.5**	***l*=3;*r*=3.5**
$\hat{\lambda}_{KLF}$		0.7137 (0.1623)	0.6683 (0.1581)	0.4311 (0.1263)	0.7808 (1.7646)	0.5697 (0.6207)	0.2406 (0.5774)
$\hat{\lambda}_{RQLF}$		0.6683 (1)	0.6726 (1)	0.6892 (1)	0.6798 (1)	0 .6894 (1)	0.6991 (1)
$\hat{\lambda}_{GELF}$	**C_1=1**	0.6921 (0.0181)	0.6964 (0.0174)	0.7122 (0.0020)	0.7034 (0.1058)	0.7124 (0.1039)	0.7214 (0.0615)
	C_1=-1	0.7159 (0.0256)	0.7201 (0.0209)	0.7352 (0.0138)	0.7269 (0.0787)	0.7353 (0.0736)	0.7437 (0.0299)
$\hat{\lambda}_{EL-SLF}$		0.7456 (0.0223)	0.7496 (0.0215)	0.7639 (0.0203)	0.7562 (0.0222)	0.7640 (0.0215)	0.7715 (0.0204)

13 Conclusion

From the above tables we conclude that the posterior risks based on loss functions and priors decrease with the increased sample size. It implies that the estimators obtained are consistent. The interesting fact to note is that in real life data as well as in simulation study the posterior risks under GELF based on Erlang prior are less, and decreases with increase in sample size. Furthermore, the performance of estimates obtained under El-sayyad loss function is also efficient as the posterior risks under GELF and El-sayyad loss function are close for different values of hyper-parameters. The other important point to note is that under mixture prior the posterior risks under RQLF remains constant in both the cases, on increasing the sample size it remains constant as well. It can also be observed that the Bayes estimates perform better under informative priors than non-informative prior.

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