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Fixed Point Theorems of Multi-value Map and Its Variation Iteration

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Abstract: In this paper, we'll give some new conclusion of fixed point related to the multi-valued map, the calculation of fixed point index, and the solutions to nonlinear equations by using VIM. We apply the effective modification of its variation iteration method to find solution to a class of integral-differential equations. Lots of examples will illustrate the effectiveness and convenience of this method.

Keywords: Operator equation; nonlinear equation; He's iterative method; initial boundary value; multi-valued map.

1 Introduction

In recent years, the fixed point theory and application has rapidly development. First, we need following some definitions and conclusion (see [3]). We always suppose that E is a Banach space, and P is a cone in E with $\operatorname{int} P \neq \Phi$ and $\leq$ is partial ordering with respect to P.

Definition 1.1 Let X be a non-empty set in E, and suppose the mapping $d: X \times X \to E$ satisfies,

(i) $0 \leq d(x, y)$ for any $x, y \in X$, and $d(x, y) = 0$ $\Leftrightarrow x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

then d is called cone distance on X, (X, d) is called a cone metric space.

It is obvious that cone metric space generalize the metric spaces.

Example 1.2 Let $E = R^2, P = \{(x, y) \in R \mid x, y \geq 0\} \subset R^2$, $X = R$ and $d: X \times X \to E$, such that

$$d(x, y) = (|x - y|, \alpha|x - y|),$$

where $\alpha \geq 0$ is a constant. Then this (X, d) is a cone metric space.

where $d(a, B) = \inf\{d(a, b), b \in B\}$ is the distance from point a to the sub-set B. An element $x \in X$ is said to be a fixed point of a multi-valued mapping $T: X \to 2^X$, if $x \in T(X)$.

Definition 1.3 Let (X, d) be a cone metric space, $x_n \in X$, and a sequence $\{x_n\}$ in X, then

(i) $\{x_n\}$ convergences to x whenever for every $c \in E$ with $0 << c$ there is a nature number N such that $d(x_n, x) << c$ for all $n \geq N$.

(ii) $\{x_n\}$ is a Cauchy sequence whenever for every $c \in E$ with $0 << c$ there is a nature number N such that $d(x_n, x_m) << c$ for all $n, m \geq N$.

Now, we give main results of this paper to fixed point problem of multi-value map in following section.

Definition 1.4 Let (X, d) is said to be a complete cone metric space, if every Cauchy sequence is convergent in X.

Let (X, d) be a metric space. We denote by $CB(X)$ the family of non-empty closed bounded subset of X. Let $H(\cdot,\cdot)$ be the Hausdorff metric on $CB(X)$. That is for

$$H(A, B) = \max\left\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b)\right\}.$$

For convenience, we first give the needing lemma as following conclusion (see the proof of theorem 2.1 [3]).

Lemma A. (see theorem 2.1[3]) Suppose that (X,d) be a cone metric space and the mapping $T:X\to CB(X)$ be the multi-value map satisfying for the sequence $\{x_n\}$ in (X,d) with following conditions hold:

$$d(x_{n+1},x_n)\le Ld(x_n,x_{n-1}), L=(a+b)/(1-a-b)<1,$$

that is

$$d(x_{n+1},x_n)\le L^n d(x_1,x_0).$$

then the sequence $\{x_n\}$ is a Cauchy sequence in (X,d).

2 Main results

In the present paper, we easy extend fixed point theorem in [3] as Theorem 2.1-2.3.

Theorem 2.1 Let (X,d) be a complete cone metric space and the mapping $T:X\to CB(X)$ be multi-valued map satisfying,

$$H(Tx,Ty)\le a\left(d(x,Tx)+d(y,Ty)\right)+b(d(x,Ty)$$
$$+d(y,Tx))+\alpha d(x,Ty)\left(1+d^2(x,Ty)\right)^{-1}d(x,y)$$

for each $x,y\in X$ (2.1)

then T has a unique fixed point in X.

Proof. For every $x_0\in X$ and $n\ge 1, x_1\in Tx_0$, $x_{n+1}\in T(x_n)$, by (1), we have

$$d(x_{n+1},x_n)\le H\left(Tx_n,Tx_{n-1}\right)$$
$$\le a(d(x_n,Tx_n)+d(x_{n-1},Tx_{n-1}))$$
$$+b\left(d(x_n,Tx_{n-1})+d(x_{n-1},Tx_n)\right)$$
$$+\left(\frac{\alpha d(x_n,Tx_{n-1})}{1+d^2(x_n,Tx_{n-1})}\right)d(x_n,x_{n-1})$$
$$\le a\left(d(x_n,x_{n+1})+d(x_{n-1},x_n)\right)$$
$$+b\left(d(x_{n-1},x_n)+d(x_n,.x_{n+1})\right).$$

Therefore, we have

$$(1-a-b)d\left(x_{n+1},x_n\right)\le(a+b)d\left(x_n,x_{n-1}\right).$$

Let $L=(a+b)/(1-a-b)$, then

$$d\left(x_{n+1},x_n\right)\le Ld\left(x_n,x_{n-1}\right)$$

and

$$d\left(x_{n+1},x_n\right)\le L^n d\left(x_1,x_0\right).$$

By lemma A, we known sequence $\{x_n\}$ is a Cauchy sequence in (X,d). Because (X,d) is a complete cone metric space, then there exists $p\in X$ such that $x_n\to p$ .We will prove this $p\in X$ is the fixed point of T, that is $p\in Tp$.

In fact, by (1) and from

$$d(Tp,p)\le H(Tx_n,Tp)+d(Tx_n,p)$$
$$\le a\left(d(x_n,Tx_n)+d(Tp,p)\right)+b(d(x_n,Tp)$$
$$+d(Tx_n,p))+d(x_{n+1},p)$$
$$\le a\left(d(x_n,x_{n+1})+d(Tp,p)\right)+b(d(x_n,Tp)$$
$$+d(x_{n+1},p))+d(x_{n+1},p)$$
$$\le a\left(d(x_n,x_{n+1})+d(Tp,p)\right)+b(d(x_n,p)$$
$$+d(p,Tp)+d(x_{n+1},p))+d(x_{n+1},p)$$

So,

$$(1-a-b)d(Tp,p)\le(a+b)d(x_n,p)$$
$$+(a+b)d(x_{n+1},p)+d(x_{n+1},p)$$

Let $k=a+b, k'=1-k$,

$$k'd(Tp,p)\le kd(x_n,p)+kd(x_{n+1},p)+d(x_{n+1},p).$$

Therefore, from $d(x_n,p)<<c(1-L)/3$, and $d(x_n,p)<<c(1-k)/3$, we have

$$d(Tp,p)\le\frac{1}{1-k}\left(d(x_n,p)+d(x_{n+1},p)+d(x_{n+1},p)\right),$$
$$d(Tp,p)<<c.$$

And all $n > N_2$, $d(Tp,p) \le c/m$ for all $m \ge 1$, we get $(c/m) - d(Tp,p) \in P$, and that as $m \to \infty$ we get $c/m \to 0$ and P is closed that $-d(Tp,p) \in P$ but $d(Tp,p) \in P$, hence $d(Tp,p) = 0$, and so $p \in Tp$.

Then T has a unique fixed point in X.

Remark. Let $\alpha = 0$, that is theorem 2.1 in [3].

Theorem 2.2 Let (X,d) be a complete cone metric space and the mapping $T: X \to CB(X)$ be multi-valued map satisfying for each $x, y \in X$,

$$H(Tx,Ty) \le r\max\{d(x,Ty)d(x,y)/(1+d^2(x,Ty)),$$
$$d(x,Tx), d(y,Ty), d(Tx,Ty), d(x,Ty), d(y,Tx)\}$$

for all $x, y \in X$, and $r \in [0,1)$ (2.2)

Then T has a unique fixed point in X.

Proof. For every $x_0 \in X$ and $n \ge 1, x_1 \in Tx_0$, $x_{n+1} \in T(x_n)$, by (2.2) we get that

$$d(x_{n+1},x_n) \le H(Tx_n,Tx_{n-1})$$
$$\le r\max\{d(x_n,Tx_{n-1})(1+d^2(x_n,Tx_{n-1}))^{-1}d(x_n,x_{n-1}),$$
$$d(x_{n-1},Tx_{n-1}), d(Tx_{n-1},Tx_n), d(x_n,Tx_{n-1}), d(x_{n-1},Tx_n)\}$$
$$\le r\max\{d(x_{n-1},x_n), d(x_n,x_{n+1}), d(x_{n-1},x_{n+1})\}.$$

Case (i) If $d(x_{n+1},x_n) \le rd(x_n,x_{n-1})$, then

then we get $d(x_{n+1},x_n) \le r^n d(x_1,x_0)$.

For $n > m$,

$$d(x_n,x_m) \le d(x_{n-1},x_{n-2}) + d(x_{n-1},x_{n-2}) + \cdots + d(x_{m+1},x_m)$$
$$\le (r^{n-1} + r^{n-2} + \cdots + r^m) d(x_1,x_0)$$
$$\le r^m(1-r)^{-1}d(x_1,x_0).$$

We get

$$\|d(x_n,x_m)\| \le Kr^m(1-r)^{-1}\|d(x_1,x_0)\|.$$

As $n,m \to \infty$, $d(x_n,x_m) \to 0$. Hence $\{x_n\}$ is a Cauchy sequence. By the completeness of X, there is $p \in X$ such that $x_n \to p$ as $n \to \infty$.

$$d(Tp,p) \le H(Tx_n,Tp) + d(Tx_n,p)$$
$$\le r\max\{2^{-1}d(x_n.p), d(x_n,x_{n+1}), d(Tp,p),$$
$$d(x_n,Tp), d(x_{n+1},p)\} + d(x_{n+1},p)$$
$$\le rd(Tp,p).$$

That is $d(Tp,p) = 0$. Hence $p \in Tp$.

Case (ii) If $d(x_{n+1},x_n) \le rd(x_n,x_{n+1})$,

Clearly, here is a contrary.

Case (iii) If $d(x_{n+1},x_n) \le rd(x_{n+1},x_{n-1})$,

then we get

$$d(x_{n+1},x_n) \le r(d(x_{n+1},x_n) + d(x_n,x_{n-1}))$$
$$d(x_{n+1},x_n) \le (r/(1-r))d(x_n,x_{n-1})$$
$$d(x_{n+1},x_n) \le h(d(x_n,x_{n-1})),$$

where $h = r(1-r)^{-1} < 1$.

For $n > m$,

$$d(x_n,x_m) \le d(x_{n-1},x_{n-2}) + d(x_{n-1},x_{n-2}) + \cdots + d(x_{m+1},x_m),$$
$$\le (h^{n-1} + h^{n-2} + \cdots + h^m) d(x_1,x_0)$$
$$\le (h^m/(1-h)) d(x_1,x_0).$$

We get

$$\|d(x_n,x_m)\| \le K(h^m/(1-h))\|d(x_1,x_0)\|.$$

As $n,m \to \infty$, $d(x_n,x_m) \to 0$. Hence $\{x_n\}$ is a Cauchy sequence. By the completeness of X, there is $p \in X$ such that $x_n \to p$ as $n \to \infty$.

$$d(Tp,p) \le H(Tx_n,Tp) + d(Tx_n,p)$$
$$\le r\max\{d(x_n,Tp)(1+d^2(x_n,Tp))^{-1}d(x_n,p),$$
$$d(p,Tp), d(x_n,Tp), d(Tx_n,p)\} + d(Tx_n,p)$$
$$\le r\max\{2^{-1}d(x_n.p), d(x_n,x_{n+1}), d(Tp,p),$$

$d(x_{n+1},Tp), d(x_n,Tp), d(x_{n+1},p)\} + d\left(x_{n+1},p\right)$

$\leq rd\left(Tp,p\right).$

Then $d\left(Tp,p\right)=0,$ hence $p\in Tp$.

(Uniqueness) If there is another fixed point $q\in Tq$, then by (2.3) we have

$$d(p,q)\leq H(Tp,Tq)$$

$$\leq r\max\{d(p,Tq)(1+d^2(p,Tq))^{-1}d(p,q), d(p,Tp), d(q,Tq), d(Tp,Tq), d(p,Tq), d(q,Tp)\}$$

Hence $d(p,q)\leq rd\left(p,q\right)$, this is a contradiction.

Hence T has a unique fixed point in X.

Theorem 2.3 Let (X,d) be a cone metric space and let $\alpha\in\theta$ which satisfies $\alpha(t_n)\to 1(t_n\to 0)$ and T be multi-value map on X, and Tx is non-empty closed subset of X and a regular cone, for each $x\in X$ hold form,

$$H(T^kx,T^ky)\leq\alpha(d(x,y))d(x,y) \qquad (2.3)$$

For each $x,y\in X$, and $k>0$ is a integer.

Then T have a unique fixed point in X. (We omit the similar proof).

Remark. Let $k=1$ that is special case in [3].

3 Some new conclusion

The topological degree theory and fixed point index theory play an important role in the study of fixed points for various classes of nonlinear operator equations in Banach spaces (see [5] and [10]) and application in more fields.Now, we may extend the famous Rothe's theorem (see theorem 3 in [5]) as following theorem form.

Lemma B Assume integer $m\geq 1, t>1$ and $p>1$, the following inequality holds,

$$t^{mp}+(t+1)^{m+p}>(m+1)(t-1)^{mp}+1.$$

Proof. Let

$$f(t)=t^{mp}-1-(m+1)(t-1)^{mp}+(t+1)^{m+p}, t>1, m\geq 1,$$

then we have

$$f'(t)=mpt^{mp-1}-(m+1)mp(t-1)^{mp-1}+mp(t+1)^{mp-1}$$
$$=mp\left(t^{mp-1}-(m+1)(t-1)^{mp-1}+(t+1)^{mp-1}\right).$$

This is because

$$\frac{t^{mp-1}+(t+1)^{mp-1}}{-(m+1)(t-1)^{mp-1}}=\frac{-1}{m+1}\left(\left(\frac{t-1+1}{t-1}\right)^{mp-1}-\left(1+\frac{2}{t-1}\right)^{mp-1}\right)$$
$$<\frac{-1}{m+1}\left(\left(1+\frac{1}{t-1}\right)^{mp-1}-\left(1+\frac{1}{t-1}\right)^{mp-1}\right)$$
$$=-\left(\frac{2}{m+1}\right)\left(1+\frac{1}{t-1}\right)^{mp-1}<0.$$

According to $-2\left(t-1\right)^{mp-1}<0$, we obtain that,

$$t^{mp-1}+\left(1+t\right)^{mp-1}>(m+1)\left(t-1\right)^{mp-1}.$$

That is,

$$t^{mp-1}-(m+1)(t-1)^{mp-1}+(1+t)^{mp-1}>0.$$

Then $f'(t)>0$. Therefore $f(t)$ is a strictly monotone increasing function in $[1,+\infty)$. When $t>1$, we have $f(t)>f(1)=2^{mp}>0$.

That is,

$$t^{mp}+\left(t+1\right)^{m+p}>(m+1)\left(t-1\right)^{mp}+1,$$

where integer $m\geq 1$ and $t>1, p>1$.

Theorem 3.1 Let D be a bounded open convex subset in a real Banach space E and $\theta\in D$; Suppose that $A:\overline{D}\to E$ is a semi-closed 1-set-contractive operator such that

$$(m+1)\|Ax-x\|^{mp}-\|x+Ax\|^{mp}\geq\|Ax\|^{mp}-\|x\|^{mp}, \text{ for every } x\in\partial D, p>1. \qquad (3.1)$$

Then the operator equation $Ax=x$ has a solution in D.

Proof. By (3.1), we know that $Ax=x$ has no solution in ∂D, that is,

$$x\neq Ax, \text{ for every } x\in\partial D. \qquad (3.2)$$

We prove

$$x \neq tAx, \text{ for every } t \in (0,1), \text{for every } x \in \partial D. \quad (3.3)$$

In fact, suppose that (3.3) is not true, that is here exists a $t_0 \in (0,1)$ an $x_0 \in \partial D$ such that $x_0 = t_0 Ax_0$, that is $x_0 = t_0^{-1} Ax_0$.

Inserting $x_0 = t_0^{-1} Ax_0$ into (3.1), we obtain

$(m+1)\left\|\frac{x_0}{t_0} - x_0\right\|^{mp} - \left\|\frac{x_0}{t_0} + x_0\right\|^{mp} \geq \left\|\frac{x_0}{t_0}\right\|^{mp} - \|x_0\|^{mp}$, where $p > 1, x_0 \in \partial D.$

$$(m+1)\left\|\frac{1}{t_0} - 1\right\|^{mp} \|x_0\|^{mp} - \left\|\frac{1}{t_0} + 1\right\|^{mp} \|x_0\|^{mp} \geq \left\|\frac{1}{t_0}\right\|^{mp} \|x_0\|^{mp} - \|x_0\|^{mp},$$
$$p > 1, x_0 \in \partial D. \quad (3.4)$$

Therefore, (3.4) gives that

$$(m+1)\left(\frac{1}{t_0} - 1\right)^{mp} - \left(1 + \frac{1}{t_0}\right)^{mp} \geq \left(\frac{1}{t_0}\right)^{mp} - 1. \quad (3.5)$$

Let $t = t_0^{-1}$ as $t_0 \in (0,1)$, we obtain that $t > 1$. Hence (3.5) gives that

$$(m+1)(t-1)^{mp} - (1+t)^{mp} \geq (t)^{mp} - 1. \quad (3.6)$$

That is,

$$(m+1)(t-1)^{mp} + 1 \geq t^{mp} + (1+t)^{mp}.$$

This is a contradiction to lemma B. Thus,

$$x \neq tAx, \text{ for every } x \in \partial D, t \in (0,1). \quad (3.7)$$

By (3.2), (3.7), we know that $x \neq tAx$, $t \in (0,1]$, for every $x \in \partial D$.

By Ref.[1], we obtain that $i(A, D, X) = 1$. Then the operator equation $Ax = x$ has a solution in D. We can prove that the following theorem 3.2 holds which extends the famous Rothe's theorem (see theorem 3 in [5]) as following form along an analogous way.

Theorem 3.2(see theorem 3 in [5]) Let D be a bounded open convex subset in a real Banach space X, and $\theta \in D$; Suppose that $A : \overline{D} \to E$ is a semi-closed 1-set-contractive operator and satisfies the following condition

$$\|Ax - x_0\| \leq \|x - x_0\| \quad \text{for } \forall x \in \partial D \text{ and } x_0 \in D. \quad (3.8)$$

then the operator equation $Ax = x$ has a solution in D (we omit the proof). To illustrate the application of the obtained results, we consider the examples.

Example 1 (see example 1 in [5]) Let us consider the following integral equation which comes from information science and applied mathematics

$$\int_0^x \left(\frac{1}{2}\sin|t| + \frac{1}{4}\cos|t|\right)dt - x + 2.1 = 0, \forall x \in [-\pi, \pi] \quad (3.9)$$

It is easy to prove that this equation has a solution in $[-\pi, \pi]$.

Example 2 Similar as example 1, we consider integral equation,

$$\int_0^x \left(\frac{1}{3}\sin|t| + \frac{1}{6}\cos|t|\right)dt - x + 2.1 = 0, \forall x \in [-\pi, \pi] \quad (3.10)$$

It is easy to prove that this equation has a solution in $[-\pi, \pi]$.

In fact, let

$$Ax = \int_0^x \left(\frac{1}{3}\sin|t| + \frac{1}{6}\cos|t|\right)dt + 2.1, \forall x \in [-\pi, \pi],$$

and that $D = [-\pi, \pi], \partial D : x = \pm\pi.$

We write $\|y\| = |y|$, for every $y \in R.$

Thus, we have

$$|A(-\pi) - 2.1| = \left|\int_0^{-\pi} \left(\frac{1}{3}\sin|t| + \frac{1}{6}\cos|t|\right)dt\right|$$

$$\leq \int_{-\pi}^0 \left(\frac{1}{3}\sin|t| + \frac{1}{6}\cos|t|\right)dt$$

$$\leq \int_{-\pi}^0 \left(\frac{1}{3} + \frac{1}{6}\right)dt = \frac{\pi}{2} < |-\pi - 2.1| = \pi + 2.1.$$

And $|A(\pi)-2.1|=\left|\int_0^{\pi}\left(\frac{1}{3}\sin|t|+\frac{1}{6}\cos|t|\right)dt\right|$

$$=\left|\frac{1}{3}(-\cos t)\Big|_0^{\pi}\right|=\frac{2}{3}<|\pi-2.1|=\pi-2.1.$$

It follows that $|Ax-2.1|\le|x-2.1|$ for $\forall x\in\partial D$.

Meanwhile, A is semi-closed 1-set-contractive operator similar example 1 by theorem 3.2 that we obtain the $Ax=x$ has a solution in $[-\pi,\pi]$. That is, Eq.(3.11) has a solution in $[-\pi,\pi]$.

Remark. Let us consider the integral equation,

$$\int_0^{x}\left(\frac{1}{3n}\sin|t|+\frac{1}{6n}\cos|t|\right)dt-x+2.1=0,\forall x\in[-\pi,\pi].$$

It is easy to prove these equations have solution in $[-\pi,\pi]$. $n=1,2,\cdots,$ more case.

4 Note the effective modification

Recently, the variation method has been favorably applied to some kinds of nonlinear problems, for example, fractional differential equations, nonlinear differential equations, nonlinear thermo-elasticity, nonlinear wave equations etc. In this section, we apply the effective modification method of He's VIM to solve Integral-differential equations (see [6]). This method is a modification of the general Lagrange multiplier method, and this method an iterative, which is by correction functional.

To illustrate its basic idea of the method, we consider the following general nonlinear system

$$Lu+Ru+Nu=g(x), \tag{4.1}$$

Lu is the highest derivative and is assumed easily invertible, R is a linear differential operator of order less than L, Nu represents the nonlinear terms, and g is the source term. Applying the inverse operator L_x^{-1} to both sides of Eq. (4.1), and we obtain

$$u=f-L_x^{-1}[Ru]-L_x^{-1}[Nu], \tag{4.2}$$

The basic character of He's method is to construct a correction functional for equation (4.1), which reads,

$$u_{n+1}(t)=u_n(t)+\int_0^{t}\lambda(s)\big(Lu_n+Ru_n+Nu_n-g(s)\big)ds, \tag{4.3}$$

Or equivalently,

$$u_{n+1}(x)=f(x)-L_x^{-1}[Ru_n]-L_x^{-1}[Nu_n]. \tag{4.4}$$

According to the assumption (4.4),

$$f(x)=f_0(x)+f_1(x),$$

we construct following variation iterative formula (see [6]).

$$\begin{cases}u_0(x)=f_0(x),\\ u_1(x)=f(x)-L_x^{-1}[Rf_0(x)]-L_x^{-1}[Nf_0(x)],\\ u_{n+1}(x)=f(x)-L_x^{-1}[Ru_n(x)]-L_x^{-1}[Nu_n(x)].\end{cases} \tag{4.5}$$

Assuming the Lagrange multiplier λ has been identified.

Example 3(similar as example 1 in [3]) Consider the following nonlinear Fredholm integral equation

$$\begin{cases}u'(x)-u(x)=x\cos 3x-x\sin 3x-\sin 3x,\\ u(0)=0.\end{cases} \tag{4.6}$$

Applying the inverse operator L_x^{-1} to both side of equation (4.6) yields,

$$u(x)=x\sin 3x+\frac{x}{3}\cos 3x-\frac{1}{9}\sin 3x+L_x^{-1}\big(u(x)\big).$$

The modified method: we first divide f into two parts defined by selecting $u_0=f_0$,

$$u_0(x)=f_0(x)=x\sin x,$$

$$f_1(x)=\frac{x}{3}\cos 3x-\frac{1}{9}\sin 3x.$$

$$L_x^{-1}(x\sin x)=-\int_0^{x}xd\cos x$$

$$=-\left(x\cos x\Big|_0^{x}-\int_0^{x}\cos xdx\right)$$

$$=\sin x-x\cos x$$

$$L_x^{-1}(x\sin 3x)=-\frac{1}{3}\int_0^{x}xd\cos 3x$$

$$=\frac{1}{9}\sin 3x-\frac{x}{3}\cos 3x .$$

$$u_1(x)=x\sin 3x+\frac{x}{3}\cos x-\frac{1}{9}\sin 3x+L_x^{-1}\left(f_0(x)\right)$$

$$u_1(x)=x\sin 3x, n\geq 1.$$

By recursive calculating $u_2(x)=x\sin 3x$.

Then inductively, we have that

$$u_{n+1}(x)=x\sin 3x+\frac{x}{3}\cos x-\frac{1}{9}\sin 3x+L_x^{-1}\left(u_n(x)\right),$$

we have $u_{n+1}(x)=x\sin 3x, n\geq 1.$

The exact solution is given by $u(x)=x\sin 3x$

Example 4 Consider the following Fredholm integral equation

$$u(x)=x-\frac{5\pi}{12}+\int_0^1\left(\frac{\sqrt{3}}{3+u^2(t)}\right)dt+\int_0^1\left(\frac{1}{1+u^2(t)}\right)dt, \tag{4.7}$$

where taking $u_0(x)=f_0(x)=x,$

$$f_1(x)=-\left(\frac{\pi}{6}+\frac{\pi}{4}\right)=-\frac{5\pi}{12},$$

by recursive relationship as in same way, we have

$$f(x)=f_0(x)+f_1(x)=x-\frac{5\pi}{12},$$

$$\arctan\left(\frac{1}{\sqrt{3}}\right)=\frac{\pi}{6}, \arctan(1)=\frac{\pi}{4}.$$

So, by simple calculating that

$$u_1(x)=x-\frac{5\pi}{12}+\int_0^1\frac{\sqrt{3}}{3+f_0^2(t)}dt+\int_0^1\frac{1}{1+f_0^2(t)}dt=x,$$

and

$$u_{n+1}(x)=x-\frac{5\pi}{12}+\int_0^1\frac{\sqrt{3}}{3+u_n{}^2(t)}dt+\int_0^1\frac{1}{1+u_n^2(t)}dt=x,$$

$n\geq 1.$

Then $u(x)=x$ is evident exact solution of (4.7) by only one iteration leads to a solution.

Example 5 (see example 5 in [13]) Consider the following partial differential equation

$$u_t+uu_x=xt^k+e^x, \tag{4.8}$$

where that $k\geq 1, (or\ k>0)$.

By selecting $u_0=f_0=xt^k$, $f_1=(2k+1)xt^{2k+1}$ and

$$f(x)=f_0(x)+f_1(x), u_1(x,t)=xt^k,\cdots,u_n(x,t)=xt^k.$$

The exact solution is given by $u(x,t)=xt^k$. (And we omit that detail case, see [13])

Remark. Let $k=1$ that is example 2 in [6].

By conclusion 4.1 in [14],

$$\begin{cases} u''(t)+k^2u(t)\in f(t,u(t)), \\ u(0)=u(1), u'(0)=u'(1). \end{cases}$$

has at least one positive solution (under suite conditions, by fixed point Theorem of multi-valued operators for theorem 3.1 [14][15][16]

5. Concluding Remarks

We have expounded theorem 2.1 and 2.4 in [3] with multi-valued mappings in cone metric Spaces and inequality for some calculation of fixed point index (see[10]) in Banach space.

Because of the effectiveness of MIV, He's method obtains some exact solution to integral-differential equation which shows these methods can be used in more fields.

Recently, schrodinger equation is also a very interesting topic, and we can find it in [8] etc. In our future work, we may try to do some research in this field and may obtain some good results.

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