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Using Known Zeta-Series to Derive the Dancs-He Series for $\ln 2$ and $\zeta(2n+1)$

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Abstract: In a recent work, Dancs and He found new formulas for $\ln 2$ and $\zeta(2n+1)$, n being a positive integer, which are expressed in terms of Euler polynomials, each containing a series that apparently can not be evaluated in closed form, distinctly from $\zeta(2n)$, for which the Euler's formula allows us to write it as a rational multiple of π^{2n}. There in that work, however, the formulas are derived through certain series manipulations, by following Tsumura's strategy, which makes it *curious* — in the words of those authors themselves — the appearance of the number $\ln 2$. In this note, I show how some known zeta-series can be used to derive the Dancs-He series in an alternative manner.

Keywords: Riemann zeta function, Euler's formula, Zeta-series

1 Introduction

The Riemann zeta function is defined, for real values of s, $s > 1$, by[1]

$$\zeta(s) := \sum_{k=1}^{\infty} \frac{1}{k^s} \, . \tag{1}$$

For these values of s, the series converges according to the integral test. For integer values of s, its sum has attracted much interest since the times of J. Bernoulli, who proved that $\sum_{k=1}^{\infty} 1/k^2$ converges to a number between 1 and 2. Further, Euler (1735) proved that this sum evaluates to $\pi^2/6$, solving the so-called *Basel problem*. For greater integer values of s, Euler found the notable formula (1750)

$$\zeta(2n) = (-1)^{n-1} \, \frac{2^{2n-1} \, \pi^{2n}}{(2n)!} \, B_{2n}, \tag{2}$$

where n is a positive integer and B_{2n} are Bernoulli numbers.[2] For odd values of s, $s > 1$, on the other hand, no analogous closed-form expression is known. In fact, not even an irrationality proof is known for $\zeta(2n+1)$, the only exception being the Apéry proof that $\zeta(3)$ is irrational (1978) [2], which makes the things enigmatic.

On trying to find out a closed-form expression for $\zeta(2n+1)$ similar to that in Eq. (2), Dancs and He found a new formula containing series involving the numbers $E_{2n+1}(1)$, where $E_{2n+1}(x)$ denotes the Euler's polynomial of degree $2n+1$ [3].[3] Their main result follows from some intricate series manipulations, in the lines of those found in Tsumura's proof of Eq. (2) [1]. However, the fortuitous appearance of the numbers $\ln 2$ and $\zeta(2n+1)$ in the Dancs-He formulae, which is hard to be explained with usual series expansions, might well remain a mystery. By noting that the numbers $E_{2n+1}(1)$ can be written in terms of B_{2n+2}, and then in terms of $\zeta(2n)$, via Eq. (2), I show here in this work how the

[1] There is also a product representation due to Euler (1749), namely $\zeta(s) = \prod_p 1/(1-p^{-s})$, taken over all prime numbers p, which is the main reason for the interest of number theorists in this function. As noted by Euler, the divergence of the harmonic series, i.e. $\lim_{s \to 1^+} \zeta(s) = \infty$, implies, according to the product representation, that there is an infinitude of prime numbers.

[2] Since $B_{2n} \in \mathbb{Q}$ and π is a transcendental number, as first proved by Lindemann (1882), then Eq. (2) implies that $\zeta(2n)$ is a transcendental number.

[3] The Euler polynomials are given by the generating function $2\,\frac{e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x)\,\frac{t^n}{n!}$, with $E_0(x) = 1$.

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Dancs-He series for $\ln 2$ and $\zeta(2n+1)$ can be derived from some known zeta-series.

2 Dancs-He formula for $\ln 2$

For the positive real number $\ln 2$, Dancs and He found the following series representation (see Eq. (2.6) of Ref. [3]).

Theorem 2.1. [Dancs-He series for $\ln 2$] Let $E_{2m+1}(x)$ denote the Euler's polynomial of degree $2m+1$, m being a nonnegative integer. Then

$$\ln 2 = \frac{\pi^2}{2} \sum_{m=0}^{\infty} (-1)^m \frac{\pi^{2m}}{(2m+3)!} E_{2m+1}(1) \, .$$

Proof. Let L be the number to which the above series converges. By noting that

$$E_{2m+1}(1) = -E_{2m+1}(0) = 2\,\frac{2^{2m+2}-1}{2m+2}\,B_{2m+2} \, ,$$

it follows that

$$\begin{aligned} L &= \pi^2 \sum_{m=0}^{\infty} (-1)^m \frac{\pi^{2m}}{(2m+3)!} \left(2^{2m+2}-1\right) \frac{B_{2m+2}}{2m+2} \\ &= \sum_{m=0}^{\infty} (-1)^m \frac{(2\pi)^{2m+2} - \pi^{2m+2}}{(2m+3)!} \, \frac{B_{2m+2}}{2m+2} \\ &= -\sum_{m=0}^{\infty} (-1)^{m+1} \frac{B_{2m+2}}{(2m+2)\,(2m+3)\,(2m+2)!} \\ &\qquad \times \left[(2\pi)^{2m+2} - \pi^{2m+2}\right] . \end{aligned}$$

By substituting $n = m+1$, one finds that

$$\begin{aligned} L &= -\sum_{n=1}^{\infty} (-1)^n \frac{B_{2n}}{2n(2n+1)(2n)!} \left[(2\pi)^{2n} - \pi^{2n}\right] \\ &= -\sum_{n=1}^{\infty} (-1)^n \frac{\pi^{2n} B_{2n}}{(2n)!} \, \frac{2^{2n}-1}{2n(2n+1)} \, . \end{aligned}$$

From Euler's formula for $\zeta(2n)$ in Eq. (2), one has

$$L = \sum_{n=1}^{\infty} \left(1-2^{-2n}\right) \frac{\zeta(2n)}{n\,(2n+1)} \, . \tag{3}$$

Now, let us reduce this latter series to a simple closed-form expression. For this, let us make use of the following series representation for $\zeta(s)$ introduced recently by Tyagi and Holm (see Eq. (3.5) in Ref. [4]):

$$\begin{aligned} \frac{\zeta(s)\cdot\left(1-2^{1-s}\right)}{\pi^{s-1}\sin(\pi s/2)} &= \sum_{n=1}^{\infty} \left(2-2^{s-2n}\right) \frac{\Gamma(2n-s+1)}{\Gamma(2n+2)} \\ &\qquad \times \zeta(2n-s+1) \, , \end{aligned} \tag{4}$$

where $\Gamma(x)$ is the gamma function.[4] As the series on the right-hand side converges when we make $s=1$, all we need to do is to take the limit, as $s \to 1^+$, of the factors at the left-hand side. Since

$$\begin{aligned} &\lim_{s\to 1^+} \zeta(s)\,\left(1-2^{1-s}\right) \\ &= \lim_{s\to 1^+} \zeta(s)\,(s-1) \times \lim_{s\to 1^+} \frac{1-2^{1-s}}{s-1} = 1 \times \ln 2 \, , \end{aligned} \tag{5}$$

then

$$\frac{\ln 2}{\pi^0 \sin(\pi/2)} = 2 \sum_{n=1}^{\infty} \left(1-2^{-2n}\right) \frac{(2n-1)!}{(2n+1)!} \, \zeta(2n) \, , \tag{6}$$

which simplifies to

$$\sum_{n=1}^{\infty} \left(1-2^{-2n}\right) \frac{\zeta(2n)}{n\,(2n+1)} = \ln 2 \, .$$

From Eq. (3), one has $L = \ln 2$. $\square$.

3 Dancs-He formula for $\zeta(2m+1)$

Before presenting a general proof for the Dancs-He series for $\zeta(2n+1)$, n being a positive integer, let us tackle the lowest case, i.e. $\zeta(3)$, a number for which several series representations have been derived since the times of Euler [6]. As will be shown below, the lowest odd zeta value can be derived from some known zeta-series, independently of the general result that will be established in the next theorem. For $\zeta(3)$, Dancs and He found the following series representation (see Eq. (3.1) of Ref. [3]).

Theorem 3.1. [Dancs-He series for $\zeta(3)$]

$$\zeta(3) = \frac{\pi^2}{9} \ln 4 - \frac{2\,\pi^4}{3} \sum_{m=0}^{\infty} (-1)^m \frac{\pi^{2m}}{(2m+5)!} E_{2m+1}(1) \, .$$

Proof. Let S be the number for which the series at the right-hand side of this theorem converges. By substituting $E_{2m+1}(1) = 2\,\frac{2^{2m+2}-1}{2m+2}\,B_{2m+2}$ in this series, one has

$$\begin{aligned} \pi^2 S &= -2 \sum_{m=0}^{\infty} (-1)^{m+1} \frac{\pi^{2m+2}}{(2m+5)!} \left(2^{2m+2}-1\right) \frac{B_{2m+2}}{2m+2} \\ &= -2 \sum_{m=0}^{\infty} \frac{(-1)^{m+1}\left[(2\pi)^{2m+2} - \pi^{2m+2}\right]}{(2m+2)\,(2m+3)\,(2m+4)\,(2m+5)} \\ &\qquad \times \frac{B_{2m+2}}{(2m+2)!} \, . \end{aligned} \tag{7}$$

By substituting $n = m+1$, one finds that

$$\pi^2 S = -2 \sum_{n=1}^{\infty} \frac{(-1)^n\left[(2\pi)^{2n} - \pi^{2n}\right]}{2n\,(2n+1)\,(2n+2)\,(2n+3)} \, \frac{B_{2n}}{(2n)!} \, . \tag{8}$$

From Eq. (2), one has

$$\begin{aligned} &\frac{\pi^2}{4} S = \sum_{n=1}^{\infty} \frac{\zeta(2n)}{2n\,(2n+1)\,(2n+2)\,(2n+3)} \\ &- \sum_{n=1}^{\infty} \frac{\zeta(2n)}{2n\,(2n+1)\,(2n+2)\,(2n+3)\cdot 2^{2n}} \, , \end{aligned} \tag{9}$$

[4] Note that $\Gamma(k) = (k-1)!$ for positive integer values of k.

which is valid since the series in Eq. (8) converges absolutely. The first series can be easily evaluated from a known summation formula (see Eq. (713) in Ref. [5]), namely

$$\sum_{k=1}^{\infty} \frac{\zeta(2k)}{k\,(k+1)\,(2k+1)\,(2k+3)}\, t^{2k} = \frac{\zeta(3)}{2\pi^2}\, t^{-2} + \frac{\ln(2\pi)}{3} - \frac{11}{18} + \frac{t^{-3}}{3}\left[\zeta'(-3,1+t) - \zeta'(-3,1-t)\right], \quad (10)$$

where $\zeta(s,a)$ is the Hurwitz (or generalized) zeta function and $\zeta'(s,a)$ is its derivative with respect to s.[5] As this formula is valid for all t with $0 < |t| < 1$, it is legitimate to take the limit as $t \to 1^-$ on both sides, which yields

$$\sum_{k=1}^{\infty} \frac{\zeta(2k)}{2k\,(2k+1)\,(2k+2)\,(2k+3)} = \frac{\zeta(3)}{8\pi^2} + \frac{\ln(2\pi)}{12} - \frac{11}{72} + \frac{1}{12}\lim_{t\to 1^-}\left[\zeta'(-3,1+t) - \zeta'(-3,1-t)\right]. \quad (11)$$

The remaining limit is null because $\lim_{t\to 1^-} \zeta'(-3,1+t) = \zeta'(-3,2) = \zeta'(-3) = \lim_{t\to 1^-} \zeta'(-3,1-t)$, which reduces Eq. (11) to

$$\sum_{k=1}^{\infty} \frac{\zeta(2k)}{2k\,(2k+1)\,(2k+2)\,(2k+3)} = \frac{\zeta(3)}{8\pi^2} + \frac{\ln(2\pi)}{12} - \frac{11}{72}. \quad (12)$$

For the second series in Eq. (9), let us make use of the Wilton's formula (Eq. (38) at p. 303, in Ref. [6]; also Eq. (54) in Ref. [7]):

$$\sum_{k=1}^{\infty} \frac{\zeta(2k)}{k\,(k+1)\,(2k+1)\,(2k+3)\cdot 2^{2k}} = \frac{2\zeta(3)}{\pi^2} + \frac{\ln\pi}{3} - \frac{11}{18}, \quad (13)$$

which can be written as

$$\sum_{k=1}^{\infty} \frac{\zeta(2k)}{2k\,(2k+1)\,(2k+2)\,(2k+3)\cdot 2^{2k}} = \frac{\zeta(3)}{2\pi^2} + \frac{\ln\pi}{12} - \frac{11}{72}. \quad (14)$$

Now, by doing a member-to-member subtraction of Eqs. (12) and (14) and putting the result in Eq. (9), one finds

$$\frac{2\pi^2}{9}\ln 2 - \frac{2\pi^4}{3} S = \zeta(3),$$

which completes the proof. □.

Now, let us generalize the above result for all $\zeta(2m+1)$, m being a positive integer. The result below is found in Eq. (3.1) of Ref. [3].

[5] The Hurwitz zeta function is classically defined for $\Re(s) > 1$ as $\zeta(s,a) := \sum_{k=0}^{\infty} 1/(k+a)^s$ $(a \neq 0,-1,-2,\ldots)$, and its meromorphic continuation over the whole s-plane, with $\zeta(s,1) = \zeta(s)$, except by a simple pole at $s = 1$.

Theorem 3.2. [Dancs-He series for odd zeta-values] For any integer m, $m > 0$,

$$\left(1-2^{-2m}\right)\zeta(2m+1) = \sum_{j=1}^{m-1} \frac{(-1)^j\,\pi^{2j}}{(2j+1)!}\left(2^{2j-2m}-1\right)\zeta(2m-2j+1) - \frac{(-1)^m\,\pi^{2m}\ln 2}{(2m+1)!} + \frac{(-1)^m\,\pi^{2m+2}}{2}\sum_{k=0}^{\infty}(-1)^k\,\frac{\pi^{2k}E_{2k+1}(1)}{(2k+2m+3)!}.$$

Proof. Let $\tilde{S}$ be the number for which the above infinite series converges, i.e.

$$\tilde{S} := \sum_{k=0}^{\infty}(-1)^k\,\frac{\pi^{2k}E_{2k+1}(1)}{(2k+2m+3)!}. \quad (15)$$

By substituting $E_{2k+1}(1) = 2\,\frac{2^{2k+2}-1}{2k+2}\,B_{2k+2}$ in this series, one finds

$$\pi^2\,\tilde{S} = \sum_{k=0}^{\infty}(-1)^k\,\frac{\pi^{2k+2}}{(2k+2m+3)!}\,2\left(2^{2k+2}-1\right)\frac{B_{2k+2}}{2k+2}. \quad (16)$$

By putting $n = k+1$, one has

$$\pi^2\,\tilde{S} = -2\sum_{n=1}^{\infty}\frac{(-1)^n\pi^{2n}}{(2n+2m+1)!}\,\frac{\left(2^{2n}-1\right)}{2n}\,B_{2n}. \quad (17)$$

From Eq. (2), one has

$$\frac{\pi^2}{2}\,\tilde{S} = 2\sum_{n=1}^{\infty}\left(1-2^{-2n}\right)\frac{(2n-1)!}{(2n+2m+1)!}\,\zeta(2n) = \sum_{n=1}^{\infty}\left(2-2^{1-2n}\right)\frac{\Gamma(2n)}{\Gamma(2n+2m+2)}\,\zeta(2n). \quad (18)$$

This latter series is just the one that appears in a formula for odd zeta-values derived recently by Milgran (see Eq. (13) in Ref. [8]), namely

$$\zeta(2m+1) = \frac{(-1)^m\,\pi^{2m}}{1-2^{-2m}}\left[-\frac{\ln 2}{(2m+1)!} + \frac{\pi^2}{2}\,\tilde{S}\right] + \frac{1}{1-2^{-2m}}\sum_{n=1}^{m-1}\left(2^{2n-2m}-1\right)\left(-\pi^2\right)^n\frac{\zeta(2m-2n+1)}{(2n+1)!}. \quad (19)$$

By multiplying both sides by $1-2^{-2m}$, one finds

$$\begin{aligned}\left(1-2^{-2m}\right)\zeta(2m+1) &= (-1)^m\,\pi^{2m}\left[-\frac{\ln 2}{(2m+1)!} + \frac{\pi^2}{2}\,\tilde{S}\right] + \sum_{n=1}^{m-1}(-1)^n\left(2^{2n-2m}-1\right)\pi^{2n}\,\frac{\zeta(2m-2n+1)}{(2n+1)!} \\ &= -(-1)^m\,\pi^{2m}\,\frac{\ln 2}{(2m+1)!} + (-1)^m\,\frac{\pi^{2m+2}}{2}\,\tilde{S} + \sum_{n=1}^{m-1}\frac{(-1)^n\,\pi^{2n}}{(2n+1)!}\left(2^{2n-2m}-1\right)\zeta(2m-2n+1),\end{aligned}$$

which completes the proof. □.

4 Conclusions

In a recent paper, Dancs and He have derived some nice series expansions for $\ln 2$ and $\zeta(2n+1)$ involving the numbers $E_{2n+1}(1)$ [3]. However, the appearances of $\ln 2$ and $\zeta(2n+1)$ in their formulas is hard to be explained with usual trigonometric or power series expansions, so they could remain a mystery. On noting that the numbers $E_{2n+1}(1)$ can be written in terms of the Bernoulli numbers B_{2n+2}, and then in terms of $\zeta(2n)$, via our Eq. (2), in this note I have shown how certain known zeta-series can be used to derive alternative proofs for the Dancs-He formulas, thus clarifying the origin of $\ln 2$ and $\zeta(2n+1)$ in these formulas. As the infinite series in our theorems apparently can not be reduced to finite closed forms, some insight was given into why the odd zeta values are more difficult, but I hope my approach to these series can stimulate further investigations on this subject.

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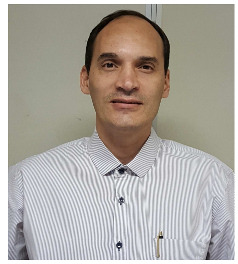

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