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Testing Exponentiality Aganist New Better Than Renwal Used in Laplace Transform Order

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Abstract: In this paper a new class of life distribution is proposed, named new better than renewal used in Laplace transform order (NBRUL). Test statistic for testing exponentiality versus (NBRUL) based on U-statistic is proposed. Pitman's asymptotic efficiencies of the test are calculated and compared with other tests. The percentiles of this test statistic are tabulated and the powers of this test are estimated for some famously alternatives distributions in reliability such as Weibull, linear failure rate (LFR) and Gamma distributions. Finally, examples in different areas are used as practical applications of the proposed test.

Keywords: Classes of life distributions, life testing, NBRUL.

1 Introduction

Statistical inferences are used to project the data from the sample to the entire population. Statistical inference based on two main branches one of them the estimation and the other is testing hypotheses. In general, we do not know the true value (claim) of population parameters - they must be estimated. However, we do have hypotheses about what the true values (claims) are. The hypothesis actually to be tested is usually given the symbol $H_\circ$, and is commonly referred to as the null hypothesis. The other hypothesis, which is assumed to be true when the null hypothesis is false, is referred to as the alternative hypothesis, and is often symbolized H_1. Both the null and alternative hypothesis should be stated before any statistical test of significance is conducted.

In this paper real data is given and we desire to test H_0: data is exponential versus the alternative hypothesis H_1: data is not exponential. To choose between H_0 and H_1 or to make a decision we need to define test statistic. The test statistic is a random variable used to determine how close a specific sample result falls to one of the hypotheses being tested. Many classes of life distributions are defined and used to construction the test statistics where the exponential distribution was a main member in these classes. The motivation for choosing the exponential distribution as null hypothesis return to the importance of this distribution in different fields. The main classes of life distributions which have been introduced in the literature are based on new better than used (NBU), new better than used failure rate (NBUFR), new better than average failure rate (NBAFR), new better than used renewal failure rate (NBURFR), new better than used average renewal failure rate (NBARFR), new better than renewal used (NBRU) and exponential better than used in Laplace transform order (EBUL). For more detailed discussion on properties and some possible applications we refer to [10,11,3,4,2,14] and [21]. Testing exponentiality against some classes of life distributions has been introduced by many researchers. For testing exponentiality versus (NBU) class see [7], [5] for (NBAFR) class, [16] for (NBARFR) class, [17] for (NBURFR) class and [1] for (RNBRUE) class. Testing exponentiality based on goodness of fit approach against many classes of life distributions was studied by some authors such as [6,8,13,18] and [20].

The rest of this paper can be organized as followes, Section 2 gives a brief knowledge about renewal classes. In Section 3 we present a procedure to test a continuous life distribution $F(x)$ is exponential versus it is NBRUL and not exponential.In Section 4 the Pitman asympototic efficiencies are calculated for our test. In Section 5 we simulate Monte Carlo null distribution critical points from the null distributions for sample size $n = 5(5)35, 39, 40(5)50$. In Section 6 the power estimates are calculated. for the test. Finally, the application of the proposed test for real data sets are discussed in Section 7.

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2 Renewal Classes

Consider a device (system or component) with life time T and a continuous life distribution $F(t)$, is put on operation. When the failure occures the device will be replaced by a sequence of mutually independent devices. The spare devices are independent of the first device and identically distributed with the same life distribution $F(t)$. In the long run, the remaining life distribution of the system under operation at time t is given by stationary renewal distribution as follows:

$$W_f(t) = \mu_F^{-1}\int_0^t \overline{F}(u)\,du, \qquad 0 \le t < \infty, \tag{1}$$

where $\mu_F = \mu = \int_0^\infty \overline{F}(u)\,du$.

The corresponding renewal survival function is

$$\overline{W_F}(t) = \mu_F^{-1}\int_t^\infty \overline{F}(u)\,du, \qquad 0 \le t < \infty.$$

For more details, see [4] and [9]. In the following, definitions of some renewal classes of life distributions are given.

Definition (2.1)

If X a random variable with survival function $\bar{F}(x)$, then X is said to have new better (worse) than renewal used property, denoted by NBRU (NWRU), if

$$\overline{W_F}(x|t) \le (\ge)\overline{F}(x|0),\ x \ge 0,\ t \ge 0,$$

or

$$\overline{W_F}(x+t) \le (\ge)\overline{W_F}(t)\,\overline{F}(x),\ x \ge 0,\ t \ge 0.$$

This definition is introduced by [2]. New definition of a new renewal class of life distributions is proposed in the following definition.

Definition (2.2)

X is said to be NBRUL (NWRUL) if

$$\int_0^\infty e^{-sx}\overline{W_F}(x+t)\,dx \le (\ge)\overline{W_F}(t)\int_0^\infty e^{-sx}\overline{F}(x)\,dx,\ x \ge 0,\ t \ge 0 \text{ and } s \ge 0.$$

It is obvious that NBRUL contains the NBRU class, i. e. NBRU $\sqsubseteq$ NBRUL. Also,

$$NBRU \Rightarrow NBRUL \Rightarrow NBRUE$$

3 Testing Against NBRUL Class

In this section, we test the null hypothesis $H_0 : F$ is exponential versus the alternative hypothesis $H_1 : F$ is $NBRUL$ and not exponential. The following lemma is needed.

Lemma (3.1)

If X is a random variable with distribution function F and F belongs to $NBRUL$ class, then

$$\frac{1}{s^3}\phi(s) + \frac{1}{s^2}\mu - \frac{1}{s^3} \ge \frac{1}{2s}\mu_{(2)}\phi(s)\ ,s \ge 0, \tag{2}$$

where $\phi(s) = Ee^{-sX} = \int\limits_0^\infty e^{-sx}dF(x)$ and $\mu_{(2)} = 2\int_0^\infty x\overline{F}(x)\,dx$.

Proof.

Since F is $NBRUL$ then

$$\int_0^\infty e^{-sx}\overline{W_F}(x+t)\,dx \le \overline{W_F}(t)\int_0^\infty e^{-sx}\overline{F}(x)\,dx,, \quad x,t \ge 0.$$

Integrating both sides with respect to t over $[0,\infty)$, gives

$$\int_0^\infty\int_0^\infty e^{-sx}\overline{W_F}(x+t)\,dxdt \le \int_0^\infty\int_0^\infty e^{-sx}\overline{F}(x)\,\overline{W_F}(t)\,dxdt, \tag{3}$$

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setting

$$I_1=\int_0^\infty\int_0^\infty e^{-sx}\overline{W_F}(x+t)\,dxdt.$$

I_1 can be rewritten as

$$I_1=E\int_0^X[\frac{1}{s}X+\frac{1}{s^2}e^{-sX}e^{st}-\frac{1}{s^2}-\frac{1}{s}t]dt.$$

So,

$$I_1=\frac{1}{2s}\mu_{(2)}-\frac{1}{s^3}\phi(s)-\frac{1}{s^2}\mu+\frac{1}{s^3}. \quad (4)$$

Similary if we set

$$I_2=\int_0^\infty\int_0^\infty e^{-sx}\overline{F}(x)\,\overline{W_F}(t)\,dxdt,$$

then

$$I_2=\frac{1}{2s}\mu_{(2)}-\frac{1}{2s}\mu_{(2)}\phi(s) \quad (5)$$

Substituting (4) and (5) into (3), we get

$$\frac{1}{s^3}\phi(s)+\frac{1}{s^2}\mu-\frac{1}{s^3}\geq\frac{1}{2s}\mu_{(2)}\phi(s),$$

which completes the proof.
Let $X_1,X_2,...,X_n$ be a random sample from a population with distribution F. Using the previous Lemma we may use $\delta(s)$ as a measure of departure from exponentiality where

$$\delta(s)=(\frac{1}{s^3}-\frac{1}{2s}\mu_{(2)})\phi(s)+\frac{1}{s^2}\mu-\frac{1}{s^3}. \quad (6)$$

It is noted that , under $H_0,\delta(s)=0$,while it is positive under H_1. The empirical estimate $\delta_n(s)$ of $\delta(s)$ can be obtained as

$$\delta_n(s)=\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}[(\frac{1}{s^3}-\frac{1}{2s}X_i^2)e^{-sX_j}+\frac{1}{s^2}X_i-\frac{1}{s^3}].$$

To make the test invarient, let $\Delta_n(s)=\frac{\delta_n(s)}{\overline{X}^2}$,where $\overline{X}=\sum_{i=1}^{n}\frac{X_i}{n}$ is the sample mean. Then

$$\Delta_n(s)=\frac{1}{n^2\overline{X}^2}\sum_{i=1}^{n}\sum_{j=1}^{n}[(\frac{1}{s^3}-\frac{1}{2s}X_i^2)e^{-sX_j}+\frac{1}{s^2}X_i-\frac{1}{s^3}]. \quad (7)$$

It is easy to show that $E[\Delta_n(s)]=\delta(s)$.
Now, set

$$\phi(X_i,X_j)=(\frac{1}{s^3}-\frac{1}{2s}X_i^2)e^{-sX_j}+\frac{1}{s^2}X_i-\frac{1}{s^3}. \quad (8)$$

The following theorem summarizes the asymptotic properties of the test statistic $\Delta_n(s)$.
Theorem (3.1)
As $n\to\infty$, $[\Delta_n(s)-\delta(s)]$ is asymptotically normal with mean 0 and variance $\sigma^2(s)/n$, where

$$\sigma^2(s)=Var\left[(\frac{1}{s^3}-\frac{1}{2s}X^2)E\left(e^{-sx}\right)+\frac{1}{s^2}X+\frac{1}{s^3}e^{-sX}-\frac{1}{2s}e^{-sX}\mu_{(2)}+\frac{1}{s^2}\mu-\frac{2}{s^3}\right]. \quad (9)$$

Under H_0, the variance reduces to

$$\sigma_0^2(s)=\frac{s+5}{(s+1)^3(2s+1)}. \quad (10)$$

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Proof.

Using standard U-statistic theory [15]

$$\sigma^2 = V\{E[\phi(X_1,X_2) \mid X_1] + E[\phi(X_1,X_2) \mid X_2]\},$$

recall the definition of $\phi(X_i,X_j)$ in (8), thus it is easy to show that

$$E(\phi(X_1,X_2) \mid X_1) = (\frac{1}{s^3} - \frac{1}{2s}X_1^2)\int_0^\infty e^{-sx}dF(x) + \frac{1}{s^2}X_1 - \frac{1}{s^3},$$

and

$$E(\phi(X_1,X_2) \mid X_2) = \frac{1}{s^3}e^{-sX_2} - \frac{1}{2s}e^{-sX_2}\int_0^\infty x^2 dF(x) + \frac{1}{s^2}\int_0^\infty x dF(x) - \frac{1}{s^3},$$

therefore

$$\sigma^2(s) = Var\left[(\frac{1}{s^3} - \frac{1}{2s}X^2)E\left(e^{-sx}\right) + \frac{1}{s^2}X + \frac{1}{s^3}e^{-sX} - \frac{1}{2s}e^{-sX}\mu_{(2)} + \frac{1}{s^2}\mu - \frac{2}{s^3}\right].$$

Under H_0

$$\sigma_0^2(s) = \frac{s+5}{(s+1)^3(2s+1)}.$$

4 The Pitman Asymptotic Efficiency (PAE)

To judge on the quality of this procedure, Pitman asymptotic efficiencies (PAEs) are computed and compared with some other tests for the following alternative distributions:

(i)The Weibull distribution: $\bar{F_1}(x) = e^{-x^\theta}, x \geq 0, \theta \geq 1$.
(ii)The linear failure rate distribution (LFR): $\bar{F_2}(x) = e^{-x-\frac{\theta}{2}x^2}, x \geq 0, \theta \geq 0$.
(iii)The Makeham distribution: $\bar{F_3}(x) = e^{-x-\theta\left(x+e^{-x}-1\right)}, x \geq 0, \theta \geq 0$.

Note that for $\theta = 1$ $\bar{F_1}$ goes to the exponential distribution and for $\theta = 0$ $\bar{F_2}$ and $\bar{F_3}$ reduce to the exponential distribution. The PAE is defined by:

$$PAE\ (\Delta_n(s)) = \frac{1}{\sigma_\circ(s)}\left|\frac{d}{d\theta}\delta(s)\right|_{\theta\to\theta_\circ}.$$

When $s = 2.5$, this leads to:

$$PAE\ [\Delta_n(2.5), Weibull] = 1.17243, \quad PAE[\Delta_n(2.5), LFR] = 0.956183 \text{ and}$$
$$PAE\ [\Delta_n(2.5), Makeham] = 0.278887, \text{where } \sigma_0(2.5) = 0.170747.$$

Table 1. Comparison between the PAE of our test and some other tests

Test	Weibull	LFR	Makeham
Kango [14]	0.132	0.433	0.144
Mugdadi and Ahmad [22]	0.170	0.408	0.039
Abdl-Aziz [1]	0.223	0.535	0.184
Mahmoud and Abdul Alim [18]	0.050	0.217	0.144
Our test $\Delta_n(2.5)$	1.172	0.956	0.279

From Table1. it is obvious that $\Delta_n(5)$ is better than the other tests based on PAEs.

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5 Monte Carlo Null Distribution Critical Points

In this section the Monte Carlo null distribution critical points of $\Delta_n(2.5)$ are simulated based on 10000 generated samples of size $n = 5(5)35, 39, 40(5)50$. from the standard exponential distribution by using Mathematica 8 program.
Table 2. gives the upper percentile points of statistic $\Delta_n(2.5)$ for different significance levels $\alpha = 0.1,\ 0.5$ and 0.01.

Table 2. Critical values of the statistic $\Delta_n(2.5)$

n	90%	95%	99%
5	0.0673307	0.0729016	0.0800246
10	0.0547539	0.0607981	0.0701405
15	0.0472103	0.0536102	0.0629472
20	0.0423262	0.0486695	0.0570500
25	0.0382311	0.0443267	0.0544547
30	0.0357435	0.0414481	0.0506311
35	0.0332073	0.0391771	0.0480114
39	0.0320280	0.0373616	0.0460094
40	0.0313255	0.0371479	0.0458000
45	0.0300729	0.0356608	0.0441959
50	0.0280767	0.0334059	0.0420647

From Table 2, it is obvious that the critical values are decreasing as the samples size increasing and they are increasing as the confidence levels increasing.

6 Power Estimates of The Test $\Delta_n(2.5)$

In this section the power of our test $\Delta_n(2.5)$ will be estimated at $(1-\alpha)\%$ confidence level, $\alpha = 0.05$ with suitable parameters values of θ at $n = 10, 20$ and 30 with respect to three alternatives Weibull, linear failure rate (LFR) and Gamma distributions based on 10000 samples.

Table 3. The power estimates of $\Delta_n(2.5)$

n	θ	Weibull	LFR	Gamma
10	2	0.8043	0.2541	0.2737
	3	0.9972	0.2850	0.7407
	4	0.9999	0.2821	0.9149
20	2	0.9851	0.4897	0.5689
	3	1.0000	0.5571	0.9502
	4	1.0000	0.5892	0.9976
30	2	0.9996	0.6864	0.7393
	3	1.0000	0.7682	0.9929
	4	1.0000	0.8033	0.9997

Table 3. shows that the power estimates of our test $\Delta_n(2.5)$ are good for all alternatives and it increases when the value of the parameter θ and the sample sizes increase.

7 Applications to Real Data

In this section, we apply our test to some real data-sets at 95% confidence level.
Application 1: Consider the data in [19] which represent 39 liver cancers patients taken from Elminia cancer center Ministry of Health – Egypt, which entered in (1999). The ordered life times (in days)

10	14	14	14	14	14	15	17	18	20
20	20	20	20	23	23	24	26	30	30
31	40	49	51	52	60	61	67	71	74
75	87	96	105	107	107	107	116	150	

In this case, $\Delta_n(2.5) = 0.0032709$ which is less than the corresponding critical value in Table 2, it is evident to accept H_0, which states that the data set have exponential property.
Application 2: Consider the real data-set given in [12] and have been used in [23]. This data set gives the times between arrivals of 25 customers at a facility.

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1.80	2.89	2.93	3.03	3.15	3.43	3.48	3.57	3.85	3.92
3.98	4.06	4.11	4.13	4.16	4.23	4.34	4.37	4.53	4.62
4.65	4.84	4.91	4.99	5.17					

Since $\Delta_n(2.5) = 0.0361609$ and this value is greater than the corresponding critical value in Table 2, then we conclude that this data have not the exponential property i.e it has the $NBRUL$ property.

8 Conclusion

A new class of life distribution is added to the family of renewal classes of life distribution and called (NBRUL). Quality criteria of the test is shown by the famous criterion which is Pitman asymptotic efficiency. The upper percentiles and the power of the proposed test are calculated and tabulated. Our test is applied to some real data to show the usefulness of the test .

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